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SOME NILPOTENT H-SPACES

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0. Introduction

In this note we give two generalisations, (Proposition 1.2 & Theorem 1.3), of Stasheff's criterion for homotopy commutativity of H-spaces, [11, Theorem 1.9], and apply them to produce examples of nilpotent H-spaces and to demonstrate the vanishing of certain Samelson-Whitehead products.

In §1.2 we give a necessary and sufficient condition for the vanishing of the Samelson-Whitehead product of $f:SA\to Y$ and $g:SB\to Y$. In Theorem 1.3 a criterion for the vanishing of the j-th iterated commutator map in an H-space, X, is given in terms of a space, $X(j)$. As a corollary it is shown that if the projective plane of X, (resp. the space X), has a finite Postnikov system then X, (resp. ΩX), is nilpotent. In §2 the nilpotency of loop spaces of spheres and projective spaces is discussed. Many of the results of §2 are known to other authors and I am grateful to G.J. Porter for drawing my attention to the results of T. Ganea, [3]. However, for completeness, the results of [3] have been included here, as corollaries of Proposition 1.2. The nilpotency of ΩS^{2n} and ΩCP^{2n} do not appear in [3] although the former was previously known to M.G. Barratt, I. Berstein and T. Ganea. Since our estimate of the nilpotency of ΩCP^{2n} is large we include a corollary of Theorem 1.3 on the vanishing of a family of triple Samelson-Whitehead products on CP^{2n}.

I am grateful to Peter Jupp for helpful conversations about homotopy operations.

In this paper we work in the category of based, countable CW complexes. A connected complex in this category is called special. The following notation is used:—

$X\wedge Y$=smash product of X and Y.,

$\overset{j}{\vee}X$, $\overset{j}{\wedge}X$ and X^j are respectively the j-fold wedge, smash and product of X,

$I=[0, 1]$ with basepoint, $*=0$,

$SX=S^1\wedge X$, ΩX=the space of loops on X,

and (eval: $S\Omega X\to X$)=the evaluation map.

1. Let X be a homotopy associative H-space and let ϕ_2: $X\times X\to X$ be the cummutator map.

DEFINITION 1.1. For $(n>2)$ put ϕ_n: $X^n=X^{n-1}\times X\to X$ be $\phi_2\circ(\phi_{n-1}\times 1)$, then the *nilpotency of* X is the least integer, n, such that ϕ_{n+1} is nullhomotopic. Nilpotency of X is denoted by nil (X).

Proposition 1.2. *Let f: $A\to\Omega Y$ and g: $B\to\Omega Y$ be maps. Then $\phi_2\circ(f\times g)$: $A\times B\to\Omega Y$ is nullhomotopic if and only if $adj(f)\vee adj(g)$: $SA\vee SB\to Y$ ($adj=$ adjoint) extends to a map $SA\times SB\to Y$.*

Theorem 1.3. *For $(j\geq 2)$ there exist complexes, $X(j)$, and inclusions i_j: $\overset{j}{\vee} SX\to X(j)$ satisfying the following properties.*

(i) $X(2)=SX\times SX$.

(ii) $X(j)/(\overset{j}{\vee}SX)\cong S^2\wedge(\overset{j}{\wedge}X)$.

(iii) *If $(fold)_j$: $\overset{j}{\vee}SX\to\overset{j-1}{\vee}SX$ is the map which folds the j-th factor onto the $(j-1)$-st factor there is a commutative diagram*

$$\begin{array}{ccc} X(j) & \xrightarrow[\Gamma_j]{} & X(j-1) \\ {\scriptstyle i_j}\uparrow & & \uparrow{\scriptstyle i_{j-1}} \\ \overset{j}{\vee}SX & \xrightarrow[(fold)_j]{} & \overset{j-1}{\vee}SX \end{array} \qquad (j>2) .$$

(iv) $\Gamma_j^*(H^*(X(j-1), \overset{j-1}{\vee}SX; \pi))=0$, $(j>2)$.

(v) *There exists a map Δ_j: $(X\times Y)(j)\to X(j)\times Y(j)$, $(j\geq 2)$, such that the k-th factor $S(X\times Y)$ is mapped to (k-th SX)$\times$(k-th SY) by*

$$\Delta_j\circ i_j([t, x, y]) = (i_j[t, x], i_j[t, y]) ,$$
$$(t\in I, x\in X, y\in Y) .$$

(vi) *If X is an H-space let $XP(2)$ be the projective plane of X and w; $X\to\Omega XP(2)$ be the H-map of* [11, *Proposition* 3.5.]. *If ϕ_j is nullhomotopic then $\overset{j}{\vee}adj(w)$: $\overset{i}{\vee}SX\to XP(2)$ extends over $X(j)$. The converse is true if X is homotopy associative and right translation is a homotopy equivalence.*

(vii) *The commutator ϕ_j: $(\Omega Y)^j\to\Omega Y$ is nullhomotopic if and only if*

$$\overset{j}{\vee}(eval)\colon \overset{j}{\vee}S\Omega Y\to Y$$

extends over $(\Omega Y)(j)$.

