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COMPLEX POWERS OF HYPOELLIPTIC PSEUDO-DIFFERENTIAL OPERATORS WITH APPLICATIONS

Dedicated to Professor Yukinari Tôki on his 60th birthday

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Introduction.

Complex powers of a pseudo-differential operator have been defined by Seeley [15] and Burak [2] for the elliptic case, and defined by Nagase-Shinkai [12] and Hayakawa-Kumano-go [5] for a more general case containing semi-elliptic operators.

In the present paper we shall construct complex powers of a hypoelliptic system of pseudo-differential operators, and apply those powers to the generalized Dirichlet problem and the index theory.

The plan of the paper is as follows. In Section 1 we describe well-known results on the theory of pseudo-differential operators which has been developed in Hörmander [6], [7], Kumano-go [9] and Grushin [4]. In Section 2 the strong (or uniform) continuity and the analyticity of pseudo-differential operators with respect to a parameter are examined by means of their symbols. In Section 3 we construct complex powers P_z of a hypoelliptic system P which belongs to a subclass of Hörmander's in [6], p. 164 (c.f. also Šubin [16]).

Section 4 treats the generalized Dirichlet problem for an operator P which admits complex powers P_z. The Sobolev space $H_{s,P}$ associated with P is defined, and a subspace V of $H_{\frac{1}{2},P}$ is defined as the completion of $C_0^\infty(\Omega)$ in the norm of $H_{\frac{1}{2},P}$ for an open set Ω of R^n. We seek the solution of $Pu=f$ for $f\in L^2(\Omega)$ in the space V. Then, the Lax-Milgram theorem can be applied effectively.

Finally Section 5 is the supplement to the first author's paper [10] where the vanishing theorem of the index is proved when an operator P is slowly varying in the sense of [4] and has complex powers.

We try here to reduce the index theory of a hypoelliptic operator Q of order m to an elliptic operator of order 0 (studied in [4]) when the symbol $\sigma(Q)(x, \xi)$ is equally strong to the symbol $\sigma(P)(x, \xi)$ of an operator P which admits complex powers.

Throughout the present paper we shall treat strict algebras of pseudo-differential operators, and investigate the topology of the symbol class precisely in Sections 2 and 3. The analyticity of complex powers P_z with respect to z is used essentially in order to determine the domain of the adjoint operator P_z^*. The symbols of complex powers are defined by the Dunford integral for the symbols of parametrices $R(\zeta)$ for $P-\zeta I$. We have to note that for a scalar operator P we can give complex powers of P in the concrete form as in [12], if the argument of the symbol $\sigma(P)(x, \xi)$ is well defined. This fact is interesting when we recall the proof of the vanishing theorem of the index by Seely [14] and Nirenberg [13] for an elliptic operator on a compact manifold.

1. Notation and definitions

Let $x=(x_1, \cdots, x_n)$ be a point of the n-dimensional Euclidean space R_x^n, and let $\mathcal{S}$ denote the space of C^∞-functions which together with all their derivatives decrease faster than any power of $|x|=(\sum_{j=1}^n x_j^2)^{1/2}$ as $|x|\to\infty$. By $S_{\rho,\delta}^m(0\leqq\delta<\rho\leqq 1)$ we denote the set of all C^∞-symblos $p(x, \xi)$ in $R_x^n\times R_\xi^n$ satisfying, for any multi-index $\alpha=(\alpha_1, \cdots, \alpha_n)$ and $\beta=(\beta_1, \cdots, \beta_n)$,

$$(1.1)\qquad |p_{(\beta)}^{(\alpha)}(x, \xi)|\leqq C_{\alpha,\beta}\langle\xi\rangle^{m-\rho|\alpha|+\delta|\beta|} \text{ on } R_x^n\times R_\xi^n$$

for a constant $C_{\alpha,\beta}$, wehre

$$p_{(\beta)}^{(\alpha)}(x, \xi) = \partial_\xi^\alpha D_x^\beta p(x, \xi),\ \partial_\xi^\alpha = \partial_{\xi_1}^{\alpha_1}\cdots\partial_{\xi_n}^{\alpha_n},$$
$$D_x^\beta = (-i\,\partial/\partial x_1)^{\beta_1}\cdots(-i\,\partial/\partial x_n)^{\beta_n},\ \langle\xi\rangle = (1+\sum_{j=1}^n \xi_j^2)^{1/2},$$

and for a $p(x, \xi)\in S_{\rho,\delta}^m$ we define a pseudo-differential operator $P=p(x, D_x)$, denoted also by $P\in S_{\rho,\delta}^m$, with the symbol $\sigma(P)(x, \xi) = p(x, \xi)$ by

$$Pu(x) = \int e^{ix\cdot\xi} p(x, \xi)\hat{u}(\xi)d\!\!{}^-\xi,\ u\in\mathcal{S}\quad (x\cdot\xi = x_1\xi_1+\cdots+x_n\xi_n),$$

where $\hat{u}(\xi)$ denotes the Fourier transform of $u(x)$ which is defined by $\hat{u}(\xi)=\int e^{-ix\cdot\xi} u(x)dx$, and $d\!\!{}^-\xi=(2\pi)^{-n}d\xi$. We set

$$S^{-\infty}=\bigcap_m S_{1,0}^m\ (=\bigcap_m S_{\rho,\delta}^m),\ S_{\rho,\delta}^\infty=\bigcup_m S_{\rho,\delta}^m.$$

For two pseudo-differential operators P and Q, $P\equiv Q(\text{mod } S^{-\infty})$ means that

$$\sigma(P)(x, \xi)-\sigma(Q)(x, \xi)\in S_{\rho,\delta}^{-\infty}.$$

For any real number s, we define a continuous operator $\wedge^s$: $\mathcal{S}\to\mathcal{S}$ by

$$\wedge^s u(x) = \int e^{ix\cdot\xi}\langle\xi\rangle^s\hat{u}(\xi)d\!\!{}^-\xi.$$

It is easy to see that $\wedge^s$ belongs to $S_{1,0}^s$ and can be extended uniquely to an operator of $\mathcal{S}'$ into itself by the relation

$$\langle\wedge^s u, v\rangle=\langle u, \wedge^s v\rangle\quad\text{for } u\in\mathcal{S}',\ v\in\mathcal{S}.$$

Let $H_s=\{u\in\mathcal{S}';\ \wedge^s u\in L^2(R^n_x)\}$ be a Hilbert space provided with the s-norm $\|u\|_s=\|\wedge^s u\|_{L^2}$ for $u\in H_s$, where $\|\cdot\|_{L^2}$ denotes the L^2-norm. We set

$$H_{-\infty}=\bigcup_s H_s,\ H_\infty=\bigcap_s H_s\,.$$

For a $p(x,\xi)\in S^m_{\rho,\delta}$, we define semi-norms $|p|_{m,k}$ by

$$(1.2)\qquad |p|_{m,k}=\max_{|\alpha+\beta|\leq k}\sup_{(x,\xi)}\{|p^{(\alpha)}_{(\beta)}(x,\xi)|\langle\xi\rangle^{-(m-\rho|\alpha|+\delta|\beta|)}\}\,,$$

then, $S^m_{\rho,\delta}$ makes a Fréchet space with these semi-norms.

DEFINITION 1.1. We say that a sequence $\{p_j(x,\xi)\}_{j=1}^\infty$ of $S^m_{\rho,\delta}$ converges to a $p(x,\xi)$ of $S^m_{\rho,\delta}$ in $S^m_{\rho,\delta}$ weakly, if $\{p_j(x,\xi)\}_{j=1}^\infty$ is a bounded set of $S^m_{\rho,\delta}$ and

$$(1.3)\qquad p_{j(\beta)}^{(\alpha)}(x,\xi)\to p^{(\alpha)}_{(\beta)}(x,\xi) \text{ as } j\to\infty \text{ uniformly on } R^n_x\times K$$

for any α, β and any compact set K of R^n_ξ. We denote it by

$$p_j(x,\xi)\xrightarrow[\text{(weak)}]{} p(x,\xi)\quad \text{in } S^m_{\rho,\delta}\quad \text{as } j\to\infty\,.$$

REMARK. If (1.3) holds for $\alpha=\beta=0$, then, we have (1.3) for any α and β. In fact, if we use a well-known inequality

$$(1.4)\qquad |f'(t_0)|^2\leqq C\max_{t\in[0,1]}(|f(t)|)\{\max_{t\in[0,1]}(|f(t)|)+\max_{t\in[0,1]}(|f''(t)|)\}\ (t_0\in[0,1])$$

for any C^2-function $f(t)$ on $[0,1]$, then, setting $f(t)=p_j(x,\xi+t\alpha)-p(x,\xi+t\alpha)$ for $|\alpha|=1$, we get

$$p_j^{(\alpha)}(x,\xi)\to p^{(\alpha)}(x,\xi)\qquad \text{as } j\to\infty \text{ uniformly on } R^n_x\times K\,,$$

and so we get

$$p_{j(\beta)}^{(\alpha)}(x,\xi)\to p^{(\alpha)}_{(\beta)}(x,\xi)\qquad \text{as } j\to\infty \text{ uniformly on } R^n_x\times K$$

for any α and β.

Lemma 1.2 (c.f. [7], p. 88). *If a sequence $\{p_j(x,\xi)\}_{j=1}^\infty$ of $S^m_{\rho,\delta}$ converges to a $p(x,\xi)$ of $S^m_{\rho,\delta}$ in $S^m_{\rho,\delta}$ weakly, then, $p_j(x,\xi)\to p(x,\xi)$ as $j\to\infty$ in the topology of $S^{m'}_{\rho,\delta}$ for any $m'>m$.*

Proof. We may assume $p(x,\xi)=0$. Then, the statement is clear from the inequality

$$\begin{aligned}&\max_{|\alpha+\beta|\leq k}\sup_{(x,\xi)}\{|p_{j(\beta)}^{(\alpha)}(x,\xi)|\langle\xi\rangle^{-(m'-\rho|\alpha|+\delta|\beta|)}\}\\ &\quad\leqq\max_{|\alpha+\beta|\leq k}\sup_{(x,\xi)\in R^n_x\times K}\{|p_{j(\beta)}^{(\alpha)}(x,\xi)|\langle\xi\rangle^{-(m'-\rho|\alpha|+\delta|\beta|)}\}\\ &\quad+\operatorname{map}_{|\alpha+\beta|\leq k}\sup_{(x,\xi)\in R^n_x\times(R^n_\xi\setminus K)}\{|p_{j(\beta)}^{(\alpha)}(x,\xi)|\langle\xi\rangle^{-(m-\rho|\alpha|+\delta|\beta|)}\}\max_{\xi\in(R^n_\xi\setminus K)}\langle\xi\rangle^{-(m'-m)}\,.\end{aligned}$$

DEFINITION 1.3. i) By $\mathring{S}^m_{\rho,\delta}$ we denote the set of all symbols $p(x, \xi)$ for which (1.1) holds for bounded functions $C_{\alpha,\beta}(x)$, instead of constants $C_{\alpha,\beta}$, such that

$$C_{\alpha,\beta}(x) \to 0 \quad \text{as } |x| \to \infty . \tag{1.5}$$

(We denote it also by $p(x, D_x) \in \mathring{S}^m_{\rho,\delta}$).
ii) We say that a symbol $p(x, \xi) (\in S^m_{\rho,\delta})$ is slowly varying, when $p_{(\beta)}(x, \xi) \in \mathring{S}^{m+\delta|\beta|}_{\rho,\delta}$ for any $\beta \neq 0$.

REMARK. In the inequality (1.4) we set $f(t) = p(x, \xi + 2^{-1}t\langle\xi\rangle^\rho\alpha)$ for $|\alpha| = 1$ (resp. $p(x + 2^{-1}t\langle\xi\rangle^{-\delta}\beta, \xi)$ for $|\beta| = 1$). Then, we have (1.5) for $|\alpha| = 1$ (resp. $|\beta| = 1$) and so for any α and β, if (1.5) holds only for $\alpha = \beta = 0$.

Lemma 1.4. *For any $p(x, \xi) \in S^m_{\rho,\delta}$ and real s we have*

$$\|p(x, D_x)u\|_s \leqq C\, |p|_{m,k} \|u\|_{s+m} \qquad \text{for } u \in H_{s+m}, \tag{1.6}$$

where C and k are constants independent of $p(x, \xi)$ and u.

Proof is omitted (c.f. Theorem 3.5 of [6] and Corollary 1 of Theorem 5.2 of [9]).

