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FLOW EQUIVALENCE OF DIFFEOMORPHISMS II

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In this paper we consider the problem of reducing the classification of dynamical systems with global cross-sections on certain manifolds to the classification of diffeomorphisms of certain manifolds.

In this paper we shall classify the dynamical systems with cross-sections on the manifolds which are homotopically equivalent to $S^1 \times S^n$, $n=2$ or $n \geqq 5$ (Theorem (7) and Theorem (8)). This is a generalization of a result obtained in [4] (Theorem 6.6).

We shall use the same definitions and notations as in [5]. The word "smooth" will mean "C^∞-". Throughout this paper, all manifolds and maps considered will be smooth.

Two diffeomorphisms f_0 and f_1 of M are called *pseudo-diffeotopic* if there is a diffeomorphism $F:[0, 1] \times M \to [0, 1] \times M$ such that $F(0, x)=(0, f_0(x))$, $F(1, x)=(1, f_1(x))$, for all $x \in$ M. The set of pseudo-diffeotopy classes of orientation-preserving diffeomorphisms of M forms a group $\mathcal{D}(M)$. If g is an orientation-preserving diffeomorphism of $S^n = D^n_- \cup D_+$ (by identifying ∂D^n_- and ∂D^α_+ by the identity map $S^{n-1} \to S^{n-1}$), then we may define a diffeomorphism Ψg of M as follows:
By an diffeotopy, make $g|D^n_- =$ identity (see [8]) and define $\Psi g(x)=x$ if $x \in M-D^n$ and $\psi g|D^n = g|D^n_+$ for an embedded closed disk $D^n \subset M^n$. By Wall ([10] §4 Hilfssatz), the pseudodiffeotopy class of Ψg depends only the pseudo-diffeotopy class of g, and Ψ : Diff $(S^n) \to$ Diff (M^n) defines a homomorphism

$$\tilde{\Psi} : \mathcal{D}(S^n) \to \mathcal{D}(M^n),$$

where Diff (M) denotes the group of orientation-preserving diffeomorphisms on M.

Let Γ_n denote the group of differentiable structures on S^n with usual p.l. structure under the connected sum operation $\#$, then $\Gamma_n \cong \mathcal{D}(S^{n-1})$. By [10] Theorem 3 and Lemma 9 (iii), we have

(1). *For any S^n_α in Γ_n, $\tilde{\Psi} : \mathcal{D}(S^n) \to \mathcal{D}(S^n_\alpha)$ is a surjective homomorphism.*

Difine a space M_f called the *mapping torus* of $f : M \to M$ by $M_f = [0, 1] \times$

M with identifications $(1, x) = (0, f(x))$ for all $x \in M$. If f is a diffeomorphism, M_f is a smooth manifold. The next result is due to Browder ([3], Lemma 1).

Let $\phi : \Gamma_{n+1} \to \mathcal{D}(S^n)$ be the usual isomorphism.

(2). *Let f be a diffeomorphism of a smooth closed manifold M. If S_γ^{n+1} is in Γ_{n+1}, then $M_f \# S_\gamma^{n+1}$ is diffeomorphic to $M_{(f\circ\psi g)}$, where g is any diffeomorphism of S^n in the pseudo-diffeomorphic class $\Phi(S_\gamma^{n+1})$.*

In [6], PD/O is defined and $\pi_n(PD/O) \cong \Gamma_n$ is shown (Corollary (1) of Theorem (6. 3)). Let $J : \pi_p(SO(n)) \to \pi_{p+n}(S^n)$ be the J-homomorphism. Let $(\beta, \alpha) \mapsto \alpha\circ\beta$ be the homotopy composition mapping.

$$\pi_{p+n}(S^n)\times\pi_n(PD/O)\to\pi_{p+n}(PD/O)$$

defined naturally by the composition

$$S^{p+n}\to S^n\to PD/O.$$

The next result is due to Schultz [9].

(3). (i). *Every smooth manifold M homotopy equivalent to $S^1\times S^n$, $n\geqq 5$, is diffeomorphic to $S^1\times S_\alpha^n \# S_\gamma^{n+1}$ for some $\alpha, \gamma\in\Gamma_n, \Gamma_{n+1}$ respectively.*

(ii) $S^1\times S_\alpha^n \# S_\gamma^{n+1}$ and $S^1\times S_{\alpha'}^n \# S_{\gamma'}^{n+1}$ are *orientation-preservingly diffeomorphic if and only if $\alpha = \pm\alpha'$ in Γ_n and $\gamma - \gamma' = \alpha\circ J(\beta)$, some $\beta\in\pi_1(SO(n))$.*