REMARK 1.4. Proposition 1.2 and Theorem 1.3 are generalisations of Stasheff's criterion for homotopy commutativity, [11, Theorem 1.9], which is

Theorem 1.3 with $j=2$. The proof of Proposition 1.2 will be omitted. It may be proved by the same method as [11, Theorem 1.9] or deduced from [13, Theorem 7] and [11, Propositions 3.5, 4.2] and is closely related to [14, Theorem 3]. The proof of Theorem 1.3 is postponed to §3. Of course, Theorem 1.3 has a minor generalisation to give a criterion for maps $\phi_n \circ (\prod_1^n f_i)\colon \prod_1^n A_i \to X^n \to X$ to be nullhomotopic.

Corollary 1.5. (i) *If right translation is a homotopy equivalence in X and $XP(2)$ has only n non-trivial homotopy groups then $nil(X) \leq n$.*
(ii) *If Y has only n non-trivial groups then $nil(\Omega Y) \leq n$.*

Proof. (i) Since $w\colon X \to \Omega XP(2)$ is an H-map, if $f\colon XP(2) \to E_1$ is the map to the first space in the Postnikov system, Proposition 1.2 implies an extension of

$$\overset{2}{\vee} adj(\Omega f) \circ w\colon \overset{2}{\vee} SX \to E_1$$

to $(SX)^2$. Hence the result follows, by induction up the Postnikov system, using composition with Γ_j to kill the obstructions.
Part (ii) is proved similarly.

2. In this section we consider H-spaces, ΩX. The commutator $\phi_2\colon (\Omega X)^2 \to \Omega X$ induces a map, also denoted by ϕ_2, $\phi_2\colon \overset{2}{\wedge} \Omega X \to \Omega X$. The Samelson-Whitehead operation derived from ϕ_2, [2, §4.2], will be denoted by

$$[\_, \_]\colon [A, \Omega X] \times [B, \Omega X] \to [A \wedge B, \Omega X]\,.$$

An element of $[SA, X]$ and its adjoint in $[A, \Omega X]$ will be denoted by the same symbol.

Proposition 2.1 [15; 16, Example 1.3].
If X is a special complex then $S\Omega SX \simeq \bigvee_{k=1}^{\infty} S(\overset{k}{\wedge} X)$.

Proof. From [8, §5] we have a homotopy equivalence, $\Omega SX \simeq X_\infty$, where X_∞ is the reduced product of X. In the notation of [8, §1] if X^m is the m-fold product of X let X_m denote its image in X_∞. The canonical map $X^m \to \overset{m}{\wedge} X$ factors through X_m and sends X_{m-1} to the basepoint, inducing a homeomorphism $X_m/X_{m-1} \simeq \overset{m}{\wedge} X$. The map $X_m \to (\overset{m}{\wedge} X) = (\overset{m}{\wedge} X)_1 \subset (\overset{m}{\wedge} X)_\infty$ has a continuous combinatorial extension, [8, §1.4], $\pi\colon X_\infty \to (\overset{m}{\wedge} X)_\infty$. Define π_m as the composition

$$S(X_\infty) \xrightarrow[S(\pi)]{} S((\overset{m}{\wedge} X)_\infty) \simeq S\Omega S(\overset{m}{\wedge} X) \xrightarrow[\text{eval}]{} S(\overset{m}{\wedge} X)\,.$$

Now define $\alpha: S(X_\infty) \to \bigvee_{k=1}^{\infty} S(\overset{k}{\wedge} X)$ by

$$\alpha[t, x] = \begin{cases} \pi_n[2^n \cdot t - 1, x] & (t \in [1/2^n, 1/2^{n-1}]; x \in X_\infty) \\ * & \text{otherwise.} \end{cases}$$

It is clear that α respects the obvious filtrations and induces homotopy equivalences $S(X_m/X_{m-1}) \to (\bigvee_1^m S(\overset{k}{\wedge} X))/(\bigvee_1^{m-1} S(\overset{k}{\vee} X))$.

REMARK 2.2. Using the work of May, [9], a similar proof shows that stably $S^n\Omega^n S^n X$ is homotopy equivalent to a wedge of n-fold suspensions of equivariant half-smash products.