Lemma 1.5 (Grushin [4]). i) *Let $P \in S^m_{\rho,\delta}$ and $Q \in \mathring{S}^{m'}_{\rho,\delta}$. Then, we have*

$$PQ \in \mathring{S}^{m+m'}_{\rho,\delta} \quad \text{and} \quad QP \in \mathring{S}^{m+m'}_{\rho,\delta}.$$

ii) *Let $P \in S^m_{\rho,\delta}$ and $Q \in S^{m'}_{\rho,\delta}$. Assume that P and Q are slowly varying, Then, we have that $PQ (\in S^{m+m'}_{\rho,\delta})$ is slowly varying. Moreover, if we write $PQ = R_N + R'_N$ with*

$$\sigma(R_N)(x, \xi) = \sum_{|\alpha|<N} \frac{1}{\alpha!} \sigma(P)^{(\alpha)}(x, \xi) \sigma(Q)_{(\alpha)}(x, \xi),$$

then we have

$$R'_N \in \mathring{S}^{m+m'-(\rho-\delta)N}_{\rho,\delta} . \tag{1.7}$$

Proof. i) By Theorem 1.1 in [9] we have

$$\sigma(PQ)(x, \xi) = \int \langle D_\eta \rangle^{n_0} \sigma(P)(x, \xi+\eta) \left(\int e^{-iw\cdot\eta} \langle w \rangle^{-n_0} \sigma(Q)(x+w, \xi) dw \right) d\eta \tag{1.8}$$

for any even integer $n_0 \geqq n+1$. Then, writing for large $R > 0$

$$\int e^{-iw\cdot\eta} \langle w \rangle^{-n_0} \sigma(Q)(x+w, \xi) dw$$
$$= \int_{|w|\leqq R} e^{-iw\cdot\eta} \langle w \rangle^{-n_0} \sigma(Q)(x+w, \xi) dw + \int_{|w|\geqq R} e^{-iw\cdot\eta} \langle w \rangle^{-n_0} \sigma(Q)(x+w, \xi) dw,$$

we can easily see that $PQ \in \mathring{S}^{m+m'}_{\rho,\delta}$, and also get $QP \in \mathring{S}^{m+m'}_{\rho,\delta}$ in the same way.
ii) By the similar way to i) we can see by (1.8) that PQ is slowly varying. If we write

$$\sigma(Q)(x+w, \xi) = \sigma(Q)(x, \xi)+\sum_{j=1}^{n} w_j \int_0^1 \sigma(Q)_{(j)}(x+tw, \xi)dt\,,$$

then, from (1.8) we have

$$\begin{aligned}&\sigma(R_1')(x, \xi)\\ &= \int \langle D_\eta\rangle^{n_0}\sigma(P)(x, \xi+\eta)\Big(\int e^{-iw\cdot\eta}\langle w\rangle^{-n_0}\Big(\sum_{j=1}^{n} w_j \int_0^1 \sigma(Q)_{(j)}(x+tw, \xi)dt\Big)dw\Big)d\eta\\ &= \sum_{j=1}^{n}\int \langle D_\eta\rangle^{n_0}(i\partial_{\eta_j})\sigma(P)(x, \xi+\eta)\Big(\int e^{-iw\cdot\eta}\langle w\rangle^{-n_0}\int_0^1 \sigma(Q)_{(j)}(x+tw, \xi)dtdw\Big)d\eta\,.\end{aligned}$$

Since $\sigma(Q)_{(j)}(x+tw, \xi)\to 0$ as $|x|\to\infty$ together with all their derivatives, we see that $R_1'\in \mathring{S}^{m+m'-(\rho-\delta)}_{\rho,\delta}$. If we use Taylor's expansion of order N for $\sigma(Q)(x+w,\xi)$, we get (1.7) for any N. Q.E.D.

Lemma 1.6. *Let P belong to $\mathring{S}^{m}_{\rho,\delta}$. Then, P is compact from H_{s+m} into $H_{s'}$ for any $s>s'$.*

Proof. We write $\|Pu\|_{s'}=\|\wedge^{s}Pu\|_{-(s-s')}$. Then, by Lemma 1.5, we have $Q=\wedge^{s}P\in \mathring{S}^{s+m}_{\rho,\delta}$. Take a C_0^∞-function $a(x)$ such that $a(x)=1$ $(|x|\leqq 1)$ and $a(x)=0$ $(|x|\geqq 2)$, and set $Q_\varepsilon=a(\varepsilon x)Q$ for $0<\varepsilon<1$. Then, noting $|D_x^\alpha a(\varepsilon x)|\leqq C_\alpha\langle x\rangle^{-|\alpha|}$ for a constant C_α independent of ε, we see that $\{\sigma(Q_\varepsilon)(x, \xi)\}_{0<\varepsilon<1}$ makes a bounded set in $S^{s+m}_{\rho,\delta}$ and $\sigma(Q_\varepsilon)(x, \xi)\to\sigma(Q)(x, \xi)$ in the topology of $S^{s+m}_{\rho,\delta}$ because of $Q\in\mathring{S}^{s+m}_{\rho,\delta}$. Hence, we have

$$\sigma(\wedge^{-(s-s')}Q_\varepsilon)(x, \xi)\to\sigma(\wedge^{s'}P)(x, \xi) \text{ in the topology of } S^{s'+m}_{\rho,\delta}\,.$$

Since $\wedge^{-(s-s')}Q_\varepsilon$: $H_{s+m}\to H_0$ is compact, we get by Lemma 1.4 that P: $H_{s+m}\to H_{s'}$ is compact. Q.E.D.

2. Topology of symbol class

Throughout what follows we shall often use a C_0^∞-function $\psi(\xi)$ such that

(2.1) $$0\leqq\psi(\xi)\leqq 1 \text{ and } \psi(\xi) = \begin{cases}1 & (|\xi|\leqq 1)\\ 0 & (|\xi|\geqq 2)\,.\end{cases}$$

Consider $\{\psi(\varepsilon\xi)\}$, $0\leqq\varepsilon\leqq 1$. Then we have

(2.2) $$\begin{cases}0\leqq\psi(\varepsilon\xi)\leqq 1 \text{ and } \psi(\varepsilon\xi) = \begin{cases}1 & (|\xi|\leqq\varepsilon^{-1})\\ 0 & (|\xi|\geqq 2\varepsilon^{-1})\end{cases}\\ |\partial_\xi^\alpha\psi(\varepsilon\xi)|\leqq C_\alpha\langle\xi\rangle^{-|\alpha|}\end{cases}$$

for a constant C_α independent of ε, which means that

(2.3) $$\psi(\varepsilon\xi)\xrightarrow[\text{(weak)}]{} 1 \text{ in } S^0_{1,0} \text{ as } \varepsilon\to 0\,.$$

Lemma 2.1 *Let $P_j\in S^{m}_{\rho,\delta}$, $j=1, 2, \cdots$, and $Q\in S^{m'}_{\rho,\delta}$.*

Suppose that for a $P\in S^m_{\rho,\delta}$

(2.4) $\quad \sigma(P_j)(x,\xi)\xrightarrow[\text{(weak)}]{}\sigma(P)(x,\xi) \qquad in\ S^m_{\rho,\delta}.$

Then we have

$$(2.5)\qquad \begin{cases}\sigma(P_jQ)(x,\xi)\xrightarrow[\text{(weak)}]{}\sigma(PQ)(x,\xi) & in\ S^{m+m'}_{\rho,\delta}\\ \sigma(QP_j)(x,\xi)\xrightarrow[\text{(weak)}]{}\sigma(QP)(x,\xi) & in\ S^{m+m'}_{\rho,\delta}\end{cases}$$

and

(2.6) $\quad \sigma(P_j^{(*)})(x,\xi)\xrightarrow[\text{(weak)}]{}\sigma(P^{(*)})(x,\xi) \qquad in\ S^m_{\rho,\delta},$

where $P^{(*)}$ *is defined by*

(2.7) $\quad (Pu,v)=(u,P^{(*)}v) \qquad$ *for* $u,v\in\mathcal{S}$ (c.f. [9], p. 36).

Proof. From Corollary 2 of Theorem 4.1 in [9] we see that $\sigma(P_jQ)(x,\xi)$ and $\sigma(QP_j)(x,\xi)$ are bounded in $S^{m+m'}_{\rho,\delta}$ and that $\sigma(P_j^{(*)})(x,\xi)$ is bounded in $S^m_{\rho,\delta}$. By means of Theorem 1.1 in [9] we have

$$\begin{aligned}&\sigma(P_jQ)(x,\xi)\\&=\int\langle D_\eta\rangle^{n_0}\sigma(P_j)(x,\xi+\eta)\Big(\int e^{-iw\cdot\eta}\langle w\rangle^{-n_0}\sigma(Q)(x+w,\xi)dw\Big)d\eta\end{aligned}$$

for any even integer $n_0\geqq n+1$. We write

$$\begin{aligned}&\sigma(P_jQ)(x,\xi)\\&=\int_{|\eta|\leqq R}\langle D_\eta\rangle^{n_0}\sigma(P_j)(x,\xi+\eta)\Big(\int e^{-iw\cdot\eta}\langle w\rangle^{-n_0}\sigma(Q)(x+w,\xi)dw\Big)d\eta\\&\quad+\int_{|\eta|\geqq R}\langle D_\eta\rangle^{n_0}\sigma(P_j)(x,\xi+\eta)\langle\eta\rangle^{-2l}\Big(\int e^{-iw\cdot\eta}\langle D_w\rangle^{2l}(\langle w\rangle^{-n_0}\\&\quad\cdot\sigma(Q)(x+w,\xi))dw\Big)d\eta\,.\end{aligned}$$

Then, if we take a large l such that the second term is absolutely integrable and fix a large R, we see that

$$\sigma(P_jQ)(x,\xi)\to\sigma(PQ)(x,\xi) \text{ on } R^n_x\times K \text{ uniformly}$$

for any compact set K of R^n_ξ. Hence we get the half part of (2.5). For $\sigma(QP_j)(x,\xi)$ we get the assertion in the same way. For $\sigma(P_j^{(*)})(x,\xi)$ we use the formula in [9];

$$\sigma(P_j^{(*)})(x,\xi)=\int\Big(\int e^{-iw\cdot\eta}\langle w\rangle^{-n_0}\langle D_\eta\rangle^{n_0}\sigma(P_j)(x+w,\xi+\eta)dw\Big)d\eta\,,$$

and get (2.6).

Lemma 2.2. *Let $P_j \in S^m_{\rho,\delta}$, $j=1, 2, \cdots$. Suppose that*

$$\sigma(P_j)(x, \xi) \xrightarrow[(weak)]{} \sigma(P)(x, \xi) \text{ in } S^m_{\rho,\delta} \text{ for a } P \in S^m_{\rho,\delta}.$$

Then, for any s, we have

$$(2.8) \qquad \|P_j u - Pu\|_s \to 0 \, (j \to \infty) \text{ for } u \in H_{s+m}.$$

Proof. By Lemma 2.1 we have

$$\sigma(\wedge^s(P_j - P))(x, \xi) \xrightarrow[\text{(weak)}]{} 0 \text{ in } S^{s+m}_{\rho,\delta}.$$

Then, using a function $\psi(\xi)$ of (2.1), we have

$$\begin{aligned} \|P_j u - Pu\|_s &= \|\wedge^s(P_j - P)u\|_0 \\ &\leqq \|\wedge^s(P_j - P)\psi(\varepsilon D_x)u\|_0 + \|\wedge^s(P_j - P)(1 - \psi(\varepsilon D_x))u\|_0. \end{aligned}$$

By Lemma 1.4 we have

$$\|\wedge^s(P_j - P)\psi(\varepsilon D_x)u\|_0 \leq C\,|\sigma(\wedge^s(P_j - P))(x, \xi)\cdot\psi(\varepsilon\xi)|_{s+m,l}\|u\|_{s+m}$$

and

$$\|\wedge^s(P_j - P)(1 - \psi(\varepsilon D_x))u\|_0 \leqq C\,|\sigma(\wedge^s(P_j - P))(x, \xi)|_{s+m,l}\|(1 - \psi(\varepsilon D_x))u\|_{s+m}.$$

Then, noting $|\sigma(\wedge^s(P_j - P))(x, \xi)\cdot\psi(\varepsilon\xi)|_{s+m,l} \to 0$ $(j \to \infty)$ for any fixed $\varepsilon > 0$, and

$$\begin{aligned} \|(1 - \psi(\varepsilon D_x))u\|^2_{s+m} &= \int |(1 - \psi(\varepsilon\xi))|^2 (\langle\xi\rangle^{s+m} |\hat{u}(\xi)|)^2 d\xi \\ &\leqq \int_{|\xi| \geqq \varepsilon^{-1}} \langle\xi\rangle^{2(s+m)} |\hat{u}(\xi)|^2 d\xi \to 0 \quad (\varepsilon \to 0), \end{aligned}$$

we get (2.8). Q.E.D.

Lemma 2.3. *Let $P_z \in S^m_{\rho,\delta}$ for $z \in \Omega$ (an open set of $\boldsymbol{C}$). Suppose that $\sigma(P_z)(x, \xi)$ is an analytic function of z in Ω in the topology of $S^m_{\rho,\delta}$.*

Then we have, for any $Q \in S^{m'}_{\rho,\delta}$,

i) *$\sigma(P_z Q)(x, \xi)$ and $\sigma(Q P_z)(x, \xi)$ are analytic functions of z in Ω in the topology of $S^{m+m'}_{\rho,\delta}$ for any $Q \in S^{m'}_{\rho,\delta}$.*

ii) *For $u \in H_{s+m}$, $P_z u$ is an analytic function of z in Ω in the topology of H_s.*

Proof is omitted.

3. Complex powers

DEFINITION 3.1. For an $l \times l$ matrix $P \in S^m_{\rho,\delta}(m > 0)$ we say that operators P_z, $z \in \boldsymbol{C}$, $(\in S^{\infty}_{\rho,\delta})$ are complex powers of P, when P_z satisfy the following conditions (c.f. [10]):

i) For a monotone increasing function $m(s)$ such that

$$m(s) \to -\infty \, (s \to -\infty), \; m(0) = 0, \; m(s) \to \infty \, (s \to \infty),$$

we have $P_z \in S^{m(\mathrm{Re}\, z)}_{\rho,\delta}$, where Re z denotes the real part of z.

ii) $P_0 = I$ (identity operator), $P_1 = P$ (original operator).

iii) For any real s_0 $\sigma(P_z)(x, \xi)$ is an analytic function of z (Re $z < s_0$) in the topology of $S^{m(s_0)}_{\rho,\delta}$.

iv) For any real s_0

$$\sigma(P_s)(x, \xi) \underset{(\text{weak})}{\longrightarrow} \sigma(P_{s_0})(x, \xi) \text{ in } S^{m(s_0)}_{\rho,\delta}$$

as $s \uparrow s_0$ along the real axis.

v) $P_{z_1} P_{z_2} \equiv P_{z_1+z_2}$ (mod $S^{-\infty}$) in the sense:

$\sigma(P_{z_1} P_{z_2} - P_{z_1+z_2})(x, \xi)$ is an analytic function of z_1 and z_2 in the topology of $S^{s_0}_{\rho,\delta}$ for any real s_0.

