Hikari, M.
Osaka J. Math.
10 (1973), 369-374

MULTIPLICATIVE P-SUBGROUPS OF SIMPLE ALGEBRAS

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(Received May 1., 1972)

Amitsur ([1]) determined all finite multiplicative subgroups of division algebras. We will try to determine, more generally, multiplicative subgroups of simple algebras. In this paper we will characterize p-groups contained in full matrix algebras $M_n(\Delta)$ of fixed degree n, where Δ are division algebras of characteristic 0.

All division algebras considered in this paper will be of characteristic 0.

Let Δ be a division algebra. We will denote by $M_n(\Delta)$ the full matrix algebra of degree n over Δ. By a subgroup of $M_n(\Delta)$ we will mean a multiplicative subgroup of $M_n(\Delta)$. Further let K be a subfield of the center of Δ and let G be a finite subgroup of $M_n(\Delta)$. Now we define $V_K(G)=\{\sum\alpha_i g_i \mid \alpha_i\in K, g_i\in G\}$. Then $V_K(G)$ is clearly a K-subalgebra of $M_n(\Delta)$ and there is a natural epimorphism $KG\to V_K(G)$ where KG denotes the group algebra of G over K. Hence $V_K(G)$ is a semi-simple K-subalgebra of $M_n(\Delta)$, which is a direct summand of KG. As usual $\boldsymbol{Q}$, $\boldsymbol{R}$, $\boldsymbol{C}$, $\boldsymbol{H}$ denote respectively the rational number field, the real number field, the complex number field and the quaternion algebra over $\boldsymbol{R}$.

If an abelian group G has invariants $(e_1, \cdots, e_n)$, $e_n \neq 1$, $e_{i+1} \mid e_i$, we say briefly that G has invariants of length n.

We begin with

Proposition 1. *Let n be a fixed positive integer and let G be a finite abelian group. Then there is a division algebra Δ such that $G\subset M_n(\Delta)$ if and only if G has invariants of length $\leqq n$.*

Proof. This may be well known. Here we give a proof. Suppose that there is a division algebra Δ such that $G\subset M_n(\Delta)$. An abelian group G has invariants of length $\leqq n$ whenever each Sylow subgroup of G has invariants of length $\leqq n$. Hence we may assume that G is a p-group $(\neq 1)$. Let m be the length of invariants of G. Then G contains the elementary abelian group G_0 of order p^m. We can write $\boldsymbol{Q}G_0 \cong \boldsymbol{Q}\oplus\overbrace{\boldsymbol{Q}(\varepsilon_p)\oplus\cdots\oplus\boldsymbol{Q}(\varepsilon_p)}^{1+p+\cdots+p^{m-1}}$ where ε_p denotes the primitive p-th root of unity. Since $V_{\boldsymbol{Q}}(G_0)$ is a direct summand of $\boldsymbol{Q}G_0$ and $G_0\subset V_{\boldsymbol{Q}}(G_0)$, we have $V_{\boldsymbol{Q}}(G_0)\cong\overbrace{\boldsymbol{Q}(\varepsilon_p)\oplus\cdots\oplus\boldsymbol{Q}(\varepsilon_p)}^{m}$. On the other hand, since

$V_Q(G_0) \subset M_n(\Delta)$, there exist at most n orthogonal idempotents in $V_Q(G_0)$. Thus we have $m \leqq n$. The converse is obvious. Q.E.D.

Proposition 2 *Let p be an odd prime and $0<n<p$. Let P be a finite p-group. If there exists a division algebra Δ such that $P \subset M_n(\Delta)$, then P is abelian.*

Proof. Let $V_Q(P) \cong M_{p^{l_1}}(\Delta_1) \oplus \cdots \oplus M_{p^{l_t}}(\Delta_t)$ be the decomposition of $V_Q(P)$ into simple algebras where each Δ_i is a division algebra. Then we easily see that $p^{l_1} + \cdots + p^{l_t} \leqq n$. Therefore, when $n<p$, we have $l_1 = \cdots = l_t = 0$. Since p is odd, each division algebra Δ_i is commutative ([3]). Hence $V_Q(P)$ is commutative. This conclude that P is abelian. Q.E.D.