For classes $\alpha_i \in \pi_{m_i}((\Omega S^q) \wedge S^1)$,

$$(1 \leq i \leq k;\ \sum m_i = n),$$

let $\{\alpha_1\{\alpha_2\{\cdots\{\alpha_{k-1}, \alpha_k\}\}\cdots\} \in \pi_{n-k+1}(S^q)$ be the class of the composition

$$S^1 \wedge S^{m_1-1} \wedge \cdots \wedge S^{m_k-1} \xrightarrow[\alpha_1 \wedge 1]{} \Omega S^q \wedge S^1 \wedge \cdots \wedge S^{m_k-1} \to \cdots$$
$$\xrightarrow[1 \wedge \alpha_k]{} (\overset{k}{\wedge} \Omega S^q) \wedge S^1 \xrightarrow[\phi_k]{} \wedge S^q .$$

A similar operation is defined on classes in $\pi_*(S\Omega X)$.

Lemma 2.3. *For* $(q \geq 1)$ *let* $\alpha_1 \in \pi_{m_1}(S^q)$, $\alpha_i \in \pi_{m_i}(S\Omega S^q)$, $(i=2, 3, 4)$, *and* $r = (\sum_1^3 m_i) - 2$, $s = (\sum_1^4 m_i) - 3$. *If q is odd the Whitehead product* $[\alpha_1, \{\alpha_2, \alpha_3\}] \in \pi_r(S^q)$ *is zero and if q is even* $[\alpha_1, \{\alpha_2\{\alpha_3, \alpha_4\}\}] \in \pi_s(S^q)$ *is zero.*

Proof.

Case (i): *q even.*

Let $S_1^q \vee S_2^q$ be the wedge of two coipes of S^q and let $i_t: S^q \to S_1^q \vee S_2^q$, $(t=1, 2)$, be the inclusions. The class

$$z = \{S\Omega i_1 \circ \alpha_2 \{S\Omega i_1 \circ \alpha_3, S\Omega i_2 \circ \alpha_4\}\}$$

maps to $\{\alpha_2\{\alpha_3, \alpha_4\}\}$ under the folding map $S_1^q \vee S_2^q \to S^q$. Collapsing S_t^q, $(t=1, 2)$, kills z and by [4, Theorems A and 6.6] there exist classes $\sigma \in \pi_t(S^{2q-1})$ and $\tau \in \pi_t(S^{3q-2})$, where $t = m_2 + m_2 + m_4 - 2$, such that

$$\{\alpha_2\{\alpha_3, \alpha_4\}\} = [\iota, \iota] \circ \sigma + [\iota[\iota, \iota]] \circ \tau,\ (\iota = [1_{S^q}] \in \pi_q(S^q)) .$$

Hence, by [2, §4.3 et seq; 4, Theorem 6.10; 12, §§3.2, 3.3],

$$[\alpha_1, \{\alpha_2\{\alpha_3, \alpha_4\}\}] = [\alpha_1, [\iota, \iota] \circ \sigma] .$$

Now consider $z \in \pi_t(S_1^q \vee S_2^q)$. Since the composition

$$\Omega S_1^q \wedge \Omega S_2^q \xrightarrow[\phi_2 \circ (\Omega i_1 \wedge \Omega i_2)]{} \Omega(S_1^q \vee S_2^q) \longrightarrow \Omega(S_1^q \times S_2^q)$$

is nullhomotopic the factorisation of $\{S\Omega i_1 \circ \alpha_3,\ S\Omega i_2 \circ \alpha_4\}$,

$$S^1 \wedge S^{m_3-1} \wedge S^{m_4-1} \longrightarrow S\Omega(S_1^q \vee S_2^q) \xrightarrow[eval]{} S_1^q \vee S_2^q ,$$

extends to a factorisation

$$S^1 \wedge S^{m_3-1} \wedge S^{m_4-1} \wedge I \xrightarrow{f} S\Omega(S_1^q \times S_2^q) \xrightarrow[eval]{} S_1^q \times S_2^q .$$

Hence, if ϕ_2: $\overset{3}{\wedge}\Omega(S_1^q \times S_2^q) \to \Omega(S_1^q \times S_2^q)$ is the three-fold commutator and q: $(S_1^q \times S_2^q,\ S_1^q \vee S_2^q) \to (S_1^q \wedge S_2^q,\ *)$ is the collapsing map, then the map of pairs, $q \circ \phi_3 \circ (1 \wedge f) \circ (\alpha_2 \wedge 1)$ is nullhomotpic. In the notation of [4, §6; 5, Lemma 3] this represents $\chi \cdot (d^{-1})(z)$. Hence, as in [4, Theorem 6.10, and Lemma 6.11] $[\alpha_1, [\iota, \iota] \circ \sigma]$ has two-primary order. However, by [4, Theorem 6.10], 3. $[\alpha_1, [\iota, \iota] \circ \sigma] = 0$.

Case (ii): *q odd.* This follows from [4, Theorem 6.10; 12, §§3.2, 3.3] and the fact that $\{\alpha_2, \alpha_3\} = [\iota, \iota] \circ \sigma$.