First we state a result obtained by Nagase-Shinkai [12] in a modified form for our aim.

Theorem 3.2°. *Let $P = p(x, D_x)$ be a single operator of class $S^m_{\rho,\delta}$. Assume that the symbol $p(x, \xi)$ satisfies conditions:*

A) $|p(x, \xi)| \geqq c_0 \langle \xi \rangle^{\tau m}$ *for constant* $c_0 > 0$ *and* $\tau (0 < \tau \leqq 1)$,

B) $|p^{(\alpha)}_{(\beta)}(x, \xi) p(x, \xi)^{-1}| \leqq c_{\alpha,\beta} \langle \xi \rangle^{-\rho|\alpha|+\delta|\beta|}$

and

C) *arg* $p(x, \xi)$ *(the argument of* $p(x, \xi)$*) is well-defined for large* $|\xi|$. *Then, for* $m(s) = \tau m s (s < 0)$ *and* $= ms (s \geqq 0)$, *we can define complex powers* P_z *of* P *by*

$$\begin{aligned} &\sigma(P_z)(x, \xi) \\ &= p(x, \xi)^z \{1 + \sum_{|\alpha|=|\beta|=k\geqq 2} C_{k,\alpha,\beta}(z) p(x, \xi)^{-k} p^{(\alpha^1)}_{(\beta^1)}(x, \xi) \cdots p^{(\alpha^k)}_{(\beta^k)}(x, \xi)\}\,, \end{aligned}$$

where $p(x, \xi)^z = e^{z \log p(x,\xi)}$, $\alpha = (\alpha^1, \cdots, \alpha^k)$, $\beta = (\beta^1, \cdots, \beta^k)$ *and* $C_{k,\alpha,\beta}(z)$ *are polynomials in* z.

Proof is given in [12] for, so called, λ-elliptic operators. But, we can see that the discussion there works in our case, if we note

$$|\partial^\alpha_\xi D^\beta_x p(x, \xi)^z \cdot p(x, \xi)^{-z}| \leqq C_{z,\alpha,\beta} \langle \xi \rangle^{-\rho|\alpha|+\delta|\beta|}$$

and

$$|p(x, \xi)^{-1} p^{(\alpha^j)}_{(\beta^j)}(x, \xi)| \leqq C_{\alpha^j,\beta^j} \langle \xi \rangle^{-\rho|\alpha^j|+\delta|\beta^j|}, j = 1, \cdots, k\,,$$

for large $|\xi|$.

Our main theorem of this section is stated as follows.

Theorem 3.2. *Let $p(x, \xi) = (p_{jk}(x, \xi))$ be an $l \times l$ matrix of symbols $p_{jk}(x, \xi)$ of class $S^m_{\rho,\delta}$, $m > 0$, such that for some positive constants C_0, c_0, $C_{0,\alpha,\beta}$ and $\tau (0 < \tau \leqq 1)$*

(3.1) $\quad \|(p(x, \xi) - \zeta I)^{-1}\| \leqq C_0 \langle \xi \rangle^{-\tau m}$

and

(3.2) $\qquad \|p_{(\beta)}^{(\alpha)}(x, \xi)(p(x, \xi)-\zeta I)^{-1}\| \leqq C_{0,\alpha,\beta}\langle\xi\rangle^{-\rho|\alpha|+\delta|\beta|}$

for large $|\xi|$ *uniformly on* Ξ_0, *where* $\|\cdot\|$ *denotes a matrix norm and* $\Xi_0=\{\zeta\in\boldsymbol{C};$ *dis* $(\zeta,(-\infty, 0])\leqq c_0\}$. *Then, we can construct complex powers* $P_z=p_z(x, D_x)$ *of* $P=p(x, D_x)$ *such that*

(3.3) $\qquad P_z\in S_{\rho,\delta}^{\tau m\mathrm{Re}\, z}$ *for* $\mathrm{Re}\, z<0$, $\quad S_{\rho,\delta}^{m\mathrm{Re}\, z}$ *for* $\mathrm{Re}\, z\geqq 0$,

that is, $m(s)=\tau ms$ *for* $s<0$, $=ms$ *for* $s\geqq 0$.

REMARK. We may assume that $p(x, \xi)$ satisfies conditions (3.1) and (3.2) for every ξ. In fact, if we set $p_\varepsilon(x, \xi)=p(x, \xi)+\varepsilon^{-1}\psi(\varepsilon\xi)I$ for a C_0^∞-function $\psi(\xi)$ of (2.1), then, for a small fixed $\varepsilon_0>0$, $p_{\varepsilon_0}(x, \xi)$ staisfies (3.1) and (3.2) uniformly on Ξ_0 for any ξ, and we have complex powers $P_{\varepsilon_0,z}$ of P_{ε_0}. Set $P_z=P_{\varepsilon_0,z}+z(P-P_{\varepsilon_0,1})$. Then, noting $P\equiv P_{\varepsilon_0}=P_{\varepsilon_0,1}$, we get required powers of P.

For the proof of Theorem 3.2 we need several lemmas.

Lemma 3.3. *Let* $\zeta_1(x, \xi), \cdots, \zeta_l(x, \xi)$ *be eigen-values of* $p(x, \xi)$ *which satisfies* (3.1) *for* $\zeta=0$. *Then, there exists a positive constant* C_1 *such that*

(3.4) $\qquad C_1^{-1}\langle\xi\rangle^{\tau m}\leqq|\zeta_j(x, \xi)|\leqq C_1\langle\xi\rangle^m, j=1, \cdots, l.$

Proof. We write

$$\det(p(x, \xi)-\zeta I)=(-1)^l\{\zeta^l+\cdots+q_j(x, \xi)\zeta^{l-j}+\cdots+q_l(x, \xi)\}.$$

Then, noting $|q_j(x, \xi)|\leqq C\langle\xi\rangle^{jm}, j=1, \cdots, l$, for a constant C, we get easily the right half of (3.4). The left half is proved in the same way, if we use $\det(\zeta_j^{-1}I-p(x, \xi)^{-1})=0, j=1, \cdots, l$, and $\|p(x, \xi)^{-1}\|\leqq C_0\langle\xi\rangle^{-\tau m}$. Q.E.D.

Lemma 3.4. *Let* $p(x, \xi)(\in S_{\rho,\delta}^m)$ *satisfy conditions* (3.1) *and* (3.2). *Then, for any* $A(>C_1)$ *we have*

$$\begin{aligned}&\|(p(x, \xi)-\zeta I)^{-1}\|\leqq B|\zeta|^{-1}\\ &\quad\text{on } \Xi_{\xi,A}=\{\zeta\in\boldsymbol{C}; |\zeta|\leqq A^{-1}\langle\xi\rangle^{\tau m} \text{ or } |\zeta|\geqq A\langle\xi\rangle^m\},\end{aligned}\tag{3.5}$$

for a constant B, *where* C_1 *is a constant of Lemma* 3.3.

Proof. We write

$$\det(p(x, \xi)-\zeta I)=(-1)^l\prod_{j=1}^{l}(\zeta-\zeta_j(x, \xi)).$$

By Lemma 3.3 we have

$$\begin{aligned}&|\zeta-\zeta_j(x, \xi)|\\ &\geqq\begin{cases}|\zeta_j(x, \xi)|-|\zeta|\geqq C_1^{-1}\langle\xi\rangle^{\tau m}-|\zeta|\geqq(A/C_1-1)|\zeta| & \text{for } |\zeta|\leqq A^{-1}\langle\xi\rangle^{\tau m}\\ |\zeta|-|\zeta_j(x, \xi)|\geqq|\zeta|-C_1\langle\xi\rangle^m\geqq(1-C_1/A)|\zeta| & \text{for } |\zeta|\geqq A\langle\xi\rangle^m.\end{cases}\end{aligned}$$

Hence, we have

$$|\det(p(x,\xi)-\zeta I)| \geqq C|\zeta|^l \text{ on } \Xi_{\xi,A}.$$

Noting $\|(p(x,\xi)-\zeta I)\| \leqq \text{const.}\ |\zeta|$ for $|\zeta| \geqq A\langle\xi\rangle^m$, we get $\|(p(x,\xi)-\zeta I)^{-1}\| \leqq B'|\zeta|^{-1}$ for $|\zeta| \geqq A\langle\xi\rangle^m$.

Using

$$\zeta(p(x,\xi)-\zeta I)^{-1} = p(x,\xi)^{-1}(\zeta^{-1}-p(x,\xi)^{-1})^{-1},$$

we have in the same way

$$\begin{aligned}\|(p(x,\xi)-\zeta I)^{-1}\| &\leqq \|p(x,\xi)^{-1}\|\ \|(\zeta^{-1}-p(x,\xi)^{-1})^{-1}\|\ |\zeta|^{-1} \\ &\leqq C_0\langle\xi\rangle^{-\tau m}|\zeta^{-1}|^{-1}|\zeta|^{-1} \leqq B''|\zeta|^{-1} \text{ for } |\zeta| \leqq A^{-1}\langle\xi\rangle^{\tau m}.\end{aligned}$$

Hence, we have proved (3.5) Q.E.D.

Now following Hörmander [6], p. 165, we shall construct a parametrix for $p(x,\xi)-\zeta I$. We define $q_j(\zeta; x, \xi), j=0, 1, \cdots$, inductively by

$$q_0(\zeta; x, \xi) = (p(x,\xi)-\zeta I)^{-1}, \tag{3.6}$$

$$q_N(\zeta; x, \xi) = -\left\{\sum_{j=0}^{N-1}\sum_{|\alpha|=N-j}\frac{1}{\alpha!}\partial_\xi^\alpha q_j(\zeta; x, \xi) D_x^\alpha(p(x,\xi)-\zeta I)\right\} q_0(\zeta; x, \xi). \tag{3.7}$$

Lemma 3.5. *Let $p(x,\xi) \in S^m_{\rho,\delta}(m>0)$ satisfy conditions* (3.1) *and* (3.2). *Then, $q_j(\zeta; x, \xi), j=0, 1, \cdots$, defined by* (3.6) *and* (3.7) *are analytic functions of ζ on $\Xi_0 \cup \Xi_{\xi,A}$ and belong to $S^{-\tau m-(\rho-\delta)j}_{\rho,\delta}$ for any fixed $\zeta \in \Xi_0$, moreover satisfy*

$$\|q_0(\zeta; x, \xi)\| \leqq C_0\langle\xi\rangle^{-\tau m}, \tag{3.8}$$

$$\|q_{j(\beta)}^{(\alpha)}(\zeta; x, \xi)\| \leqq C_{j,\alpha,\beta}\langle\xi\rangle^{-\tau m-\rho|\alpha|+\delta|\beta|-(\rho-\delta)j} \qquad (j=0, 1, \cdots) \tag{3.9}$$

uniformly on Ξ_0, and

$$\|q_0(\zeta; x, \xi)\| \leqq C_0'|\zeta|^{-1}, \tag{3.10}$$

$$\|q_{j(\beta)}^{(\alpha)}(\zeta; x, \xi)\| \leqq C'_{j,\alpha,\beta}|\zeta|^{-1}\langle\xi\rangle^{-\rho|\alpha|+\delta|\beta|-(\rho-\delta)j} \qquad (j=0, 1, \cdots), \tag{3.11}$$

$$\|q_{j(\beta)}^{(\alpha)}(\zeta; x, \xi)\| \leqq C''_{j,\alpha,\beta}|\zeta|^{-2}\langle\xi\rangle^{m-\rho|\alpha|+\delta|\beta|-(\rho-\delta)j} \qquad (j+|\alpha+\beta| \neq 0), \tag{3.12}$$

$$\|q_{j(\beta)}^{(\alpha)}(\zeta; x, \xi)\| \leqq C'''_{j,\alpha,\beta}|\zeta|^{-3}\langle\xi\rangle^{2m-\rho|\alpha|+\delta|\beta|-(\rho-\delta)j} \qquad (j \geqq 1) \tag{3.13}$$

uniformly on $\Xi_0 \cup \Xi_{\xi,A}$.

Proof. The estimate (3.8) is clear by (3.1), and (3.9) is proved by induction in view of (3.2). We write

$$(p(x,\xi)-\zeta I)^{-1} = \zeta^{-1}\{p(x,\xi)(p(x,\xi)-\zeta I)^{-1}-I\}.$$

Then, from (3.1) and (3.2) we get (3.10) on Ξ_0, and by Lemma 3.4 we get on $\Xi_{\xi,A}$. For $|\alpha|=1$ we have

$$\partial_\xi^\alpha q_0 = -q_0\partial_\xi^\alpha p \cdot q, \qquad D_x^\alpha q_0 = -\ q_0 D_x^\alpha p \cdot q_0$$

and so

$$q_{0(\beta)}^{(\alpha)} = \sum C_{l,\beta^1,\dots,\beta^k}^{\alpha^1,\dots,\alpha^k} q_0 p_{(\beta^1)}^{(\alpha^1)} q_0 \cdots q_0 p_{(\beta^k)}^{(\alpha^k)} q_0 , \tag{3.14}$$

where the summation is taken under the condition

$$1 \leqq k \leqq |\alpha+\beta|, \qquad \alpha^1+\cdots+\alpha^k = \alpha, \qquad \beta^1+\cdots+\beta^k = \beta .$$

Hence, using (3.1) we have (3.9), (3.11) and (3.12) for $j=0$. From (3.7) we can see that $q_{j(\beta)}^{(\alpha)}$ also have the form (3.14) and get (3.9), (3.11)-(3.13) in general.

Q.E.D.