DEFINITION. Let $P_0 = \langle g \rangle$ be a cyclic group of order p. Let P, P' be finite p-groups and let P_1', P_2', $\cdots$, P_p' be the copies of P'. We will call P *a simple* (**1**-*fold*) *p-extension of* P' if P is an extension of $P_1' \times P_2' \times \cdots \times P_p'$ by P_0 such that $P_1'^g = P_2'$, ..., $P_{p-1}'^g = P_p'$, $P_p'^g = P_1'$. It should be remarked that this extension does not always split. More generally, a finite p-group P will be called *an n-fold p-extension of a finite p-group P'*, if there exist finite p-groups, $P_0 = P'$, P_1, $\cdots$, P_{n-1}, $P_n = P$ such that, for each $0 \leqq i \leqq n-1$, P_{i+1} is a simple p-extension of P_i.

Now we set

$$T_p^{(0)} = \begin{cases} \{\text{all cyclic } p\text{-groups}\} & \text{when } p \neq 2, \\ \{\text{all generalized quaternion 2-groups}\} & \text{when } p = 2, \end{cases}$$

and $\tilde{T}_p^{(0)} = \{$all cyclic p-groups$\}$ for any prime p. An element of $T_p^{(0)}$ (resp. $\tilde{T}_p^{(0)}$) is called *a p-group of* 0-*type* (*resp.* $\tilde{0}$-*type*).

A finite p-group P is said to be of *n-type* (*resp.* *$\tilde{n}$-type*) if P is an n-fold p-extension of a p-group of 0-type (resp. $\tilde{0}$-type). We denote by $T_p^{(n)}$ (resp. $\tilde{T}_p^{(n)}$) the set of all p-groups of n-type (resp. $\tilde{n}$-type).

Our main result is given the following

Theorem. *Let n be a fixed positive integer and let P be a finite p-group. Then following conditions are equivalent*:

(1) *P is a subgroup of $M_n(\boldsymbol{H})$ (resp. $M_n(\boldsymbol{C})$).*

(2) *There is a division algebra Δ (resp. a commutative field K) such that $P \subset M_n(\Delta)$ (resp. $M_n(K)$).*

(3) *There exist non-negative integers, t, m_0, $\cdots$, m_t with $\sum_{i=0}^{t} p^i m_i \leqq n$ and $P_i^{(1)}, P_i^{(2)}, \cdots, P_i^{(m_i)} \in T_p^{(i)}$ (resp. $\tilde{T}_p^{(i)}$) for each $0 \leqq i \leqq t$ such that $P \subset \prod_{i=0}^{t} \prod_{j=1}^{m_i} P_i^{(j)}$.*

The following theorem plays an essential part in the proof of our main theorem.

Theorem (Witt-Roquette [3], [4]). *Let P be a p-group. Let K be a* *commutative field of characteristic* 0. *Suppose that one of the following hypotheses is satisfied.*

(a) $p \neq 2$,

(b) $p=2$ *and* $\sqrt{-1} \in K$.

(c) $p=2$ *and P does not contain a cyclic subgroup of index* 2.

Then if χ is a nonlinear irreducible faithful character of P there exists $P_0 \lhd P$ and a character ζ of P_0 such that $|P: P_0|=p$, $\chi=\zeta^P$ *and* $K(\chi)=K(\zeta)$.

From this theorem the following remark follows directly.

REMARK. If K is an algebraic number field in this theorem, each division algebra equivalent to a simple component of KP is an algebraic number field or a quaternion algebra.