Corollary 2.4.
(a) $nil(\Omega S^{2n+1}) \leq 2, \quad (n \geq 0).$
(b) $nil(\Omega S^{2n}\ ) \leq 3, \quad (n \geq 1).$
(c) $nil(\Omega S^n) = 1$ *if and only if* $n = 1$, 3 *or* 7.
(d) $nil(\Omega S^2) = 2$.

Proof. Parts (a) and (b) are proved using Lemma 2.3 and Proposition 1.2. For (b) it suffices to extend the map

$$(eval) \vee (eval) \circ \phi_3 \colon S\Omega S^{2n} \vee S((\Omega S^{2n})^3) \to S^{2n} \text{ over } S\Omega S^{2n} \times S((\Omega S^{2n})^3) .$$

Since $S(A \times B) \simeq SA \vee SB \vee S(A \wedge B)$, Proposition 2.1 implies that both factors are wedges of spheres. Hence the obstructions to the extension are Whitehead products. These obstructions are clearly of the form $[\alpha_1, \{\alpha_2 \{\alpha_2, \alpha_4\}\}]$. Parts (c) and (d) follow from well-known properties of Whitehead products.

Let F denote the real field, (R), the complex field, (C), or the quaternions, (H). Let d be the real dimension of F. If FP^n is the projective n-space over F, $(n \geq 1)$, let β: $S^{d-1} \to \Omega FP^n$ be the adjoint of the inclusion of FP^1 and let π: $S^{d\cdot(n+1)-1} \to FP^n$ be the canonical projection, then

$$\mu(F, n) = \beta \cdot \Omega\pi \colon S^{d-1} \times \Omega S^{d\cdot(n+1)-1} \to \Omega FP^n \times \Omega FP^n \to \Omega FP^n$$

is a homotopy equivalence, [11, Proposition 14].

Proposition 2.5. *If $F=R$ or C, $\mu(F, n)$ is an H-equivalence if and only if $n \geq 3$ and n is odd. Also $\mu(H, 24k-1)$ is an H-equivalence, $(k \geq 1)$.*

Proof. The map, $\mu(F, n)$, is an H-map if and only if β and $\Omega\pi$ have zero "commutator".

By Proposition 1.2, to demonstrate this we need only extend $adj(\beta)\vee adj(\Omega\pi)\colon S^d\vee S\Omega S^{d\cdot(n+1)-1}\to FP^n$ in the cases indicated. The obstructions to this are all Whitehead products of the third kind which are zero by [1, §4; 6, Theorem 2.1]. The converses follow from the behaviour of Whitehead products of the third kind, [1, §4].

Corollary 2.6.

(i) $nil(\Omega RP^{2n})=\infty, \quad (n\geq 1).$

(ii) $nil(\Omega RP^{2n+1})=\begin{cases}\leq 2 & (n\geq 0)\\ =1 & \textit{if and only if} \quad n=0,1 \textit{ or } 3.\end{cases}$

(iii) $nil(\Omega CP^{2n+1})=\begin{cases}\leq 2 & (n\geq 0)\\ =1 & \textit{if and only if} \quad n=1\end{cases}$

(iv) $nil(\Omega HP^{24k-1})=3, \quad (k\geq 1).$

Proof. (i) By [1, §4.1] there are arbitrarily long, non-zero iterated Whitehead products in $\pi_*(RP^{2n})$.

Parts (ii)-(iv) follow from Proposition 2.5, the behaviour of Whitehead products and the fact that $nil(S^3)=3$, [10].

For the rest of this section we concentrate on ΩCP^{2n}. Let $\mu\colon S^1\times\Omega S^{4n+1}\to\Omega CP^{2n}$ be the homotopy equivalence of [11, Proposition 1.14] and let ν be an inverse equivalence. Let β and $\Omega\pi$ be as above and let $\pi_i(i=1, 2)$ be the projections from $S^1\times\Omega S^{4n+1}$. Also denote by β and $\Omega\pi$ the compositions $\beta\circ\pi_1\circ\nu$ and $\Omega\pi\circ\pi_2\circ\nu$ respectively. In the group $[\Omega CP^{2n}, \Omega CP^{2n}]$ the homotopy class of the identity is the product $\beta\cdot\Omega\pi$. The n-fold commutator, ϕ_n, for ΩCP^{2n} is nullhomotopic if the n-fold iterated Samelson-Whitehead product of $1_{\Omega CP^{2n}}$ is zero. Before proving that ΩCP^{2n} is nilpotent we derive some preliminary results about Samelson-Whitehead products in $[\Omega CP^{2n}, \Omega CP^{2n}]$, $(n>0)$.