Let $I(S^1 \times S_\alpha^n)$ denote the inertia group of $S^1\times S_\alpha^n$, i.e. $\{S_\gamma^{n+1}\in\Gamma_{n+1} \mid S^1\times S_\alpha^n \# S_\gamma^{n+1} = S^1\times S_\alpha^n\}$. (3) (ii) implies

$$I(S^1\times S_\alpha^n) = \{\alpha\circ J(\beta) \mid \beta\in\pi_1(SO(n))\},$$

put $\tilde{\psi}\Phi(I(S^1\times S_\alpha^n)) = \mathcal{I}(S_\alpha^n)$. Since $\tilde{\psi}\Phi : \Gamma_{n+1}\to\mathcal{D}(S_\alpha^n)$ is a surjective homomorphism by (1) and since Γ_{n+1} is abelian, $\mathcal{I}(S_\alpha^n)$ is a normal subgroups of $\mathcal{D}(S_\alpha^n)$.

Let (f) denote the pseudo-diffeotopy class of f.

Proposition 4. *Let f, g be orientation-preserving diffeomorphisms of S_α^n. If f and g be conjugate,*

$$(f) = (g) \bmod \mathcal{I}(S_\alpha^n).$$

Proof. Let $S_\alpha^n = D_-^n \cup D_+^n$ with identification by $\phi : \partial D_-^n \to \partial D_+^n$ such that $(\phi) = \Phi(S_\alpha^n)$. By diffeotopies make $f \mid D_-^n = g \mid D_-^n =$ identity map. Here, by these diffeotopies the diffeomorphism classes of $(S_\alpha^n)_f$ and $(S_\alpha^n)_g$ are not altered. f,g are contained in the image of Ψ : Diff $(S^n) \to$ Diff (S_α^n). Let

$$\Psi(f_0) = f, \qquad \Psi(g_0) = g,$$

and put

$$\Phi^{-1}((f_0)) = S_\gamma^{n+1}, \quad \Phi^{-1}((g_0) = S_\delta^{n+1}.$$

By (2),

$$(S_\alpha^n)_f = (S^1\times S_\alpha^n)\# S_\gamma^{n+1}, \quad (S_\alpha^n)_g = (S^1\times S^n)\# S_\delta^{n+1}.$$

But, since f and g are conjugate, there is a natural diffeomorphism from $(S_\alpha^n)_f$ to $(S_\alpha^n)_f$. Then (3) implies

$$\gamma-\delta \in I(S^1\times S_\alpha^n).$$

Therefore,

$$(f)-(g) = \tilde{\Psi}\Phi(\gamma-\delta) \in \mathcal{I}(S_\alpha^n).$$

Lemma 5. *Suppose that M^n is a simply connected, orientable, closed manifold with $n\geqq 5$ and that f, g are orientation-preserving diffeomorphisms of M^n. If f and g are pseudo-diffeotopic, then M_f and M_g are diffeomorphic.*

Proof. There is a diffeomorphism $F: I\times M^n \to I\times M^n (I=[0,1])$ such that $F(0,x) = (0, f(x))$, $F(1,x) = (1, g(x))$ for all $x\in M^n$. Let

$$W^{n+2} = (I\times M^n)_F = I\times(I\times M^n)/(0,t,x)\sim(1, F(t,x)).$$

Then $(W; M_f, M_g)$ is a h-cobordism.

In fact, the maps $W^{n+2}\to S^1$, $M_f\to S^1$, defined by $(s,t,x)\mapsto e^{is}$, $(s,x)\mapsto e^{is}$ respectively, are fiber maps. Let $j: M_f\to W^{n+2}$ be the inclusion map, it is given by $(s,x)\mapsto(s,0,x)$. Since the diagram

$$\begin{array}{ccc} M_f & \xrightarrow{j} & W^{n+2} \\ \downarrow & & \downarrow \\ S^1 & \xrightarrow{id} & S^1 \end{array}$$

is commutative, we have the next diagram of exact sequences.

$$\begin{array}{ccccccccc} \pi_{i+1}(S^1) & \to \pi_i(M) & \to & \pi_i(M_f) & \to & \pi_i(S^1) & \to\cdots\to & \pi_1(S^1) & \to 0 \\ \downarrow & \downarrow j_* & & \downarrow j_* & & \downarrow (id)_* & & \downarrow & \\ \pi_{i+1}(S^1) & \to\pi_i(I\times M) & \to & \pi_i(W^{n+2}) & \to & \pi_i(S^1) & \to\cdots\to & \pi_i(S^1) & \to 0 \end{array}$$

Hence, we have that

$$j_*: \pi_*(M_f)\to\pi_*(W^{n+2})$$

is an isomorphism. Therefore $j: M_f\to W^{n+2}$ is a homotopy equivalence.