Lemma 3. *Let P be a finite non-abelian p-group and let Δ be a division algebra such that $P \subset M_n(\Delta)$. Suppose that $V_Q(P)=M_n(\Delta)$.*

(1) *Suppose that P is a 2-group which is not of type* 0 *and that Δ is non-commutative. Then there exists a subgroup P_0 of P of index* 2 *such that* $V_Q(P_0) \cong M_{n/2}(\Delta) \oplus M_{n/2}(\Delta)$.

(2) *Suppose that Δ is commutative. Then we have $V_C(P)=M_n(\boldsymbol{C})$ and there exists a normal subgroup P_0 of P of index p such that* $V_C(P_0) \cong \overbrace{M_{n/p}(\boldsymbol{C}) \oplus \cdots \oplus M_{n/p}(\boldsymbol{C})}^{p}$.

Proof. (a) Let M be a simple $M_n(\Delta)$-module and let E be a splitting field of Δ. Since M is a non-linear faithful $\boldsymbol{Q}P$-module by the assumption that $V_Q(P)=M_n(\Delta)$, there exists a non-linear faithful irreducible EP-module N such that $M \otimes_Q E \cong m_Q(N)(N \oplus N^\sigma \oplus \cdots)$, $\sigma \in Gal(\boldsymbol{Q}(\zeta)/Q)$, where ζ is the character of N and $m_Q(N)$ denotes the Schur index of N. Applying the Witt-Roquette's theorem to N, we can find a normal subgroup P_0 of P and an irreducible EP_0-module N_0 with character ζ_0 such that $N_0^P \cong N$ and $\boldsymbol{Q}(\zeta)=\boldsymbol{Q}(\zeta_0)$. Let χ denote the character of M. Then we have $\chi=m_Q(\zeta)(\zeta+\zeta^\sigma+\cdots)=m_Q(\zeta)(\zeta_0+\zeta_0^\sigma+\cdots)+m_Q(\zeta)(\zeta_0^g+(\zeta_0^g)^\sigma+\cdots)$ where $\{1, g\}$ are representatives of P/P_0. Since $2=m_Q(\zeta) \leqq m_Q(\zeta_0) \leqq 2$, we have $m_Q(\zeta)=m_Q(\zeta_0)=2$. Let $\chi_0=m_Q(\zeta_0)(\zeta_0+\zeta_0^\sigma+\cdots)$. Then χ_0 is a $\boldsymbol{Q}$-character of P_0. Further let M_0 be the $\boldsymbol{Q}P_0$-module corresponding to χ_0. Then we see that $M_0 \oplus M_0^g \cong \boldsymbol{Q}P \otimes_{\boldsymbol{Q}P_0} M_0 \cong \boldsymbol{Q}P \otimes_{\boldsymbol{Q}P_0} M_0^g \cong M$ as $\boldsymbol{Q}P$-module. Since $M_0 \not\cong M_0^g$ as $\boldsymbol{Q}P_0$-module, we have

$$\begin{aligned}
\Delta &\cong \mathrm{Hom}_{\boldsymbol{Q}P}(M, M) \\
&\cong \mathrm{Hom}_{\boldsymbol{Q}P}(\boldsymbol{Q}P \otimes_{\boldsymbol{Q}P_0} M_0, \boldsymbol{Q}P \otimes_{\boldsymbol{Q}P_0} M_0) \\
&\cong \mathrm{Hom}_{\boldsymbol{Q}P_0}(M_0, \mathrm{Hom}_{\boldsymbol{Q}P}(\boldsymbol{Q}P, \boldsymbol{Q}P \otimes_{\boldsymbol{Q}P_0} M_0)) \\
&\cong \mathrm{Hom}_{\boldsymbol{Q}P_0}(M_0, \boldsymbol{Q}P \otimes_{\boldsymbol{Q}P_0} M_0) \\
&\cong \mathrm{Hom}_{\boldsymbol{Q}P_0}(M_0, M_0),
\end{aligned}$$

and, similarly, $\Delta\cong \mathrm{Hom}_{QP_0}(M_0^g, M_0^g)$. Clearly $\dim_Q M_0=\dim_Q M_0^g=\frac{1}{2}\dim_Q M$; and the semi-simple subalgebra $V_Q(P_0)\subset V_Q(P)=M_n(\Delta)$ has only two simple compotents corresponding to M_0, M_0^g. Thus we get $V_Q(P_0)\cong M_{n/2}(\Delta)\oplus M_{n/2}(\Delta)$.