Proposition 2.7. *The class, $[1_{\Omega CP^{2n}}, 1_{\Omega CP^{2n}}]$, is represented by a map which factors through $\Omega\pi\colon\Omega S^{4n+1}\to\Omega CP^{2n}$.*

Proof. We have to show that

$$(S^1\times S^{4n+1})^2\xrightarrow[\mu\times\mu]{}(\Omega CP^{2n})^2\xrightarrow[\pi_1\circ\nu\circ\phi_2]{}S^1$$

is nullhomotopic. It is nullhomotopic on $S^1\times S^1$, since S^1 is abelian. However, further obstructions to extending the nullhomotopy from $S^1\times S^1$ to $(S^1\times\Omega S^{4n+1})^2$ lie in zero groups, by Proposition 2.1.

Corollary 2.8. $[[[1_{\Omega CP^{2n}}, 1_{\Omega CP^{2n}}]\Omega\pi]\Omega\pi]=0.$

Proof. By Proposition. 2.7 and Corollary 2.4(a).

Proposition 2.9. $[[\Omega\pi, \Omega\pi]\beta]=0$.

Proof. By Proposition 1.2 this is so if $\beta\vee(\phi_2\circ(\Omega\pi)^2)$ extends over $S^2\times S((\Omega S^{4n+1})^2)$. Since $S((\Omega S^{4n+1})^2)$ is a wedge of spheres the obstructions are Whitehead products of the form $[\beta, \pi\circ x]\in\pi_*(CP^{2n})$. However, by the argument of Lemma 2.3 (proof), $x\in\pi_*(S^{4n+1})$ is a Whitehead product of the form $[\sigma_1, \sigma_2]=[\iota, \iota]\circ\sigma$. Since $[\beta[\pi, \pi]]=0$, by the Jacobi identity, [4, Theorem B]; then $[\beta, \pi\circ x]=0$ by [6, Theorem 2.1].

Let $a: A\to\Omega CP^{2n}$, $b_i: B\to\Omega CP^{2n}$, $(i=1, 2)$, be maps and define $\Delta_1: A\wedge B\to B\wedge A\wedge B$, $\Delta_2: B\wedge A\to A\wedge B\wedge B$ by $\Delta_1(a\wedge b)=b\wedge a\wedge b$ and $\Delta_2(b\wedge a)=a\wedge b\wedge b$. The commutator identity in a group,

$$[x, y, z] = [x, y]\cdot[y, [x, z]]\cdot[x, z] \text{ implies, (c.f. [2, §4])},$$

$$[a, b_1\cdot b_2] = [a, b_1]. \qquad \{[b_1[a, b_2]]\circ\Delta_1\}\cdot[a, b_2]$$

and (2.10)

$$[b_1\cdot b_2, a_2] = [b_2, a]\cdot\{[[a, b_2]b_1]\circ\Delta_2\}\cdot[b_1, a]$$

Notice that if $A=B=\Omega CP^{2n}$ then

$$[[a, b_2]\beta]\circ\Delta_2 = [[a, b_2\circ\Omega\pi]\beta]\circ\Delta_2$$

and

$$[\beta[a, b_2]]\circ\Delta_1 = [\beta[a, b_2\circ\Omega\pi]]\circ\Delta_1$$

since the diagonal $S^1\to S^1\times S^1$ deforms onto $S^1\vee S^1$.

Using (2.10) and $\beta\cdot\Omega\pi=1_{\Omega CP^{2n}}$ it is straightforward to deduce the following result from Corollary 2.8 and Proposition 2.9.

Proposition 2.11. *Let x_m be the m-fold iterated Samelson-Whitehead product,*

$$x_m = [1_{\Omega CP^{2n}}[1_{\Omega CP^{2n}}[\cdots[_{\Omega CP}1_{2n}, 1_{\Omega CP^{2n}}]]\cdots]]$$

and y_m be the $(m+2)$-fold product,

$$y_m = [\beta[\beta[\cdots[\beta[1_{\Omega CP^{2n}}, 1_{\Omega CP^{2n}}]]\cdots]]. \quad \textit{Then}$$

$$x_{m+2} = [\Omega\pi, y_{m-1}]\cdot y_m, \qquad (m\geq 2)\,.$$

Proposition 2.12. *In the notation of* (2.11), $y_5=0$.

Proof. By Proposition 2.7, y_1 factors through a map $S\Omega S^{4n+1}\to\Omega CP^{2n}$. However, $S\Omega S^{4n+1}$ is a wedge of spheres, by Proposition 2.1. Hence it suffices to show that $[\beta[\beta[\beta[\beta, \alpha]]]]=0$, where $\alpha: S^{4kn+2}\to CP^{2n}$ and $\alpha=\pi\circ\xi$. From [1, §4.2],

$$[\beta[\beta[\beta[\beta, \pi]]]] = \pi\circ\eta\circ S\eta\circ S^2\eta\circ S^3\eta,\ (0\neq\eta\in\pi_{4n+2}(S^{4n+1}))\,,$$

which is zero by [7, pp. 328–331]. Now if i_t, $(t=1, 2)$ are the inclusions of the factors in the wedge $S^2\vee S^{4n+1}$ then $[\beta[\beta[\beta[\beta,\pi\circ\xi]]]]=(\beta\vee\pi)\circ[i_1[i_1[i_1[i_1,i_2\circ\xi]]]]$. It is now straightforward to show $[\beta[\beta[\beta[\beta,\alpha]]]]=0$, using [1, §4.2; 12, §§3.2 and 3.3].