Since $\pi_1(W) = Z$, by [2], the Whitehead torsion $\tau(W, M_f) = 0$. Therefore, by s-cobordism theorem ([1] Corollary (6, 3) or [7]), we have a desired diffeomorphism.

Let M, N be diffeomorphic manifolds and $h : M \to N$ be a diffeomorphism. For a diffeomorphism f of M, we may correspond it to a diffeomorphism hfh^{-1} of N. If we correspond the conjugate class of f to the conjugate class of hfh^{-1}, the correspondence is independent of the choice of the diffeomorphism h. By this correspondence, we shall identity the conjugate classes of diffeomorphisms on diffeomorphic manifolds. We shall denote this conjugate class containing f by $[f]$.

If we denote the element of $\mathcal{D}(S^n_\alpha)/\mathcal{J}(S^n_\alpha)$ containing the pseudo-diffeotopy class of f by $\langle(f)\rangle$ or $\langle f\rangle$, by Proposition 4, the notation $\langle[f]\rangle$ can be well-defined by $\langle[f]\rangle = \langle([f])\rangle$.

Proposition 6. (i) *Every somooth manifold M which is homotopically equivalent to $S^1\times S^n$, $n \geqq 5$, is diffeomorphic to $(S^n_\alpha)_f$ for some $\alpha\in\Gamma_n$ and some diffeomorphism f of S^n_α.*

(ii). *$(S^n_\alpha)_f$ and $(S^n_{\alpha'})_{f'}$ are orientation-preservingly diffeomorphic if and only if $\alpha = \pm\alpha'$ in Γ_n and $\langle[f]\rangle = \langle[f']\rangle$.*

Proof. By (3) and (2), (i) is obvious.

Next we prove (ii). Let $S^n_\alpha = D^n_- \cup D^n_+$ with identification by $\phi : \partial D^n_- \to \partial D^n_+$, where $(\phi) = \Phi(S^n_\alpha)$. Similarly, let $S^n_{\alpha'} = D^n_- \cup D^n_+$. If $(S^n_\alpha)_f$ and $(S^n_{\alpha'})_{f'}$ are orientation preservingly diffeomorphic, as the proof of Proposition 4 we can show that S^n_α and $S^n_{\alpha'}$ are diffeomorphic and

$$\langle[f]\rangle=\langle[f']\rangle.$$

Conversely, suppose S^n_α and $S^n_{\alpha'}$ are diffeomorphic and $\langle[f]\rangle = \langle[f']\rangle$. Here, we are identifying $[f']$ with $[hf'h^{-1}]$, where h is an any diffeomorphism of $S^n_{\alpha'}$ onto S^n_α.

There are γ and γ' in $\mathcal{D}(S^n)$ such that

$$\tilde{\Psi}(\gamma) = ([f]), \quad \tilde{\Psi}(\gamma')=([f'])$$
$$\gamma-\gamma'\in I(S'\times S^n_\alpha),$$

identifying $I(S^1\times S^n_\alpha)$ with $\Phi(I(S^1\times S^n_\alpha)$. In fact, since $([f])-([f'])\in\mathcal{J}(S^n_\alpha)$ and $\tilde{\Psi}$ maps $I(S^1\times S^n)$ onto $\mathcal{J}(S^n_\alpha)$, there is $\gamma_0\in I(S^1\times S^n_\alpha)$ such that $\tilde{\Psi}(\gamma_0) = ([f])-([f'])$. Since $\tilde{\Psi}$ maps $\mathcal{D}(S^n)$ onto $\mathcal{D}(S^n_\alpha)$, there is $\gamma'\in\mathcal{D}(S^n)$ such that $\tilde{\Psi}(\gamma') = ([f'])$. Put $\gamma = \gamma_0+\gamma'$. Then we have $\tilde{\Psi}(\gamma) = ([f])$.