(b) Since Δ is commutative by the assumption, we have $\boldsymbol{C}\otimes_\Delta V_Q(P)\cong \boldsymbol{C}\otimes_\Delta M_n(\Delta)\cong M_n(\boldsymbol{C})$. From this it follows directly that $V_C(P)=M_n(\boldsymbol{C})$. Let M be a simple $V_C(P)$-($\boldsymbol{C}P$-)module and let χ be the character of M. According to the Witt-Roquette's theorem, there exists a normal subgroup P_0 of P of index p and an irreducible $\boldsymbol{C}P_0$-module M_0 such that $M\cong M_0^P$. Hence, along the same line as in the case (a), we can show that $V_C(P_0)\cong\overbrace{M_{n/p}(\boldsymbol{C})+\cdots+M_{n/p}(\boldsymbol{C})}^{p}$.

Q.E.D.

Lemma 4. *Let P be a finite p-group. Suppose one of the following conditions:*

(a) *$p=2$ and P is a subgroup of $M_{2^n}(\Delta)$ such that $V_Q(P)=M_{2^n}(\Delta)$ where Δ is a quaternion algebra.*

(b) *P is a subgroup of $M_{p^n}(\boldsymbol{C})$ such that $V_C(P)=M_{p^n}(\boldsymbol{C})$. Then P is a subgroup of a p-group of n-type. Further, in the case* (b) *P is a subgroup of a p-group of $\tilde{n}$-type.*

Proof. We will give the proof only in the case (*a*), because the proof in the case (*b*) can be done similarly. This will be done by induction on n. In case $n=0$ this is obvious. Hence we assume that $n\geq 1$. By Lemma 3, there exists a normal subgroup P_0 of P of index 2 such that $V_Q(P_0)=A_1\oplus A_2$ where $A_i\cong M_{2^{n-1}}(\Delta)$. Let M_i be a simple A_i-module and let $\{1, g\}$ be representatives of P/P_0. Then $M_2\cong M_1^g$ as $\boldsymbol{Q}P_0$-module. Let P_i denote the image of P_0 by the projection on A_i. Then $V_Q(P_i)=M_{2^{n-1}}(\Delta)$. Hence, by induction, each P_i is a subgroup of a 2-group of $(n-1)$-type. We regard M_i as $\boldsymbol{Q}P_0$-module by the projection $P_0\rightarrow P_i$ and so, since $M_2\cong M_1^g$, we have $P_2=P_1^g$ and the following commutative diagram:

$$\begin{array}{ccc} P_0 & \xrightarrow{\ g\ } & P_0 \\ \downarrow & & \downarrow \\ P_1\times P_2 & \xrightarrow{(g,\, g)} & P_2\times P_1 \end{array}$$

On the other hand, we can find 2-groups $\tilde{P}_1$, $\tilde{P}_2$ of (n-1)-type such that $\tilde{P}_1\cong\tilde{P}_2$. Here we may assume that the restriction of the isomorphism $\tilde{P}_1\xrightarrow{\sim}\tilde{P}_2$ on P_1 coincides with g: $P_1\xrightarrow{\sim}P_2$. We denote this isomorphism from $\tilde{P}_1$ onto $\tilde{P}_2$ by σ. Put $h=g^2$. Then the map $(1, h)$; $\tilde{P}_2\times\tilde{P}_1\rightarrow\tilde{P}_2\times\tilde{P}_1$ is an isomorphism and so $(\sigma, h\sigma^{-1})$: $\tilde{P}_1\times\tilde{P}_2\rightarrow\tilde{P}_2\times\tilde{P}_1$ is an isomorphism, too. Since the restriction of $h\sigma^{-1}$ on P_2 coincides with $hg^{-1}=g$, we get the following commutative diagram:

$$\begin{array}{ccc} P_0 & \xrightarrow{\ g\ } & P_0 \\ \downarrow & & \downarrow \\ P_1\times P_2 & \xrightarrow{(g,\, g)} & P_2\times P_1 \\ \downarrow & & \downarrow \\ \tilde{P}_1\times\tilde{P}_2 & \xrightarrow{(\sigma,\, h\sigma^{-1})} & \tilde{P}_2\times\tilde{P}_1 \end{array}$$

Let $\langle u\rangle$ be a cyclic group of order 2. The automorphism $(\sigma, h\sigma^{-1})$ and the factor set $\{(1, 1)=(u, 1)=(1, u)=1, (u, u)=h\}$ define a group $\tilde{P}$ with normal subgroup $\tilde{P}_1\times\tilde{P}_2$ and $\tilde{P}/\tilde{P}_1\times\tilde{P}_2\cong\langle u\rangle$, because $(h\sigma^{-1}, \sigma)\cdot(\sigma, h\sigma^{-1})=(h, \sigma h\sigma^{-1})=(h, h^{\sigma^{-1}})=(h, h^{g^{-1}})=(h, h)$. Then the group $\tilde{P}$ is clearly a 2-group of n-type which contains P. Thus the proof of the lemma is completed.

Lemma 5. *If $P\in T_2^{(n)}$ (resp. $\tilde{T}_p^{(n)}$), P is a subgroup of $M_{2^n}(\boldsymbol{H})$ (resp. $M_{p^n}(\boldsymbol{C})$) and $V_{\boldsymbol{R}}(P)=M_{2^n}(\boldsymbol{H})$ (resp. $V_{\boldsymbol{C}}(P)=M_{p^n}(\boldsymbol{C})$).*

Proof. We will prove this in the case $P\in T_2^{(n)}$.

(a) $n=0$. Since P is a generalized quaternion group, P is a subgroup of $\boldsymbol{H}$ and $V_{\boldsymbol{R}}(P)=\boldsymbol{H}$ ([1], [2]).

(b) $n>0$. We proceed by induction on n. By the definition of $T_2^{(n)}$, there exist 2-groups P_1, $P_2\in T_2^{(n-1)}$ such that $P_1\times P_2$ is a subgroup of P of index 2 and that $P_1^g=P_2$, where g is a representative of a generator of $P/P_1\times P_2$. By the induction hypothesis each P_i is a subgroup of $M_{2^{n-1}}(\boldsymbol{H})$ and $V_{\boldsymbol{R}}(P_i)=M_{2^{n-1}}(\boldsymbol{H})$. Let M_1 be a simple $V_{\boldsymbol{R}}(P_1)$-($\boldsymbol{R}P_1$-)module. Put $M=M_1\otimes_{\boldsymbol{R}(P_1\times P_2)}\boldsymbol{R}P$. Since $P_1^g=P_2$, M_1^g is a simple $\boldsymbol{R}P_2$-module. It follows that $M_1\not\cong M_1^g$ as $\boldsymbol{R}(P_1\times P_2)$-module and therefore $\mathrm{Hom}_{\boldsymbol{R}P}(M, M)\cong\mathrm{Hom}_{\boldsymbol{R}(P_1\times P_2)}(M_1, M_1\oplus M_1^g)\cong\mathrm{Hom}_{\boldsymbol{R}(P_1\times P_2)}(M_1, M_1)=\boldsymbol{H}$. We see that the simple component of $\boldsymbol{R}P$ corresponding to M is $M_{2^n}(\boldsymbol{H})$. Because M is a faithful $\boldsymbol{R}P$-modlue, P is a subgroup of $M_{2^n}(\boldsymbol{H})$ and $V_{\boldsymbol{R}}(P)\cong M_{2^n}(\boldsymbol{H})$.