Corollary 2.13. $3\leq nil(\Omega CP^{2n})\leq 7$, $(n\geq 1)$.

Since the upper bound in Corollary 2.13 is large we prove the vanishing of another triple product.

Proposition 2.14. *Let n_1, n_2 be integers and let $n_1\colon S^1\to S^1$, $n_2\colon S^{4n+1}\to S^{4n+1}$ be maps of those degrees. Let x be represented by the composition*

$$\Omega CP^{2n}\xrightarrow[\nu]{} S^1\times\Omega S^{4n+1}\xrightarrow[(\beta\circ n_1)\times\Omega(\pi\circ n_2)]{}(\Omega CP^{2n})^2\xrightarrow[m]{}\Omega CP^{2n},$$

where m is the multiplication, and $(n>1)$.

If $n_1\cdot n_2\equiv 0(mod\ 2)$ then $0=[[x,x]x]\in[\overset{3}{\wedge}(\Omega CP^{2n}),\Omega CP^{2n}]$. In particular $[[\beta\cdot 1_{\Omega CP^{2n}},\beta\cdot 1_{\Omega CP^{2n}}]\beta\cdot 1_{\Omega CP^{2n}}]=0$.

Proof. By Theorem 1.3 (iii) and (vi) we have a map $\gamma\colon S^1(3)\to S^1P(2)=CP^2\subset CP^{2n}$ extending $\overset{3}{\vee}(\beta\circ n_1)$ on $(\overset{3}{\vee}S^2)$. Consider the problem of extending $\gamma\vee\pi\circ n_2$ over $S^1(3)\times S^{4n+1}$. This map extends over $E=(\overset{3}{\vee}S)\times S^{24n+1}\cup S^1(3)\vee S^{4n+1}$, since the obstructions are Whitehead products, $[\beta\circ n_1,\pi\circ n_2]$, which are zero by [1, §4.2]. By Theorem 1.3 (ii), the only other obstruction lies in

$$H^{4n+6}(S^1(3)\times S^{4n+1},E;\pi_{4n+5}(CP^{2n}))=0\,.$$

If $\delta\colon S^1(3)\times S^{4n+1}\to CP^{2n}$ is the extension, consider $\delta\circ(1\times f)\circ\Delta_3$ where Δ_3 is as in Theorem 1.3(v) and f is derived from Theorem 1.3(vii) and Corollary 2.4(a). Since the map $g\colon S(S^1\times\Omega S^{4n+1})\to S^2\times S^{4n+1}$ given by $g([t,(z,h)])=([t,z],h(t))$ is homotopic to the map, g_1, given by

$$g_1([t,(z,h)])=\begin{cases}([2t,z],*) & (0\leq t\leq 1/2)\\ (*,h(2t-1)) & (1/2\leq t\leq 1)\end{cases}$$

then $(\delta\circ(l\times f)\circ\Delta_2|\overset{3}{\vee}S(S^1\times\Omega S^{4n+1}))$ is homotopic to $\overset{3}{\vee}(\beta\circ n_1).(\Omega(\pi\circ n_2))$. Hence, by Theorem 1.3 (vii) and Remark 1.4, $[[x,x],x]=0$.

3. The spaces, $X(j)$

Let $\{m_j, j\geq 1\}$ be the sequence of integers $m_1=1$, $m_{j+1}=2.(m_j+1)$. Let P_j, $(j\geq 2)$, be the 2-disc represented as a regular (plane) m_j-gon with vertices $a_1,\cdots,a_{m_j}$ and base point $a_1=*$. If S is a finite set in the plane let $ch(S)$ denote its closed convex hull. Write

$$P_j=Q_j\cup R_j\cup Q_j',\ (j>2),\text{ where } Q_j=ch(a_1,\cdots,a_{1+m_{j-1}})\,,$$

$$Q_j' = ch(a_{2+m_{j-1}}, \cdots, a_{m_j}) \text{ and } R_j = ch(a_1, a_{1+m_{j-1}}, a_{2+m_{j-1}}, a_{m_j}) .$$