Now we construct a parametrix $r(\zeta; x, D_x)(\in S_{\rho,\delta}^{-\tau m})$ of $p(x, D_x)-\zeta I$ as follows: Let $\varphi(\xi)$ be a C_0^∞-function in R_ξ^n such that

$$\varphi(\xi) = 0 \quad (|\xi| \leqq 1) \quad \text{and} \quad \varphi(\xi) = 1 \quad (|\xi| \geqq 2), \tag{3.15}$$

and set as in Theorem 2.7 of [6]

$$r(\zeta; x, \xi) = q_0(\zeta; x, \xi) + \sum_{j=1}^{\infty} \varphi(t_j^{-1}\xi) q_j(\zeta; x, \xi) \tag{3.16}$$

for an appropriate increasing sequence $t_j \to \infty$. Then, by Lemma 3.5, we have

$$r(\zeta; x, \xi) \in S_{\rho,\delta}^{-\tau m} \text{ for } \zeta \in \Xi_0 , \tag{3.17}$$

and moreover we have

$$\| r_{(\beta)}^{(\alpha)}(\zeta; x, \xi) \| \leqq C_{\alpha,\beta} \langle\xi\rangle^{-\tau m-\rho|\alpha|+\delta|\beta|} \text{ unifomly on } \Xi_0 , \tag{3.18}$$

and

$$\| r_{(\beta)}^{(\alpha)}(\zeta; x, \xi) \| \leqq C'_{\alpha,\beta} |\zeta|^{-1} \langle\xi\rangle^{-\rho|\alpha|+\delta|\beta|} , \tag{3.19}$$

$$\| r_{(\beta)}^{(\alpha)}(\zeta; x, \xi) \| \leqq C''_{\alpha,\beta} |\zeta|^{-2} \langle\xi\rangle^{m-\rho|\alpha|+\delta|\beta|} , \quad |\alpha+\beta| \neq 0 , \tag{3.20}$$

$$\| r_{(\beta)}^{(\alpha)}(\zeta; x, \xi) - q_{0(\beta)}^{(\alpha)}(\zeta; x, \xi) \| \leqq C'''_{\alpha,\beta} |\zeta|^{-3} \langle\xi\rangle^{2m-(\rho-\delta)-\rho|\alpha|+\delta|\beta|} \tag{3.21}$$

uniformly on $\Xi_0 \cup \Xi_{\xi,A}$.

Let A be a positive number of Lemma 3.4 such that $A^{-1} < c_0$ for a constant c_0 of Theorem 3.2, and let $\Gamma_{\xi,A}$ be a counterclockwisely oriented curve defined by

$$\begin{aligned} \Gamma_{\xi,A} = &\{\zeta \in \boldsymbol{C};\ |\zeta| = A\langle\xi\rangle^m \text{ or } = A^{-1}\langle\xi\rangle^{\tau m},\ \mathrm{dis}\,(\zeta;(-\infty, 0]) \geqq A^{-1}\} \\ &\cup \{\zeta = \zeta_1 \pm iA^{-1};\ -R_1 \leqq \zeta_1 \leqq -R_2\} , \end{aligned} \tag{3.22}$$

where R_1 and R_2 are positive numbers satisfying

$$|-R_1+iA^{-1}| = A\langle\xi\rangle^m \text{ and } |-R_2+iA^{-1}| = A^{-1}\langle\xi\rangle^{\tau m}$$

respectively. Then, we have

Lemma 3.6. *For a complex number z we define symbols $p_z(x, \xi)$ by*

$$p_z(x, \xi) = \frac{1}{2\pi i} \int_{\Gamma_{\xi,A}} \zeta^z r(\zeta; x, \xi) d\zeta . \tag{3.23}$$

Then, for a function $m(s)=\tau ms\,(s<0)$ *and* $=ms\,(s\geqq 0)$, *we have* i)—iv) *of Definition* 3.1 *for* $p_z(x,\xi)$.

Proof. Since

$$p_{z(\beta)}^{(\alpha)}(x,\xi)=\frac{1}{2\pi i}\int_{\Gamma_{\xi,A}}\zeta^z r_{(\beta)}^{(\alpha)}(\zeta;x,\xi)d\zeta\,,$$

we have by (3.19)

$$\|p_{z(\beta)}^{(\alpha)}(x,\xi)\|\leqq\frac{C'_{\alpha,\beta}}{2\pi}e^{2\pi|\mathrm{Im}\,z|}\int_{\Gamma_{\xi,A}}|\zeta|^{\mathrm{Re}\,z-1}\langle\xi\rangle^{-\rho|\alpha|+\delta|\beta|}|d\zeta|\,.$$

Then, estimating the cases: Re $z<0$ and Re $z\geqq 0$ separately, and noting

$$p_s(x,\xi)\to p_{s_0}(x,\xi)\text{ uniformly on }R_x^n\times K\text{ as }s\uparrow s_0$$

for any compact set K of R_ξ^n, we have i) and iv). Next, we write

$$p_z(x,\xi)=\frac{1}{2\pi i}\int_{\Gamma_{\xi,A}}\zeta^z q_0(\zeta)d\zeta+\frac{1}{2\pi i}\int_{\Gamma_{\xi,A}}\zeta^z(r(\zeta)-q_0(\zeta))d\zeta\,.$$

Then, by (3.21) we see that the second term can be deformed to

$$\frac{1}{2\pi i}\int_{\Gamma_0}\zeta^z(r(\zeta)-q_0(\zeta))d\zeta\qquad\text{when Re }z<2\,,$$

and vanishes for $z=0$ and $=1$, where

$$(3.24)\quad \Gamma_0=\{\zeta\in\boldsymbol{C};\ \mathrm{dis}\,(\zeta;(-\infty,0])=A^{-1}\}\,.$$

Hence, noting that the first term defines $p(x,\xi)^z$ we get ii) of Definition 3.1. Since

$$\frac{d}{dz}p_{z(\beta)}^{(\alpha)}(x,\xi)=\frac{1}{2\pi i}\int_{\Gamma_{\xi,A}}\log\zeta\cdot\zeta^z r_{(\beta)}^{(\alpha)}(\zeta;x,\xi)d\zeta\,,$$

we get the last assertion in the same way. Q.E.D.

Lemma 3.7. *Let* $R(\zeta)=r(\zeta;x,D_x)(\zeta\in\Xi_0)$ *be the parametrix of* $P=p(x,D_x)$ *defined by* (3.16). *Then we have for* $\zeta_1\neq\zeta_2$

$$(3.25)\quad R(\zeta_1)R(\zeta_2)=(\zeta_2-\zeta_1)^{-1}(R(\zeta_2)-R(\zeta_1))+(\zeta_2-\zeta_1)^{-1}K(\zeta_1,\zeta_2)\,,$$

where $K(\zeta_1,\zeta_2)\in S^{-\infty}$ *is a pseudo-differential operator with the symbol* $k(\zeta_1,\zeta_2;x,\xi)$ *which satisfies, for any real number* s *and multi-index* α,β,

$$(3.26)\quad \|k_{(\beta)}^{(\alpha)}(\zeta_1,\zeta_2;x,\xi)\|\leqq C_{\alpha,\beta,s}|\zeta_1|^{-1}|\zeta_2|^{-1}\langle\xi\rangle^s\,.$$

Proof. For some $K_1(\zeta_1)$, $K_2(\zeta_2)$ of class $S^{-\infty}$ we have

$$R(\zeta_1)(P-\zeta_1 I)=I+K_1(\zeta_1)\text{ and }(P-\zeta_2 I)R(\zeta_2)=I+K_2(\zeta_2)\,.$$

Then, we have

$$R(\zeta_1)R(\zeta_2)(\zeta_2-\zeta_1)=R(\zeta_2)-R(\zeta_1)+K(\zeta_1,\zeta_2)\,,$$

where $K(\zeta_1, \zeta_2)=K_1(\zeta_1)R(\zeta_2)-R(\zeta_1)K_2(\zeta_2)$. Hence, by (3.19) we have only prove for symbols $k_j(\zeta_j; x, \xi)$ of $K_j(\zeta_j)$, $j=1, 2$,

$$\|k_{j(\beta)}^{(\alpha)}(\zeta_j; x, \xi)\| \leqq C_{j,\alpha,\beta,s}|\zeta_j|^{-1}\langle\xi\rangle^s \text{ for any } \alpha, \beta, s. \tag{3.27}$$

By Theorem 1.1 of [9] we can write for any integer N

$$\begin{aligned} k_1(\zeta_1; x, \xi) &= \sigma(R(\zeta_1)(P-\zeta_1 I))(x, \xi)-I \\ &= \sum_{|\alpha|<N} \frac{1}{\alpha!} \partial_\xi^\alpha r(\zeta_1; x, \xi) D_x^\alpha(p(x, \xi)-\zeta_1 I)+R_N(\zeta_1; x, \xi)-I \\ &\equiv I_N(\zeta_1; x, \xi)+R_N(\zeta_1; x, \xi), \end{aligned} \tag{3.28}$$

where

$$\begin{aligned} R_N(\zeta_1; x, \xi) = \int\langle D_\eta\rangle^{n_0} N \sum_{|\gamma|=N} \frac{\eta^\gamma}{\gamma!}\Big(\int_0^1(1-t)^{N-1}\partial_\xi^\gamma r(\zeta_1; z, \xi+t\eta)dt\Big) \\ \cdot\Big(\int e^{-iw\cdot\eta}\langle w\rangle^{-n_0}(p(x+w, \xi)-\zeta_1 I)dw\Big)d\eta \end{aligned} \tag{3.29}$$

for any even number $n_0 \geqq n+1$. Using (3.16) and interchanging the order of summation, we can write

$$\begin{aligned} I_N &= \sum_{|\alpha|<N} \sum_{j+|\alpha|<N} \frac{1}{\alpha!} \partial_\xi^\alpha q_j D_x^\alpha(p-\zeta_1 I)-I \\ &+ \sum_{|\alpha|<N} \sum_{\substack{j+|\alpha|<N \\ j\geqq 1}} \frac{1}{\alpha!} \partial_\xi^\alpha((\varphi_j(\xi)-1)q_j) D_x^\alpha(p-\zeta_1 I) \\ &+ \sum_{|\alpha|<N} \sum_{\substack{j+|\alpha|\geqq N \\ N>j\geqq 1}} \frac{1}{\alpha!} \partial_\xi^\alpha(\varphi_j(\xi)q_j) D_x^\alpha(p-\zeta_1 I) \\ &+ \sum_{|\alpha|<N} \sum_{j=N}^\infty \frac{1}{\alpha!} \partial_\xi^\alpha(\varphi_j(\xi)q_j) D_x^\alpha(p-\zeta_1 I) \equiv I_1+I_2+I_3+I_4. \end{aligned} \tag{3.30}$$

From (3.6) and (3.7) we have

$$I_1 = 0. \tag{3.31}$$

Using (3.12), we have

$$\|\partial_\xi^\alpha D_x^\beta I_2\| \leqq \text{const. } \langle\xi\rangle^s|\zeta_1|^{-2}(\langle\xi\rangle^m+|\zeta_1|) \leqq \text{const. } |\zeta_1|^{-1}\langle\xi\rangle^{m+s} \tag{3.32}$$

for any real number s, and

$$\begin{aligned} \|\partial_\xi^\alpha D_x^\beta I_3\| &\leqq \text{const. } |\zeta_1|^{-2}\langle\xi\rangle^{-(\rho-\delta)N}(\langle\xi\rangle^m+|\zeta_1|\langle\xi\rangle^{m-\rho|\alpha|+\delta|\beta|}) \\ &\leqq \text{const. } |\zeta_1|^{-1}\langle\xi\rangle^{2m-(\rho-\delta)N-\rho|\alpha|+\delta|\beta|}. \end{aligned} \tag{3.33}$$

Similarly we have

$$\|\partial_\xi^\alpha D_x^\beta I_4\| \leqq \text{const. } |\zeta_1|^{-1}\langle\xi\rangle^{2m-(\rho-\delta)N-\rho|\alpha|+\delta|\beta|}. \tag{3.34}$$

Finally we have to estimate $R_N(\zeta_1; x, \xi)$.

Since

$$\langle D_\eta\rangle^{n_0}(\eta^\gamma\partial_\xi^\gamma r(\zeta_1;x,\xi+t\eta))=\sum_{|\beta_1+\beta_2|\leqq n_0} C_{\beta_1,\beta_2}t^{|\beta_2|}\eta^{\gamma-\beta_1}\partial_\xi^{\gamma+\beta_2}r(\zeta_1;x,\xi+t\eta)$$

and

$$\eta^{\gamma-\beta_1}e^{-iw\cdot\eta}=(i\partial_w)^{\gamma-\beta_1}e^{-iw\cdot\eta},$$

integrating by parts we have only to estimate

$$\int\{\partial_\xi^{\gamma+\beta_2}r(\zeta_1;x,\xi+t\eta)(\int e^{-iw\cdot\eta}\partial_w^{\gamma-\beta_1}(\langle w\rangle^{-n_0}(p(x+w,\xi)-\zeta_1 I))dw)\}d\eta$$

$$=\int_{|\eta|\leqq\langle\xi\rangle/2}\{\partial_\xi^{\gamma+\beta_2}r(\zeta_1;x,\xi+t\eta)(\int e^{-iw\cdot\eta}\partial_w^{\gamma-\beta_1}(\langle w\rangle^{-n_0}(p(x+w,\xi)-\zeta_1 I))dw)\}d\eta$$

$$+\int_{|\eta|\geqq\langle\xi\rangle/2}\{\langle\eta\rangle^{-2l}\partial_\xi^{\gamma+\beta_2}r(\zeta_1;x.\ \xi+t\eta)$$

$$\cdot(\int e^{-iw\cdot\eta}\langle D_w\rangle^{2l}\partial_w^{\gamma-\beta_1}(\langle w\rangle^{-n_0}(p(x+w,\xi)-\zeta_1 I))dw)\}d\eta\equiv J_1+J_2\,.$$

Then, noting $C^{-1}\langle\xi\rangle\leqq\langle\xi+t\eta\rangle\leqq C\langle\xi\rangle$ for a constant $C>0$ when $|\eta|\leqq\langle\xi\rangle/2$ and $0\leqq t\leqq 1$, we have by (3.20)

$$\begin{aligned}||J_1(\zeta_1;x,\xi)||&\leqq \text{const. } |\zeta_1|^{-2}\langle\xi\rangle^{m-\rho(N+|\alpha|)+n}(\langle\xi\rangle^{m+\delta N}+|\zeta_1|)\\&\leqq \text{const. } |\zeta_1|^{-1}\langle\xi\rangle^{2m+n-(\rho-\delta)N}\,.\end{aligned}$$

Taking a large integer l we have

$$\begin{aligned}||J_2(\zeta_1;x,\xi)||&\leqq \text{const. } |\zeta_1|^{-2}\langle\xi\rangle^{m-2l+n}(\langle\xi\rangle^{2l\delta+N}+|\zeta_1|)\\&\leqq \text{const. } |\zeta_1|^{-1}\langle\xi\rangle^{m-2l(1-\delta)+n+N}\,.\end{aligned}$$

Hence, fixing l such as $m-2l(1-\delta)+N\leqq 2m-(\rho-\delta)N$, we have

$$||R_N(\zeta_1;x,\xi)||\leqq \text{const. } |\zeta_1|^{-1}\langle\xi\rangle^{2m+n-(\rho-\delta)N}$$

and also have

$$(3.35)\qquad ||R_{N(\beta)}^{(\alpha)}(\zeta_1;x,\xi)||\leqq \text{const. } |\zeta_1|^{-1}\langle\xi\rangle^{2m+n-(\rho-\delta)N-\rho|\alpha|+\delta|\beta|}\,.$$

Consequently from (3.28)–(3.35) we have (3.27) for $j=1$ for a large N, and for $j=2$ analogously, which completes the proof. Q.E.D.