We will omit the proof in the case $P\in\tilde{T}_p^{(n)}$, because we can prove it along the same line as in the case $P\in T_2^{(n)}$. Q.E.D.

Now we give the proof of our main theorem.

Proof of the main theorem. The implication (1)⇒(2) is obvious and therefore it suffices to show the implications (2)⇒(3)⇒(1).

(*a*) (2)⇒(3). Assume $P\subset M_n(\Delta)$. Let $V_{\boldsymbol{Q}}(P)\cong M_{p^{l_1}}(\Delta_s)\oplus\cdots\oplus M_{p^{l_s}}(\Delta_s)$ be the decomposition of $V_{\boldsymbol{Q}}(P)$ into simple algebras where each Δ_i is a division algebra. Here we easily see that $p^{l_1}+\cdots+p^{l_s}\leqq n$. Let P_i be the image of P by the projection to $M_{p^{l_i}}(\Delta_i)$, for each $1\leqq i\leqq s$. Then P can be identified with a subgroup of $\prod_{i=1}^{s} P_i$ and, for each $1\leqq i\leqq s$, $V_{\boldsymbol{Q}}(P_i)\cong M_{p^{l_i}}(\Delta_i)$. According to the remark on the Witt-Roquette's theorem, Δ_i is a quaternion algebra or an algebraic number field. Further if Δ_i is a quaternion algebra for some $1 \leqq i \leqq s$, $p=2$ ([3]). If Δ_i is an algebraic number field, by Lemma 3 (2) $V_C(P_i) \cong M_{p^{l_i}}(\boldsymbol{C})$. Applying Lemma 4, it follows that each P_i is a subgroup of a p-group of l_i-type. Here (3) is concluded in this case.

Assume $P \subset M_n(K)$. Let L be the algebraic closure of K and let $L' = \boldsymbol{C} \cap L$. Since K is commutative, we have $L \otimes_K M_n(K) \cong M_n(L)$. From this it follows directly that $V_{L'}(P) \subset M_n(L)$. In addition, each division algebra equvalent to a simple component of $L'P$ conicides with L'([3]). Let $V_{L'}(P) \cong M_{p^{l_1}}(L') \oplus \cdots \oplus M_{p^{l_s}}(L')$ be the decomposition of $V_{L'}(P)$ into simple algebras. Then $p^{l_1} + \cdots + p^{l_s} \leqq n$. If P_i is the image of P by the projection to $M_{p^{l_i}}(L')$, P_i is a subgroup of $M_{p^{l_i}}(\boldsymbol{C}) \cong M_{p^{l_i}}(L') \otimes_{L'} \boldsymbol{C}$ and $V_C(P_i) \cong M_{p^{l_i}}(\boldsymbol{C})$. It follows from Lemma 4 that P_i is a subgroup of $\tilde{l}_i$-type. On the other hand P can be identified with a subgroup of $\prod_{i=1}^{s} P_i$ and so we conclude (3).

(b) (3)$\Rightarrow$(1). Since $P_i^{(j)}$ is a p-group of i-type (resp. $\tilde{i}$-type), by Lemma 5, $P_i^{(j)}$ is a subgroup of $M_{p^i}(\boldsymbol{H})$ (resp. $M_{p^i}(\boldsymbol{C})$) and so $\prod_i \prod_{j=1}^{m_i} P_i^{(j)} \subset \sum_{i,j}^{\oplus} M_{p^i}(\boldsymbol{H}) \subset M_n(\boldsymbol{H})$ (resp. $\prod_i \prod_{j=1}^{m_i} P_i^{(j)} \subset M_n(\boldsymbol{C})$) by $\sum_{i=0}^{t} p^i m_i \leqq n$. Since P is a subgroup of $\prod_i \prod_{j=1}^{m_i} P_i^{(j)}$, P is a subgroup of $M_n(\boldsymbol{H})$ (resp. $M_n(\boldsymbol{C})$). Q.E.D.

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