Let $k_j: Q_j' \to Q_j$ be the linear homeomorphisms given by $k_j(a_{1-r+m_j}) = a_r$. Also let $\gamma_j: (Q_j, ch(a_1, a_{1+m_{j-1}})) \to (P_{j-1}, *)$ be a relative homeomorphism such that $\gamma_j(a_i) = a_i$, $(1 \leq i \leq m_{j-1})$, and γ_j is linear on each edge. Put $P_2 = I^2$ with vertices $a_1 = (0, 0)$, $a_2 = (0, 1)$, $a_3 = (1, 1)$ and $a_4 = (1, 0)$. Let $h_j: R_j \to I^2$ be the linear homeomorphism given by

$$h_j(a_1) = a_1, \ h_j(a_{1+m_{j-1}}) = a_2, \ h_j(a_{2+m_{j-1}}) = a_3 \text{ and } h_j(a_{m_j}) = a_4 .$$

Now let $\{X_i, i \geq 1\}$ be an indexed set of copies of a space X. Define $\delta: I^2 \times (X_1 \vee X_2) \to SX_1 \vee SX_2$ by

$$\delta(t, s, *, y) = [s, y]_2, \ \delta(t, s, x, *) = [t, x]_1$$

($x, y \in X$; $s, t \in I$ and the suffix indicates the wedge factor).

We now inductively construct the spaces, $X(j)$, $(j \geq 2)$. Put $X(2) = I^2 \times X_1 \times X_2 \cup_{\beta_2} (SX_1 \vee SX_2)$ where

$$\beta_2: I^2 \times (X_1 \vee X_2) \cup \partial I^2 \times X_1 \times X_2 \to SX_1 \vee SX_2 \text{ is given by}$$
$$\beta_2(t, \varepsilon, x, y) = [t, x]_1, \ \beta_2(\varepsilon, s, x, y) = [s, y]_2, \ (\varepsilon = 0 \text{ or } 1) ,$$

and $\beta_2 = \delta$ otherwise. Thus $X(2) = SX_1 \times SX_2$.

Now let
$$\pi_1: \mathop{\times}_1^j X_i \to \mathop{\times}_1^{j-1} X_i, \ \pi_2: \mathop{\times}_1^j X_i \to X_j ,$$
$$i_1: \bigvee_1^{j-1} SX_i \to \bigvee_1^j SX_i, \ i_2: SX_j \to \bigvee_1^j SX_i$$

be the canonical projections and inclusions. Define

$$X(j) = P_j \times (\mathop{\times}_1^j X_i) \cup_{\beta_j} (\bigvee_1^j SX_i), \qquad (j > 2) ,$$

where $\beta_j: \partial P_j \times (\mathop{\times}_1^j X_i) \cup P_j \times (\bigvee_1^j X_i) \to \bigvee_1^j SX_i$ is defined by the following compositions:—

$$\beta_j | (\partial P_j \cap Q_j) \times (\mathop{\times}_1^j X_i) = i_1 \circ \beta_{j-1} \circ (\gamma_j \times \pi_1) ,$$

$$\beta_j | (\partial P_j \cap Q_j') \times (\mathop{\times}_1^j X_i) = i_1 \circ \beta_{j-1} \circ ((\gamma_j \circ k_j) \times \pi_1) ,$$

$$\beta_j | (\partial P_j \cap R_j) \times (\mathop{\times}_1^j X_i) = i_2 \circ (\delta | I^2 \times (X_j \vee *)) \circ (h_j \times \pi_2) ,$$

$$\beta_j | R_j \times (\bigvee_1^{j-1} X_i) = * = \beta_j | (Q_j \cup Q_j') \times X_j ,$$

$$\beta_j | R_j \times X_j = i_2 \circ (\delta | I^2 \times (X_j \vee *)) \circ (h_j \times 1) ,$$

$$\beta_j | Q_j \times (\bigvee_1^{j-1} X_i) = i_1 \circ \beta_{j-1} \circ (\gamma_j \times 1) ,$$

and $$\beta_j | Q_j' \times (\bigvee_1^{j-1} X_i) = i_1 \circ \beta_{j-1} \circ ((\gamma_j \circ k_j) \times 1)\,.$$

The map Δ_j of Theorem 1.3 (v) is induced by

$$P_j \times X \times Y \xrightarrow[\Delta \times 1 \times 1]{} P_j{}^2 \times X \times Y \cong P_j \times X \times P_j \times Y\,.$$

We now prove Theorem 1.3 (vi); part (vii) is similar. Consider the problem of extending $\bigvee_1^j adj(w) \colon \bigvee_1^j SX_i \to XP(2)$ over $X(j)$. The map $(\bigvee_1^j adj(w)) \circ \beta_j$ sends $\partial P_j \vee (\times_1^j X_i)$ to the basepoint and induces

$\mu_j \colon \partial P_j \wedge (\times_1^j X_i) = S(\times_1^j X_i) \to XP(2)$ with adjoint