Proof of Theorem 3.2. Let $P_z=p_z(x,D_x)$ be operators defined by (3.23). Then, by Lemma 3.6 we have i)–iv) of Definition 3.1. For the proof of v) we consider the case: Re $z_j<0$, $j=1,2$.

Set

$$\begin{aligned}\Gamma_1&=\{\zeta\in\boldsymbol{C};\ \text{dis}\,(\zeta,(-\infty,0])=c_0/2\}\,,\\ \Gamma_2&=\{\zeta\in C;\ \text{dis}\,(\zeta,(-\infty,0])=c_0/3\}\,.\end{aligned}$$

Then, by means of (3.19) and Lemma 3.7 we have

$$P_{z_1}P_{z_2}u(x)$$

$$
\begin{aligned}
&= \int e^{ix\cdot\xi}\Big\{\frac{1}{2\pi i}\int_{\Gamma_1}\zeta_1^{z_1} r(\zeta_1; x, \xi)d\zeta_1\Big\}P_{z_2}u(\xi)d\xi \\
&= \frac{1}{2\pi i}\int_{\Gamma_1}\zeta_1^{z_1}R(\zeta_1)P_{z_2}u(x)d\zeta_1 \\
&= \Big(\frac{1}{2\pi i}\Big)^2\int_{\Gamma_1}\int_{\Gamma_2}\zeta_1^{z_1}\zeta_2^{z_2}R(\zeta_1)R(\zeta_2)u(x)d\zeta_2 d\zeta_1 \\
&= \frac{1}{2\pi i}\int_{\Gamma_2}\zeta_2^{z_1+z_2}R(\zeta_2)u(x)d\zeta_2 \\
&\quad + \Big(\frac{1}{2\pi i}\Big)^2\int_{\Gamma_1}\int_{\Gamma_2}\zeta_1^{z_1}\zeta_2^{z_2}\frac{K(\zeta_1, \zeta_2)u(x)}{\zeta_2-\zeta_1}d\zeta_2 d\zeta_1 \\
&= P_{z_1+z_2}u(x) + \Big(\frac{1}{2\pi i}\Big)^2\int_{\Gamma_1}\int_{\Gamma_2}\zeta_1^{z_1}\zeta_2^{z_2}\frac{K(\zeta_1, \zeta_2)u(x)}{\zeta_2-\zeta_1}d\zeta_2 d\zeta_1 .
\end{aligned}
$$

Hence, we get iv) when $\operatorname{Re} z_j<0$, $j=1, 2$.

Next we consider P_zP-P_{z+1}. For any N, using (3.16), we write

$$
\begin{aligned}
\sigma(P_zP)(x, \xi) &= \sum_{|\alpha|<N}\frac{1}{\alpha!}p_z^{(\alpha)}(x, \xi)p_{(\alpha)}(x, \xi)+r_{z,N}(x, \xi) \\
&= \frac{1}{2\pi i}\Big\{\sum_{|\alpha|<N}\sum_{j+|\alpha|<N}\frac{1}{\alpha!}\int_{\Gamma_{\xi,A}}\zeta^z q_j^{(\alpha)}p_{(\alpha)}d\zeta \\
&\quad + \sum_{|\alpha|<N}\sum_{\substack{j+|\alpha|<N\\ j\geq 1}}\frac{1}{\alpha!}\int_{\Gamma_{\xi,A}}\zeta^z\partial_\xi^\alpha((\varphi_j(\xi)-1)q_j)p_{(\alpha)}d\zeta \\
&\quad + \sum_{|\alpha|<N}\sum_{\substack{j+|\alpha|\geq N\\ N>j\geq 1}}\frac{1}{\alpha!}\int_{\Gamma_{\xi,A}}\zeta^z\partial_\xi^\alpha(\varphi_j(\xi)q_j)p_{(\alpha)}d\zeta \\
&\quad + \sum_{|\alpha|<N}\sum_{j=N}^{\infty}\frac{1}{\alpha!}\int_{\Gamma_{\xi,A}}\zeta^z\partial_\xi^\alpha(\varphi_j(\xi)q_j)p_{(\alpha)}d\zeta\Big\}+r_{z,N} \\
&\equiv \frac{1}{2\pi i}\int_{\Gamma_{\xi,A}}\zeta^z(I_1+I_2+I_3+I_4)d\zeta+r_{z,N} ,
\end{aligned}
$$

where $r_{z,A}\in S_{\rho,\delta}^{m(\operatorname{Re} z)+m-(\rho-\delta)N}$ and, by the similar way to the estimation of $R_N(\zeta_1; x, \xi)$ in the proof of Lemma 3.7, is an analytic function of z ($\operatorname{Re} z<s_0$) in the topology of $S_{\rho,\delta}^{m(s_0)+m-(\rho-\delta)N}$ for any s_0. Using (3.7) we have

$$
\begin{aligned}
I_1 &= \sum_{\mu=0}^{N-1}\sum_{j=0}^{\mu}\sum_{|\alpha|=\mu-j}\frac{1}{\alpha!}q_j^{(\alpha)}p_{(\alpha)} \\
&= \sum_{\mu=0}^{N-1}\Big\{\sum_{j=1}^{\mu-1}\sum_{|\alpha|=\mu-j}\frac{1}{\alpha!}q_j^{(\alpha)}p_{(\alpha)}+q_\mu(p-\mu I)+\zeta q_\mu\Big\} \\
&= \sum_{\mu=0}^{N-1}\zeta q_\mu .
\end{aligned}
$$

It is clear that $\int_{\Gamma_{\xi,A}}I_2 d\zeta\in S^{-\infty}$, and is an analytic function of z in the topology of $S_{\rho,\delta}^{s_0}$ for any s_0. By the similar way to the proof of Lemma 3.6, we see that

$\int_{\Gamma_{\xi,A}}\zeta^z I_3 d\zeta$ and $\int_{\Gamma_{\xi,A}}\zeta^z I_4 d\zeta$ belong to $S_{\rho,\delta}^{m(\mathrm{Re}\,z)+m-(\rho-\delta)N}$ and are analytic in z ($\mathrm{Re}\,z<s_0$) in $S_{\rho,\delta}^{m(s_0)+m-(\rho-\delta)N}$ for any s_0. Now we write

$$p_{z+1}(x,\xi)=\frac{1}{2\pi i}\int_{\Gamma_{\xi,A}}\sum_{j=0}^{N-1}\zeta^{z+1}q_j d\zeta+r'_{z+1,N}(x,\xi)\,.$$

Then, by (3.11) we see that $r'_{z+1,N}(x,\xi)$ belongs to $S_{\rho,\delta}^{m(\mathrm{Re}\,z+1)-(\rho-\delta)N}$ and is analytic in z ($\mathrm{Re}\,z<s_0$) in $S_{\rho,\delta}^{m(s_0+1)-(\rho-\delta)N}$ for any s_0. Consequently we see, by taking large N, that $\sigma(P_zP-P_{z+1})(x,\xi)$ is analytic in z in the topology of $S_{\rho,\delta}^{s_0}$ for any s_0. Then, we see that, for any positive integer k,

$$\begin{aligned}&\sigma(P_zP^k-P_{z+k})(x,\xi)\\&=\sigma((P_zP-P_{z+1})P^{k-1})(x,\xi)+\cdots+\sigma(P_{z+k-1}P-P_{z+k})(x,\xi)\end{aligned}$$

is analytic in z in the topology of $S_{\rho,\delta}^{s_0}$ for any s_0. Hence, for any z_1 and z_2, if we fix a positive integer k such that $\mathrm{Re}\,z_j-k<0$, $j=1,2$, then writing

$$\begin{aligned}P_{z_1}P_{z_2}-P_{z_1+z_2}&=P_{z_1}(P_{z_2}-P_{z_2-2k}P^{2k})+(P_{z_1}-P_{z_1-k}P^k)P_{z_2-2k}P^{2k}\\&+P_{z_1-k}P^k(P_{z_2-2k}-P_{-k}P_{z_2-k})P^{2k}+P_{z_1-k}(P^kP_{-k}-I)P_{z_2-k}P^{2k}\\&+(P_{z_1-k}P_{z_2-k}-P_{z_1+z_2-2k})P^{2k}+(P_{z_1+z_2-2k}P^{2k}-P_{z_1+z_2})\end{aligned}$$

we see that $\sigma(P_{z_1}P_{z_2}-P_{z_1+z_2})(x,\xi)$ is analytic in z_1 and z_2 in the topology of $S_{\rho,\delta}^{s_0}$ for any s_0. Thus the proof is complete. Q.E.D.

4. Generalized Dirichlet problem

Let $p(x,\xi)$ be an $l\times l$ matrix of symbols $p_{jk}(x,\xi)$ which satisfies the assumption of Theorem 3.2, and let $P_z=p_z(x,D_x)$ be complex powers of P defined there.

We define a Hilbert space $H_{s,P}$ by

$$H_{s,P}=\{u\in H_{-\infty};\,P_su\in L^2\}$$

provided with the norm: $||u||_{s,P}=\{||P_su||_0^2+||\Phi(D_x)u||_0^2\}^{1/2}$, where $\Phi(\xi)$ is a fixed function of $\mathcal{S}$ such that $\Phi(\xi)>0$ in R_ξ^n.

Then we have

Theorem 4.1. *For any real number s, there exist constants C_s and C'_s such that*

$$\text{(4.1)}\qquad\begin{cases}C'_s||u||_{\tau ms}\leqq||u||_{s,P}\leqq C_s||u||_{ms}\ \textit{for}\ s\geqq0\,,\\ C'_s||u||_{ms}\leqq||u||_{s,P}\leqq C_s||u||_{\tau ms}\ \textit{for}\ s<0\,.\end{cases}$$

Proof. Noting $P_s\in S_{\rho,\delta}^{ms}(s\geqq0)$, $P_s\in S_{\rho,\delta}^{\tau ms}(s<0)$ and $\Phi(D_x)\in S^{-\infty}$, we have the right halves of (4.1) by means of Lemma 1.4. For $s\geqq0$ we write

$$||u||_{\tau ms}=||\wedge^{\tau ms}u||_0=||\wedge^{\tau ms}(P_{-s}P_s-K_s)u||_0\,,$$

where $K_s \in S^{-\infty}$ which is defined by $P_{-s}P_s = I + K_s$. Then noting $\wedge^{\tau m s}P_{-s} \in S^0_{\rho,\delta}$ and $\wedge^{\tau m s}K_s \in S^{-\infty}$, we have by Lemma 1.4

$$||u||_{\tau ms} \leqq ||\wedge^{\tau ms}P_{-s}(P_s u)||_0 + ||\wedge^{\tau ms}K_s u||_0 \leqq C''_s(||P_s u||_0 + ||u||_{\tau ms-1}).$$

On the other hand, for any $\varepsilon > 0$, there exists a constant C_ε such that

$$||u||_{\tau ms-1} \leqq \varepsilon ||u||_{\tau ms} + C_\varepsilon ||\Phi(D_x)u||_0,$$

so, if we fix $\varepsilon_0 > 0$ such that $C''_s \varepsilon_0 < 1/2$, we have

$$\frac{1}{2}\, ||u||_{\tau ms} \leqq C''_s(||P_s u||_0 + C_{\varepsilon_0}||\Phi(D_x)u||_0).$$

Hence, we have $C'_s||u||_{\tau ms} \leqq ||u||_{s,P}$ for $s \geqq 0$. Writing $||u||_{ms} = ||\wedge^{ms}(P_{-s}P_s - K_s) u||_0$, we can also prove the statement for $s < 0$ in this manner. Q.E.D.