Let f_0, f_0' be orientation-preserving diffeomorphisms of S^n included in the pseudo-diffeotopy classes γ, γ' respectively. Since $\tilde{\Psi}(f_0) = (\Psi(f_0))$ and $\tilde{\Psi}(f_0') = (\Psi(f'_0))$, $\Psi(f_0)$ and $\Psi(f'_0)$ are pseudo-diffeotopic to f and $hf'h^{-1}$ respectively. By Lemma 5, $(S^n_\alpha)_f$ and $(S^n_{\alpha'})_{hfh^{-1}}$ are diffeomorphic to $(S^n_\alpha)_{\Psi(f_0)}$ and $(S^n_\alpha)_{\Psi(f_0')}$ respectively. Hence, by (2),

$$(S^n_\alpha)_f = (S^1 \times S^n_\alpha) \# S^{n+1}_\gamma$$
$$(S^n_\alpha)_{hf'h^{-1}} = (S \times S^n_\alpha) \# S^n_{\gamma'} ,$$

where $S^{n+1}_\gamma = \Phi^{-1}((f_0))$, $S^{n+1}_{\gamma'} = \Phi^{-1}((f_0'))$. Since $\gamma - \gamma' \in I(S^1 \times S^n_\alpha)$, $(S^n_\alpha)_f = (S^n_\alpha)_{hf'h^{-1}}$. Therefore, since $(S^n_\alpha)_{hf'h^{-1}}$ is diffeomorphic to $(S^n_{\alpha'})_{f'}$, $(S^n_\alpha)_f$ and $(S^n_{\alpha'})_{f'}$ are diffeomorphic with preserving natural orientations.

This completes the proof of Proposition (6).

Theorem 7. *Let $n \geqq 5$ or $n = 2$.*

(i). *The set of the smooth equivalence classes of dynamical systems with cross-sections on a manifold which is homotopically equivalent to $S^1 \times S^n$ has an one-to-one correspondence to the set of the smooth conjugate classes of $\{(S^n_\alpha, f) \mid S^n_\alpha \in \Gamma_n, f \in$ Diff $(S^n_\alpha)\}$. This correspondence is given by associated diffeomorphisms of dynamical system or suspensions of diffeomorphisms.*

Theorem 8. *Suppose that (M, ϕ), (M', f') are dynamical systems as in Theorem (7) and that they correspond to the conjugate classes of (S^n_α, f), $(S^{n'}_\alpha, f')$ respectively. Then M and M' are diffeomorphic if and only if $\alpha = \pm\alpha'$ in Γ_n and $([f]) = ([f'])$ mod $\mathcal{J}(S^n_\alpha)$, where $[f]$ denotes the conjugate class of f and $([f])$ denotes the pseudo-diffeotopy class of $[f]$, and $\mathcal{J}(S^n_\alpha) = \tilde{\Psi}\Phi(I(S^1 \times S^n_\alpha)) = \tilde{\Psi}\Phi\{\alpha \circ J(\beta) \mid \beta \in \pi_1 (SO(n))\}$.*

Proof of theorem 7. Let $(M, \phi ; X)$ be a dynamical system with a cross-section X such that M is homotopically equivalent to $S^1 \times S^n$. There is a fiber map $M \to S^1$ with fiber X. By the homotopy exact sequence of the fiber map, we see that X is a homotopy n-sphere. Hence, $X = S^n_\alpha \in \Gamma_n (\Gamma_2 = \{S^2\})$. Let f denote the associated diffeomorphism of $(M, \phi ; X')$. If f' is the associated diffeomorphism of $(M, \phi ; X')$ for any other cross-section X' of (M, ϕ), by [5] Corollary (5. 6), $f : S^n_\alpha \to S^n_\alpha$ and $f' : X' \to X'$ are conjugate. Therefore, a unique conjugate class correspond to (M, ϕ).

Conversely, if the associated diffeomorphisms of (M, ϕ) and (M', ϕ') are conjugate, by [5] Corollary (5. 6), (M, ϕ) and $(M' \phi')$ are equivalent.
This proves Theorem (7).

Proof of Theorem 8. By [5] (2. 2), M and M' are diffeomorphic to $(S^n_\alpha)_f$ and $(S^n_{\alpha'})_{f'}$ respectively. Therefore Proposition (6) (ii) implies Theorem (7).

The following corollary is a previously obtained result ([4] Theorem (6. 6)).

Corollary 9. *The set of the smooth equivalence classes of dynamical systems with cross-sections on $(S^1 \times S^n \# S^{n+1}_\gamma)$, $n \geqq 5$ or $n = 2$, has an one-to-one correspondence to the set of the conjugate classes of diffeomorphisms which are in the pseudodiffeotopy class $\phi(S^{n+1}_\gamma) \in \mathcal{D}(S^n)$.*

Proof. $\tilde{\Psi} : \mathcal{D}(S^n) \to \mathcal{D}(S^n)$ is the identity map. Since $I(S^1 \times S^n) = 0$ by

(3) (ii), $\mathcal{J}(S^n)=\tilde{\Psi}\Phi(I(S^1\times S^n))=0$. Hence, the corollary follows from (2), Theorem (7) and Theorem (8).

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