$\mu_j \colon \times_1^j X_i \to \Omega XP(2)$. Let $f \colon C(\partial P_j) \xrightarrow{\cong} P_j$ be a cone-wise homeomorphism which is the identity on ∂P_j. Also let f have cone-point, $z_0 \in P_j$, such that

$((\bigvee_1^j adj(w)) \circ \beta_j)\,(z_0 \times \bigvee_1^j X_i) = *$, (if $j=2$) this can be arranged by altering $adj(w)$ by a homotopy). Suppose that μ_j is nullhomotopic then there exists a nullhomotopy,

$G_u(u \in I)$, of $(\bigvee_1^j adj(w)) \circ \beta_j$ such that

$G_u(q, x) = ((\bigvee_1^j adj(w) \circ \beta_j)\,(f[u, q], x)$, $(q \in \partial P_j;\ x \in \times_1^j X_i)$.

Thus defining $H \colon P_j \times (\times_1^j X_i) \to XP(2)$ by $H(q, x) = G_u(q', x)$, where $f([u, q']) = q (q' \in \partial P_j;\ q \in P_j;\ u \in I;\ x \in \times_1^j X_i)$, induces a map $X(j) \to XP(2)$ extending $\bigvee_1^j adj(w)$. Conversely, if $\bigvee_1^i adj(w)$ extends, we have

$H \colon P_j \times (\times_1^j X_i) \to XP(2)$ extending $(\bigvee_1^j adj(w)) \circ \beta_j$ and we may assume $H(P_j \times *) = *$. Now let $G_u \colon \partial P_j \to P_j$ be a based homotopy from the inclusion to the constant map. Thus

$H \circ (G \times 1) \colon I \times \partial P_j \times (\times_1^j X_i) \to XP(2)$ induces a nullhomotopy of μ_j. However, the map $\mu_j \colon \times_1^j X_i \to XP(2)$ is the composition of $(\times_1^j w)$ and the j-fold commutator on $\Omega XP(2)$, Since w is an H-map we have $w \circ \phi_j \simeq \mu_j$. Thus if ϕ_j is nullhomotopic the extension exists. If right translation is a homotopy equivalence in X there exists a map $r \colon \Omega XP(2) \to X$, [11, Lemma 4.2], such that $r \circ w \simeq 1$.

The maps, Γ_j, of Theorem 1.3 (iii) are induced by maps $G_j \colon P_j \times (\times_1^j X_i \to P_{j-1} \times (\times_1^{j-1} X_i)$ which are defined in the following manner. Let $proj \colon R_j \to R_{j-1}$ be such that $h_{j-1} \circ proj \circ (h_j)^{-1}$ is projection on the first factor in I^2 and let p_2 be $p_2 \colon \times_1^j X_i \xrightarrow[\pi_2]{} X_j = X = X_{j-1}$.

Put

$$G_j \mid Q_j \times (\overset{j}{\underset{1}{\times}} X_i) = \gamma_j \times \pi_1 ,$$

$$G_j \mid Q_j' \times (\overset{j}{\underset{1}{\times}} X_i) = (\gamma_j \circ k_j) \times \pi_1 \text{ and}$$

$$G_j \mid R_j \times (\overset{j}{\underset{1}{\times}} X_i) = proj \times p_2 .$$

It is clear that there exist homeomorphisms

$$\begin{aligned} X(j)/(\overset{j}{\vee} SX) &\cong D^2 \times X^j/(\partial D^2 \times X^j \cup D^2 \times (\overset{j}{\vee} X)) \\ &\cong S^2 \times X^j/(* \times X^j \cup S^2 \times (\overset{j}{\vee} X)) . \end{aligned}$$

Also $G_j(R_j \times (\overset{j}{\underset{1}{\times}} X_i)) \subset (\partial P_{j-1} \cap R_{j-1}) \times X_{j-1}$ which goes to the basepoint in $X(j-1)/(\overset{j-1}{\vee} SX)$. Let $q\colon S^2 \to \overset{3}{\vee} S^2$ be the standard pinching map and put $A_j\colon S^2 \wedge X^j \to S^2 \wedge X^{j-1}$ as the composition (fold $\wedge 1) \circ ((1 \vee * \vee -1) \wedge \pi_1) \circ (q \wedge 1)$. We have a commutative diagram in which the rows are cofibrations

$$\begin{array}{ccccc} S^2 \wedge (\overset{j}{\vee} X) \to S^2 \wedge X^j & \longrightarrow & S^2 \times X^j/(* \times X^j \cup S^2 \times (\overset{j}{\vee} X)) \\ \downarrow & & \downarrow \\ S^2 \wedge (\overset{j-1}{\vee} X) \to S^2 \wedge X^{j-1} & \longrightarrow & S^2 \times X^{j-1}/(* \times X^{j-1} \cup S^2 \times (\overset{j-1}{\vee} X)) . \end{array}$$

Hence Theorem 1.3 (iv) is proved.

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