Lemma 4.2. *Let $P(\in S^m_{\rho,\delta})$ be a formally self-adjoint in the sense*

$$(Pu, v) = (u, Pv) \qquad \textit{for } u, v \in \mathcal{S},$$

and satisfy the condition of Theorem 3.2, and let P_z be complex powers of P defined there. Then, we have

$$(4.2) \qquad P_z^{(*)} \equiv P_{\bar z} \ (\textit{mod } S^{-\infty}),$$

where $P_z^{(*)}(\in S^m_{\rho,\delta})$ is defined by

$$(P_z u, v) = (u, P_z^{(*)}v) \qquad \textit{for } u, v \in \mathcal{S}.$$

Proof. By the assumption it is clear that $(P^k)^{(*)} = P^k$ for any positive integer k. If we can prove

$$(4.3) \qquad P_z^{(*)} \equiv P_{\bar z} \text{ for } \mathrm{Re}\, z < 0,$$

then, by v) of Definition 3.1, it follows that for $k(\mathrm{Re}\, z < k)$

$$\begin{aligned} P_z^{(*)} &\equiv (P_k P_{z-k})^{(*)} = P_{z-k}^{(*)}P_k^{(*)} \equiv P_{\bar z - k}P_k^{(*)} \\ &\equiv P_{\bar z-k}(P^k)^{(*)} = P_{\bar z-k}P^k \equiv P_{\bar z-k}P_k \equiv P_{\bar z} \ (\text{mod } S^{-\infty}). \end{aligned}$$

Hence, we have only to prove (4.3). Let $R(\zeta) = r(\zeta; x, D_x)$ be the parametrix of $P - \zeta I$. Since $I \equiv ((P-\zeta I)R(\zeta))^{(*)} = R(\zeta)^{(*)}(P - \bar\zeta I)$, $R(\zeta)^{(*)}$ is the parametrix of $P - \bar\zeta I$. Now, using the path Γ_0 of (3.24), we have for $u, v \in \mathcal{S}$

$$\begin{aligned} (P_z u, v) &= \left(\int e^{ix\cdot\xi}\left(\frac{1}{2\pi i}\int_{\Gamma_0}\zeta^z r(\zeta; x, \xi)d\zeta\right)\hat u(\xi)d\xi, v\right) \\ &= \frac{1}{2\pi i}\int_{\Gamma_0}\zeta^z(R(\zeta)u, v)d\zeta = \frac{1}{2\pi i}\int_{\Gamma_0}\zeta^z(u, R(\zeta)^{(*)}v)d\zeta \end{aligned}$$

$$= \int u(x)\Big(\frac{1}{2\pi i}\int_{\Gamma_0} \zeta^z \overline{R(\zeta)^{(*)}v(x)}d\zeta\Big)dx\,.$$

Then we get

$$P_z{}^{(*)}v = \overline{\frac{1}{2\pi i}\Big(\int_{\Gamma_0} \zeta^{\bar z}\overline{R(\zeta)^{(*)}v(x)}d\zeta\Big)}$$

$$= -\frac{1}{2\pi i}\int e^{ix\cdot\xi}\hat{v}(\xi)\Big(\overline{\int_{\Gamma_0} \zeta^z \overline{r^{(*)}(\zeta\,;x,\xi)}d\zeta}\Big)d\xi\,,$$

so that we have

$$\sigma(P_z{}^{(*)}) = -\frac{1}{2\pi i}\overline{\Big(\int_{\Gamma_0} \zeta^{z}\overline{r^{(*)}(\zeta\,;x,\xi)}d\zeta\Big)} = \frac{1}{2\pi i}\int_{\Gamma_0} \zeta^{\bar z} r^{(*)}(\zeta\,;x,\xi)d\zeta\,.$$

Noting $r^{(*)}(\zeta\,;x,\xi)$ is a parametrix of $P-\zeta I$, we have (4.3). Q.E.D.

Theorem 4.3. *Let L be an $l\times l$ matrix of pseudo-differential operators of class $S^m_{\rho,\delta}(m>0)$, and set*

$$P = (L+L^{(*)})/2,\ Q = (L-L^{(*)})/2\,.$$

Assume that $\sigma(P)(x,\xi)$ satisfies the assumption of Theorem 3.2 and $P_{-\frac12}QP_{-\frac12}\in S^0_{\rho,\delta}$, where P_z is complex powers defined by Theorem 3.2. Then, there exist constants C and λ_0 such that

(4.4) $\quad |(Lu,v)|\leqq C||u||_{\frac12,P}||v||_{\frac12,P} \qquad$ *for $u,v\in\mathcal{S}$*

and

(4.5) $\quad Re\,(Lu,u)\geqq ||u||^2_{\frac12,P}-\lambda_0||u||^2_0 \qquad$ *for $u\in\mathcal{S}$.*

REMARK 1°. i) Assume that $Q\in S^{\tau m}_{\rho,\delta}$. Then, we have

$$P_{-\frac12}QP_{-\frac12}\in S^0_{\rho,\delta},\quad \text{since } P_{-\frac12}\in S^{-\tau m/2}_{\rho,\delta}\,.$$

ii) For the single case we assume that Re $\sigma(L)(x,\xi)$ satisfies
A)′ Re $\sigma(L)(x,\xi)\geqq c_0\langle\xi\rangle^{\tau m}$,
B)′ $|\partial_\xi^\alpha D_x^\beta\sigma(L)(x,\xi)\cdot(\text{Re}\,\sigma(L)(x,\xi))^{-1}|\leqq c_{\alpha,\beta}\langle\xi\rangle^{-\rho|\alpha|+\delta|\beta|}$
and
C′) are Re $\sigma(L)(x,\xi)$ is well-defined
for large $|\xi|$ instead of conditions A)–B) of Theorem 3.2°. Then, by using the asymptotic expansion fomula of $\sigma(P_z)(x,\xi)$, we can see that the operator L satisfies the conditions of Theorem 4.3.

REMARK 2°. The inequality (5.4) is a generalization of Gårding's inequality to hypoelliptic operators, which is different form [3], [9], [11], [17] where the positivity as in A)′ is not assumed, but the space is limited to the usual Sobolev space.

Proof of Theorem 4.3. We can write for $u,v\in\mathcal{S}$

$$(4.6)\quad \begin{aligned}(Lu, v) &= (Pu, v)+(Qu, v)\\ &= (P_{\frac{1}{2}}u, P_{\frac{1}{2}}^{(*)}v)+(P_{-\frac{1}{2}}QP_{-\frac{1}{2}}(P_{\frac{1}{2}}u), P_{\frac{1}{2}}^{(*)}v)+(Ku, v)\end{aligned}$$

for some $K\in S^{-\infty}$. Then, from Lemma 4.2 and the assumption $P_{-\frac{1}{2}}QP_{-\frac{1}{2}}\in S^{0}_{\rho,\delta}$, we have

$$(4.7)\quad |(Lu, v)| \leqq C||u||_{\frac{1}{2},P}||v||_{\frac{1}{2},P} \text{ for } u, v\in \mathcal{S}$$

for a constant C. On the other hand, using Lemma 4.2 again and noting $\mathrm{Re}\,(Qu, u)=0$, we have

$$(4.8)\quad \mathrm{Re}\,(Lu, u) = (Pu, u)\geqq ||u||^2_{\frac{1}{2},P}-\lambda_0||u||^2_0$$

for a constant λ_0. Q.E.D.

Now, let V be the closure of $C_0^\infty(\Omega)$ in $H_{\frac{1}{2},P}$ for an open set Ω of R^n_x, and set

$$(4.9)\quad B_\lambda[u, v] = (P_{\frac{1}{2}}u, P_{\frac{1}{2}}^{(*)}v)+(P_{-\frac{1}{2}}QP_{-\frac{1}{2}}(P_{\frac{1}{2}}u), P_{\frac{1}{2}}^{(*)}v)+(Ku, v)+\lambda(u, v) \quad \text{for } u, v\in V\,.$$

Then, we have

Theorem 4.4 (Generalized Dirichlet problem). *Let L be a matrix of operators of class $S^m_{\rho,\delta}$ $(m>0)$ which satisfies conditions of Theorem 4.3. Then, for any $f\in L^2(\Omega)$, we can find a unique element $u\in V$ such that*

$$(L+\lambda)u = f \qquad \textit{in } \Omega$$

for any $\lambda\geqq\lambda_0$, where λ_0 is a constant determined in Theorem 4.3.

Proof. Consider $B_\lambda[u, v]$ for $u, v\in V$. Then, from (4.6)–(4.9) we have

$$(4.10)\quad \begin{cases}|B_\lambda[u, v]| \leqq C_\lambda||u||_{\frac{1}{2},P}||v||_{\frac{1}{2},P}\,,\\ \mathrm{Re}\ B_\lambda[u, u]\geqq ||u||^2_{\frac{1}{2},P} \qquad \text{for } u, v\in V\,.\end{cases}$$

Then, by means of the Lax-Milgram theorem (see, for example, [1], p. 98), we have a unique element $u\in V$ such that

$$B_\lambda[u, v] = (f, v) \qquad \text{for any } v\in V.$$

In particular for $v\in C_0^\infty(\Omega)$ we have from (4.6) and (4.9)

$$B_\lambda[u, v] = (Lu, v)+\lambda(u, v)$$

Hence, we have $(L+\lambda)u=f$ in Ω. Q.E.D.

REMARK. Consider a neighborhood $U(x_0)$ of a point x_0 on the boundary $\partial\Omega$ of Ω. Assume that $\partial\Omega$ is smooth and P is elliptic of order m_0 (>0) in $U(x_0)$ in the sense

$$(4.11)\quad \begin{cases} |\sigma(P)(x,\xi)|\geqq C_0\langle\xi\rangle^{m_0}, \\ |\sigma(P)^{(\alpha)}_{(\beta)}(x,\xi)|\leqq C_{\alpha,\beta}\langle\xi\rangle^{m_0-\rho|\alpha|+\delta|\beta|} \qquad \text{in } U(x_0)\end{cases}$$

for large $|\xi|$. Then, for any $a(x)\in C_0^\infty(U(x_0))$, we have

$$(4.12)\quad au\in H_{\frac{1}{2}m_0}$$

and concerning the trace of au, we have

$$(4.13)\quad \partial_n^j(au)|_{\partial\Omega}=0,\ 0\leqq j<(m_0-1)/2\,,$$

where ∂_n denotes the normal derivative for $\partial\Omega$. In fact, we can write for some $K\in S^{-\infty}$

$$au=aP_{-\frac{1}{2}}(P_{\frac{1}{2}}u)+aKu=(aP_{-\frac{1}{2}}\wedge^{\frac{1}{2}m_0})(\wedge^{-\frac{1}{2}m_0}P_{\frac{1}{2}}u)+aKu\,.$$

Then, noting $P_{\frac{1}{2}}u\in L^2$ we have $\wedge^{-\frac{1}{2}m_0}P_{\frac{1}{2}}u\in H_{\frac{1}{2}m_0}$, and in view of (4.11) we have $aP_{-\frac{1}{2}}\wedge^{\frac{1}{2}m_0}\in S^0_{\rho,\delta}$. Consequently we have (4.12), and noting supp $u\subset\bar{\Omega}$, we get (4.13).

EXAMPLE. Consider a single operator

$$L=a(x)\wedge^m+(1-a(x))\wedge^{m'}\,,$$

where $m, m'(m>m')$ are positive number and $a(x)$ is a C^∞-function such that

$$a(x)=0(|x|\leqq 1/2),\ =1(|x|\geqq 1),\ 0<a(x)<1\,(1/2<|x|<1)$$

and for a fixed $\sigma\geqq 1$

$$|D_x^\alpha a(x)/a(x)|\leqq C_\alpha\left|\,|x|-\frac{1}{2}\,\right|^{-\sigma|\alpha|} \text{ for any } \alpha\,.$$

Then, setting $\tau=m'/m$, we can see that $\sigma(L)(x,\xi)$ satisfies A) and B) of Definition 3.2° for any $0<\delta<1$ and $\rho=1$, so that Theorem 4.3 is applied to this operator L.

5. Index theory

First we describe results obtained in [10] with complete proofs. Let P be a system of pseudo-differential operators of class $S^m_{\rho,\delta}$, which maps $H_{-\infty}$ into itself, more precisely H_{s+m} into H_s boundedly for any real s.

Consider P as the closed operator of $L^2(=H_0)$ into itself with the domain $\mathcal{D}(P)$ defined by

$$(5.1)\quad \mathcal{D}(P)=\{u\in L^2;\, Pu\in L^2\}\,.$$

Then, the adjoint operator $P^*: L^2\to L^2$ is defined as follows. For a $v\in L^2$, if there exists $g\in L^2$ such that

$$(5.2)\quad (Pu,v)=(u,g) \qquad \text{for any } u\in\mathcal{D}(P)\,,$$

we say that v belongs to the domain $\mathcal{D}(P^*)$ of P^* and define $P^*v=g$. On the other hand we have defined the formal adjoint $P^{(*)}$ of class $S^m_{\rho,\delta}$ by

(5.3) $\quad (Pu, v) = (u, P^{(*)}v) \qquad$ for any $u, v \in \mathcal{S}$.

Then, considering $P^{(*)}$ as the closed operator L^2 into itself as above, we have

(5.4) $\quad \mathcal{D}(P^{(*)}) = \{v \in L^2; P^{(*)}v \in L^2\}$.

Concerning P^* and $P^{(*)}$ we have

Lemma 5.1. *Let P be a system of operators of class $S^m_{\rho,\delta}$. Then, as the operator of L^2 into itself, the operator $P^{(*)}$ is an extension of P^*, so that we have*

(5.5) $\quad \mathcal{D}(P^*) \subset \mathcal{D}(P^{(*)})$.

Proof. Assume $v \in \mathcal{D}(P^*)$. Then, noting $\mathcal{D}(P) \supset \mathcal{S}$, we have

$$(u, P^*v) = (Pu, v) = (u, P^{(*)}v).$$

In the above the right half is guaranteed, if we take a sequence $v_j (\in \mathcal{S}) \rightarrow v$ in L^2 and, considering u as an element of H_m, apply Lemma 1.4. Then, we have $P^*v = P^{(*)}v \in L^2$, which means that $v \in \mathcal{D}(P^{(*)})$. Q.E.D.

Lemma 5.2. *Let $P(\in S^m_{\rho,\delta})$ have complex powers P_z in the sense of Definition* 3.1. *Then, we have, for any $z_0 \in \boldsymbol{C}$, $P_{z_0}{}^{(*)} = P_{z_0}{}^*$ as the operator of L^2 into itself.*

Proof. By means of Lemma 5.1 we have only to prove

(5.6) $\quad (P_{z_0}u, v) = (u, P_{z_0}{}^{(*)}v)$ for $u \in \mathcal{D}(P_{z_0})$, $v \in \mathcal{D}(P_{z_0}{}^{(*)})$.

By i) of Definition 3.1 for a large N we have $P_z u \in H_{m(\mathrm{Re}\, z)}$ for $u \in \mathcal{D}(P_{z_0})$ so, using Lemma 1.4, we have

$$(5.7) \quad \begin{aligned} &(P_z u, P_{z_0}{}^{(*)}v) = (P_{z_0}P_z u, v) = (P_z P_{z_0} u, v) \\ &\quad + ((P_{z_0}P_z - P_z P_{z_0})u, v) \text{ for } u \in \mathcal{D}(P_{z_0}),\ v \in \mathcal{D}(P_{z_0}{}^{(*)})\ (\mathrm{Re}\, z < -N). \end{aligned}$$

From Lemma 2.3 and iii) of Definition 3.1 we have $(P_z u, P_{z_0}{}^{(*)}v)$ is analytic in z when $\mathrm{Re}\, z < 0$, and from Lemma 2.2 and iv) of Definition 3.1 we have $\lim_{s \to -0} (P_s u, P_{z_0}{}^{(*)}v) = (u, P_{z_0}{}^{(*)}v)$. Since $P_{z_0}u \in L^2$, we also have that $(P_z P_{z_0} u, v)$ is analytic in z when $\mathrm{Re}\, z < 0$ and $\lim_{s \to -0} (P_s P_{z_0} u, v) = (P_{z_0}u, v)$. Setting $s_0 = 0$ in v) of Definition 3.1 and writing $P_{z_0}P_z - P_z P_{z_0} = (P_{z_0}P_z - P_{z_0+z}) + (P_{z_0+z} - P_z P_{z_0})$, we can see that $((P_{z_0}P_z - P_z P_{z_0})u, v)$ is analytic in z and $\lim_{s \to -0} ((P_{z_0}P_s - P_s P_{z_0})u, v) = 0$. Then. letting $z \rightarrow -0$ on the real line in (5.7), we get (5.6). Q.E.D.

Lemma 5.3. *Let $p_j(x, \xi)$, $j = 0, 1, 2, \cdots$, be a sequence of slowly varying* *symbols of class* $S^{m_j}_{\rho,\delta}$ *(resp.* $\overset{\circ}{S}{}^{m_j}_{\rho,\delta}$*) such that* $m_j \downarrow -\infty$ *as* $j\to\infty$. *Then we can construct a slowly varying symbol* $p(x,\xi)\in S^{m}_{\rho,\delta}$*(resp.* $\overset{\circ}{S}{}^{m}_{\rho,\delta}$*) such that*

$$(5.8)\qquad p(x,\xi)-\sum_{j=1}^{N-1}p_j(x,\xi)\in S^{m_N}_{\rho,\delta},\ (resp.\ \overset{\circ}{S}{}^{m_N}_{\rho,\delta})$$

and is slowly varying for any N (c.f. [4]).

Proof. Take C^∞-functions $\varphi(\xi)$ and $\psi(x,\xi)$ such that

$$(5.9)\qquad \begin{cases}\varphi(\xi)=0(|\xi|\leqq 1),=1(|\xi|\geqq 2),\\ \psi(x,\xi)=0(|x|+|\xi|\leqq 1),=1(|x|+|\xi|\geqq 2).\end{cases}$$

Then, setting $p(x,\xi)=p_0(x,\xi)+\sum_{j=1}^{\infty}\varphi(t_j^{-1}\xi)\psi(t_j^{-1}x,t_j^{-1}\xi)p_j(x,\xi)$ for an appropriate $t_j\to\infty(j\to\infty)$, we get a required symbol. Q.E.D.

Lemma 5.4 (c.f. Prop. 2.1 of [8]). *Let* $\{P_t\}_{t\in[0,1]}$ *be a family of operators of class* $S^{m}_{\rho,\delta}$ *such that* $\sigma(P_t)(x,\xi)$ *is a continuous function of* t *in* $S^{m}_{\rho,\delta}$. *Suppose there exist two families* $\{Q_t\}_{t\in[0,1]}$ *and* $\{K_t\}_{t\in[0,1]}$ *in* $S^{0}_{\rho,\delta}$ *such that* $Q_tP_t=I+K_t$, Q_t *is strongly continuous in* t, *and* K_t *is uniformly continuous in* t *and compact as operators from* L^2 *into itself. Then, it follows that*

$$\textit{dim ker } P_t<\infty \textit{ and } \mathrm{Re}\, P_t \textit{ is closed}$$

and that

$$\textit{index } P_t\equiv \textit{dim ker } P_t\textit{-codim } \mathrm{Re}\, P_t$$

is upper semi-continuous in t, *where ker* P_t *denotes the kernel of* P_t *and* Re P_t *denotes the range of* P_t.

Proof. For $u\in\ker P_t$ we have

$$0=Q_tP_tu=u+K_tu\,.$$

Then, we can easily see that dim ker $P_t<\infty$, sicne K_t is compact. If we write $L^2=\ker P_t\oplus(\ker P_t)^\perp$, then, for the closedness of Re P_t we have only to prove

$$(5.10)\qquad \|u\|_0\leqq C_t\|P_tu\|_0 \text{ for } u\in\mathcal{D}(P_t)\frown(\ker P_t)^\perp$$

for a constant C_t.

Assume that there exists a sequence $\{u_\nu\}_{\nu=1}^\infty$ of $\mathcal{D}(P_t)\frown(\ker P_t)^\perp$ such that $1=\|u_\nu\|_0\geqq\nu\|P_tu_\nu\|_0$. Then, we have

$$0\leftarrow Q_tP_tu_\nu=u_\nu+K_tu_\nu\,.$$

Since K_t is compact, by taking a subsequence we may assume that

$$K_tu_\nu\to v \text{ in } L^2 \text{ for a } v\in L^2\,.$$

Then we have $v\in\ker P_t$ and consequently $0=(v,u_n)\to\|v\|^2=1$, which derives the contradiction.

For the proof of the upper semi-continuity of index P_t we first get the statement:

(5.11) If $t_\nu \to t_0 \in [0, 1]$, $u_\nu \to u_0$ in L^2, $P_{t_\nu} u_\nu \to f_0$ in L^2, then, $P_{t_0} u_0 = f_0$,

which means that the graph $\{(t, u, P_t u); t \in I, u \in \mathcal{D}(P_t)\}$ is closed. For any $v \in H_m$ we have

$$(P_{t_0} u_0, v) = (u_0, P_{t_0}{}^{(*)} v) = \lim_{\nu\to\infty} (u_\nu, P_{t_\nu}{}^{(*)} v) = \lim_{\nu\to\infty} (P_{t_\nu} u_\nu, v) = (f_0, v),$$

since $u_\nu \to u_0$ in L^2 and $P_{t_\nu}{}^{(*)} v \to P_{t_0}^{(*)} v$ in $L^2 = H_0$ by Lemma 1.4 and the continuity of $\sigma(P_t)(x, \xi)$ in $S^m_{\rho,\delta}$. Hence we get (5.11).

Now let W be a finite dimensional subspace of L^2 and set $\Delta_t = \{u \in \mathcal{D}(P_t); P_t u \in W\}$. Then we can easily get

(5.12) $\|P_t u\|_0 \leqq C\|u\|_0$ for $u \in \Delta_t$

for a constant C independent of $t \in [0, 1]$.

Assume there exist sequences $\{t_\nu\}_{\nu=1}^\infty$ and orthonormal systems $\{u_1^{(\nu)}, \cdots, u_l^{(\nu)}\}$ of Δ_{t_ν} for a fixed l such that $t_\nu \to t_0 \in [0, 1]$. Then, writing $Q_{t_\nu} P_{t_\nu} u_j^{(\nu)} = u_j^{(\nu)} + (K_{t_\nu} - K_{t_0}) u_j^{(\nu)} + K_{t_0} u_j^{(\nu)}$, $j = 1, \cdots, l$, we may assume that $K_{t_0} u_j^{(\nu)} \to v_j$ and $P_{t_\nu} u_j^{(\nu)} \to w_j \in W$ for $j = 1, \cdots, l$ by taking a subsequence, since K_{t_0} is compact and $P_{t_\nu} u_j^{(\nu)} \in W$ (finite dimensional) with (5.12). Hence from (5.11) we have $P_{t_0} u_j = w_j$ for $u_j = -v_j + Q_{t_0} w_j$. It is clear that $u_1, \cdots, u_l$ is orthonormal, which means that $\dim \Delta_t$ is upper simi-continuous in t. Then, for any $W_0 \subset (\mathrm{Re}\, P_{t_0})^\perp$, we have

$$\begin{aligned}\dim \Delta_{t_0} &\geqq \overline{\lim_{t\to t_0}} \dim \Delta_t = \overline{\lim_{t\to t_0}} \{\dim \ker P_t + \dim (\mathrm{Re}\, P_t) \cap W_0\} \\ &\geqq \lim_{t\to t_0} \{\dim \ker P_t + \dim W_0 - \dim (\mathrm{Re}\, P_t)^\perp\}.\end{aligned}$$

Since $\dim \Delta_{t_0} = \dim \ker P_{t_0}$, this means that index $P_{t_0} \geqq \overline{\lim_{t\to t_0}}$ inex P_t. Q.E.D.

Theorem 5.5. *Let P be an $l \times l$ matrix of operators of class $S^m_{\rho,\delta}(m>0)$ such that $\sigma(P)(x, \xi)$ satisfies conditions* (3.1) *and* (3.2) *for large $|x|+|\xi|$ uniformly on Ξ_0. Assume that $\sigma(P)(x, \xi)$ is slowly varying and that, for $\beta \neq 0$,* (3.2) *holds with a bounded function $C_{0,\alpha,\beta}(x)$ such as $C_{0,\alpha,\beta}(x) \to 0 (|x| \to \infty)$. Then, we can construct complex powers P_z such that $\sigma(P_z)(x, \xi)$ is slowly varying and*

(5.13) $\sigma(P_{z_1} P_{z_2} - P_{z_1+z_2})(x, \xi) \in \mathring{S}^{-\infty} (= \bigcap_s \mathring{S}^s_{\rho,\delta})$.

REMARK. We may assume that $\sigma(P)(x, \xi)$ satisfies (3.1) and (3.2) for every x and ξ. In fact, for a C_0^∞-function $\gamma(x, \xi)$ such that $0 \leqq \gamma(x, \xi) \leqq 1$, and $\gamma(x, \xi) = 1$ $(|x|+|\xi| \geqq 1)$, $=0$ $(|x|+|\xi| \leqq 2)$, We set $P_\varepsilon = P + \varepsilon^{-1} \gamma(\varepsilon x, \varepsilon D_x) I$, Then, for a small fixed $\varepsilon_0 > 0$, $\sigma(P_{\varepsilon_0})(x, \xi)$ satisfy conditions (3.1) and (3.2) for every x and ξ, and has complex powers $P_{\varepsilon_0, z}$. We set $P_z = P_{\varepsilon_0, z} + z(P - P_{\varepsilon_0, 1})$. Then, noting

$P-P_{\varepsilon_0,1}=P-P_{\varepsilon_0}=\varepsilon_0^{-1}\gamma(\varepsilon_0 x, \varepsilon_0 D_x)I\in \overset{\circ}{S}{}^{-\infty}$, we see that P_z are required powers.

Proof. Instead $r(\zeta; x, \xi)$ of (3.16) we consider, using functions $\varphi(\xi)$ and $\psi(x, \xi)$ of (5.9),

$$(5.14)\quad r(\zeta; x, \xi) = q_0(\zeta; x, \xi)+\sum_{j=1}^{\infty}\varphi(t_j^{-1}\xi)\psi(t_j^{-1}x, t_j^{-1}\xi)q_j(\zeta; x, \xi)$$

for an appropriate increasing sequence $\{t_j\}_{j=1}^{\infty}$. Then, we may assume that $p_z(x, \xi)$ defined by (3.23) is slowly varying and that

$$(5.15)\quad \sigma(P_z)(x, \xi)-\sigma(P)(x, \xi)^z\in \overset{\circ}{S}{}^{m(\mathrm{Re}\, z)-(\rho-\delta)}_{\rho,\delta}.$$

Now, for any N, we define $R_{z_1,z_2,N}\in S^{m(\mathrm{Re}\, z_1)+m(\mathrm{Re}\, z_2)}_{\rho,\delta}$ by $(R_{z_1,z_2,N})(x, \xi)=\sum_{|\alpha|<N}\frac{1}{\alpha!}\sigma(P_{z_1})^{(\alpha)}(x, \xi)\sigma(P_{z_2})_{(\alpha)}(x, \xi)$. Then, by ii) of Lemma 1.5, we have

$$(5.16)\quad P_{z_1}P_{z_2}-R_{z_1,z_2,N}\in \overset{\circ}{S}{}^{m(\mathrm{Re}\, z_1)+m(\mathrm{Re}\, z_2)-(\rho-\sigma)N}_{\rho,\delta}.$$

Noting $\sigma(P)(x, \xi)^{z_1}\sigma(P)(x, \xi)^{z_2} = \sigma(P)(x, \xi)^{z_1+z_2}$, we have

$$(5.17)\quad \sigma(R_{z_1,z_2,N})(x, \xi)-\sigma(P)(x, \xi)^{z_1+z_2}\in \overset{\circ}{S}{}^{m(\mathrm{Re}\, z_1)+(\mathrm{Re}\, z_2)-(\rho-\delta)}_{\rho,\delta}.$$

Hence, if we write

$$(S^{-\infty}\ni)\, P_{z_1}P_{z_2}-P_{z_1+z_2} = (P_{z_1}P_{z_2}-R_{z_1,z_2,N})+(R_{z_1,z_2,N}-P_{z_1+z_2}),$$

then, using (5.16), (5.17) and (5.15) for $z=z_1+z_2$, we get (5.13). Q.E.D.

Theorem 5.6. *Let P be an $l\times l$ matrix of operators of class $S^m_{\rho,\delta}$, $m>0$, which are slowly varying. Assume that the symbol $\sigma(P)(x, \xi)$ satisfies conditions* (3.1) *and* (3.2) *for large $|x|+|\xi|$ uniformly on Ξ_0. Then, the operator P as the map from L^2 into itself with the domain $\mathcal{D}(P)=\{u\in L^2; Pu\in L^2\}$ is Fredholm type and we have*

$$(5.18)\quad \textit{index } P\equiv \textit{dim ker } P-\textit{codim } \mathrm{Re}\, P = 0.$$

Proof. Let P_z be complex powers of P defined in Theorem 5.5. For $t\in[0, 1]$, consider $\{P_t\}_{t\in I}$ and set $Q_t=P_{-t}$. Then, by iv) of Definition 3.1, Q_t is strongly continuous in t as L^2-operators. Moreover, if we write $Q_tP_t=P_{-t}P_t=I+K_t$, then, by means of (5.13), $K_t\in \overset{\circ}{S}{}^{-\infty}$ and consequently, by Lemma 1.4 and Lemma 1.6, K_t is uniformly continuous in t and compact as operators from L^2 into itself. Hence, we can apply Lemma 5.4 and we have that index P_t is upper semi-continuous in t. Now, using Lemma 5.2, we note that $\ker P_t=(\mathrm{Re}\, P_t^*)^{\perp}=(\mathrm{Re}\, P_t^{(*)})^{\perp}$, $(\mathrm{Re}\, P_t)^{\perp}=\ker P_t^*=\ker P_t^{(*)}$, so that index $P_t=-$ index $P_t^{(*)}$. Since $(P_tP_{-t})^{(*)}=P_{-t}^{(*)}P_t^{(*)}$, setting $Q_t=P_{-t}^{(*)}$, we have also that index $P_t^{(*)}$ is upper semi-continuous in t. Hence we get that index P_t is continuous, so is constant in $[0, 1]$. Then, index $P=$ index P_t, $t\in[0, 1]$, $=$ index $I=0$.

Q.E.D.

Lemma 5.7. *Let P and Q be $l\times l$ matrices of operators of class $S^m_{\rho,\delta}$ such that P has complex powers P_z and Q has the parametrix Q_{-1}. Assume that QP_{-1} and PQ_{-1} are of class $S^0_{\rho,\delta}$. Then, for $P_z'=QP_{-1+z}$, we have*

$$(5.19)\quad P_z'^{*}=P_z'^{(*)}.$$

Proof. We write

$$P_z\equiv PP_{-1+z}\equiv(PQ_{-1})P_z' \pmod{S^{-\infty}} \text{ and } P_z'\equiv(QP_{-1})P_z \pmod{S^{-\infty}},$$

then we can see that

$$(5.20)\quad P_zu\in L^2 \text{ if and only if } P_z'u\in L^2 \text{ for } u\in H_{-\infty}.$$

If we write, for some $K\in S^{-\infty}$, $P_z'=(QP_{-1})P_z+K$, then we have

$$(5.21)\quad P_z'^{(*)}=P_z^{(*)}(QP_{-1})^{(*)}+K^{(*)}.$$

Now we assume that $v\in\mathcal{D}(P_z'^{(*)})$, i.e., $v\in L^2$ and $P_z'^{(*)}v\in L^2$. Since $\sigma(QP_{-1})^{(*)}\in S^0_{\rho,\delta}$, by means of (5.21) we have

$$(QP_{-1})^{(*)}v\in L^2 \text{ and } P_z^{(*)}(QP_{-1})^{(*)}v\in L^2.$$

Then, noting $P_z^{(*)}=P_z^*$ by Lemma 5.2, we have $(QP_{-1})^{(*)}v\in\mathcal{D}(P_z^*)$, so that, for any $u\in\mathcal{D}(P_z')$, we have, noting $u\in\mathcal{D}(P_z)$ by (5.20),

$$\begin{aligned}(u, P_z'^{(*)}v)&=(u, P_z^{(*)}(QP_{-1})^{(*)}v)+(u, K^{(*)}v)\\&=(P_zu, (QP_{-1})^{(*)}v)+(u, K^{(*)}v)\\&=(QP_{-1}P_zu, v)+(Ku, v)=(P_z'u, v),\end{aligned}$$

which means that $v\in\mathcal{D}(P_z'^*)$. Hence, by Lemma 5.1 we have $P_z'^{(*)}=P_z'^*$.

Q.E.D.

DEFINITION 5.8. For $l\times l$ matrices P and Q of class $S^m_{\rho,\delta}$ we say that $\sigma(P)(x, \xi)$ and $\sigma(Q)(x, \xi)$ are equally strong, when they satisfy with each other

$$(5.22)\quad \|\sigma(Q)^{(\alpha)}_{(\beta)}(x, \xi)\sigma(P)(x, \xi)^{-1}\|\leqq C_{\alpha,\beta}(x)\langle\xi\rangle^{-\rho|\alpha|+\delta|\beta|}$$

and

$$(5.23)\quad \|\sigma(P)^{(\alpha)}_{(\beta)}(x, \xi)\sigma(Q)(x, \xi)^{-1}\|\leqq C'_{\alpha,\beta}(x)\langle\xi\rangle^{-\rho|\alpha|+\delta|\beta|}$$

for large $|x|+|\xi|$, where we assume that, for $\beta\neq 0$, $C_{\alpha,\beta}(x)\to 0$ and $C'_{\alpha,\beta}(x)\to 0$ as $|x|\to\infty$.

Then we have

Lemma 5.9. *Let P and Q be $l\times l$ matrices of class $S^m_{\rho,\delta}(m>0)$. Assume that $\sigma(P)(x,\xi)$ and $\sigma(Q)(x,\xi)$ satisfy conditions* (3.1) *and* (3.2) *for $\zeta=0$ and are equally strong. Then, for parametrices P_{-1} of P and Q_{-1} of Q (which can be defined by* (3.6), (3.7) *and* (3.16) *by setting $\zeta=0$, c.f. also* [6]), *we have that $\sigma(P_{-1})(x,\xi)$ and $\sigma(Q_{-1})(x,\xi)$ are slowly varying and that*

$$QP_{-1}\in S^0_{\rho,\delta} \text{ and } PQ_{-1}\in S^0_{\rho,\delta}\,.$$

Proof. We expand for large N

$$\sigma(QP_{-1})(x,\xi)=\sum_{|\alpha|<N}\frac{1}{\alpha!}\sigma(Q)^{(\alpha)}(x,\xi)\sigma(P_{-1})_{(\alpha)}(x,\xi)+R_N(x,\xi)$$

such that $R_N(x,\xi)\in S^0_{\rho,\delta}$. Then, noting the form (3.14) and using (5.22) we see that $\sigma(QP_{-1})(x,\xi)\in S^0_{\rho,\delta}$. Analogously, using (5.13), we get $\sigma(PQ_{-1})(x,\xi)\in S^0_{\rho,\delta}$. Q.E.D.

Theorem 5.10. *Let P and Q be $l\times l$ matrices of class $S^m_{\rho,\delta}$ $(m>0)$. Assume that $\sigma(P)(x,\xi)$ and $\sigma(Q)(x,\xi)$ are slowly varying and equally strong, and that P has complex powers P_z. Then, $QP_{-1+t}(0\leqq t\leqq 1)$ is Fredholm type as the L^2-operator, and we have*

(5.24) $\quad \textit{index } Q=\textit{index } QP_{-1+t}=\textit{index } QP_{-1}\,.$

Moreover we have

(5.25) $\quad \textit{index } Q=\textit{index } Q_0\,,$

where Q_0 is defined by

$$\sigma(Q_0)(x,\xi)=\psi(c^{-1}x,c^{-1}\xi)\sigma(Q)\Big(\frac{cx}{\langle x\rangle},\frac{c\xi}{\langle\xi\rangle}\Big)\sigma(P)\Big(\frac{cx}{\langle x\rangle},\frac{c\xi}{\langle\xi\rangle}\Big)^{-1}$$

with the function $\psi(x,\xi)$ of (5.9) *and a large fixed constant $c>0$, which is an elliptic operator of class $S^0_{1,0}$ and is slowly varying* (c.f. [4]).

Proof. Set $P_t'=QP_{-1+t}$ and let Q_{-1} be a parametrix of Q. Then, $Q_t'=P_{1-t}Q_{-1}$ is a parametrix of P_t' and belongs to $S^0_{\rho,\delta}$. If we write $Q_t'P_t'=I+K_t'$, then by Lemma 1.6 we have $K_t'\in S^{-\infty}$. By Lemma 5.7 we have $P_t'^*=P_t'^{(*)}=P_{-1+t}^{(*)}Q^{(*)}$ and $Q_t'^{(*)}=Q_{-1}^{(*)}P_{1-t}^{(*)}$ is a parametrix of $P_t'^{(*)}$. Then, in the same way to the proof of Theorem 5.6, we get (5.24). By means of Lemma 1.5 we can write for large N

$$\sigma(QP_{-1})(x,\xi)=\sum_{|\alpha|<N}\frac{1}{\alpha!}\sigma(Q)^{(\alpha)}(x,\xi)\sigma(P_{-1})_{(\alpha)}(x,\xi)+r_N(x,\xi)$$

such that $r_N(x,\xi)\in \mathring{S}^{-(\rho-\delta)}_{\rho,\delta}$. Then, noting that

$$\sigma(Q)(x,\xi)(\sigma(P_{-1})(x,\xi)-\psi(c^{-1}x,c^{-1}\xi)\sigma(P)(x,\xi)^{-1})\in\mathring{S}_{\rho,\delta}^{-(\rho-\delta)}$$

and

$$\sigma(Q)^{(\alpha)}(x,\xi)\sigma(P_{-1})_{(\alpha)}(x,\xi)\in\mathring{S}_{\rho,\delta}^{-(\rho-\delta)} \qquad \text{for } |\alpha|\geqq 1\,,$$

we have

$$\sigma(QP_{-1})(\mathrm{x},\xi)=\psi(c^{-1}x,c^{-1}\xi)\sigma(Q)(x,\xi)\sigma(P)(x,\xi)^{-1}+R_0(x,\xi)\,,$$

where $R_0(x,\xi)\in\mathring{S}_{\rho,\delta}^{-(\rho-\delta)}$. Since by Lemma 1.6 $R_0(x,D_x)$ is compact on L^2, we have index QP_{-1}=index P_0', where P_0' is defined by

$$\sigma(P_0')(x,\xi)=\psi(c^{-1}x,c^{-1}\xi)\sigma(Q)(x,\xi)\sigma(P)(x,\xi)^{-1}\,.$$

Now consider a family of symbols

$$\sigma(Q_\varepsilon)(x,\xi)=\psi(c^{-1}x,c^{-1}\xi)\sigma(Q)\left(\left(\frac{c}{\langle x\rangle}\right)^{1-\varepsilon}x,\left(\frac{c}{\langle\xi\rangle}\right)^{1-\varepsilon}\xi\right)\sigma(P)$$
$$\left(\left(\frac{c}{\langle x\rangle}\right)^{1-\varepsilon}x,\left(\frac{c}{\langle\xi\rangle}\right)^{1-\varepsilon}\xi\right).$$

It is easy to see that $\{\sigma(Q_\varepsilon)(x,\xi)\}_{0\leqq\varepsilon\leqq1}$ makes a bounded set in $S^0_{\rho,\delta}$ and $Q_1=P_0'$. Furthermore we have with a constant $C>0$

$$C^{-1}\leqq|\det\sigma(Q_\varepsilon)(x,\xi)|\leqq C \qquad \text{for large } |x|+|\xi|\,.$$

As the regularizers for Q_ε we adopt operators $Q_{-\varepsilon}$ defined by $\sigma(Q_{-\varepsilon})(x,\xi)=\psi(c_1^{-1}x,c_1^{-1}\xi)\sigma(Q_\varepsilon)(x,\xi)^{-1}(\in S^0_{\rho,\delta})$ for a large constant $c_1>0$. For a fixed $u\in L^2$ we wtite

$$Q_{-\varepsilon}u-Q_{-\varepsilon_0}u=Q_{-\varepsilon}(1-\psi_\delta)u+(Q_{-\varepsilon}\psi_\delta u-\psi_\delta Q_{-\varepsilon}u)$$
$$+\psi_\delta(Q_{-\varepsilon}-Q_{-\varepsilon_0})u+(\psi_\delta Q_{-\varepsilon_0}u-Q_{-\varepsilon_0}\psi_\delta u)+Q_{-\varepsilon_0}(\psi_\delta-1)u\,,$$

where $\psi_\delta(x)=\psi(\delta x)$, $\delta>0$, with a function $\psi(\xi)$ of (2.1). Then by Lemma 2.2 we have for any fixed $\delta>0$

$$\|\psi_\delta(Q_{-\varepsilon}-Q_{-\varepsilon_0})u\|_0\to 0 \text{ as } \varepsilon\to\varepsilon_0\,,$$

and other terms tend to zero in L^2 as $\delta\downarrow 0$ uniformly in ε. Hence we see that $Q_{-\varepsilon}$ is strongly continuous in L^2 and by Lemma 5.4 we have

index P_0' = index Q_ε = index Q_0. Q.E.D.

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