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THE WHITNEY JOIN AND ITS DUAL

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The notion of the Whitney join is a generalization of the Whitney sum of vector bundles and has been treated by various authors [1], [8], [18].

In this paper we show that the Whitney join can be used to remove in Serre's theorem on relative fibrations the usual assumption that the fibration should be orientable (cf. [15], p. 476). This, applied to arbitrary group extensions, permits us to extend by one term the homology exact sequence which has been deduced by many authors [4], [6], [11], [16], [17]. The resulting exact sequences (see Theorems 3.1 and 3.3) reduce to those of T. Ganea [7] in the case of a central extension. In section 4 we examine a relationship between maps with left homotopy inverse and monomorphisms in homotopy theory (see Theorem 4.1) and, in section 5 we give a generalization of an exact sequence due to E. Thomas [19]. Duality suggests that there may be an exact sequence involving a principal cofibration, and we establish it in the final section (see Theorem 6.3).

Throughout we will work in the category of spaces with base point (denoted by$*$) which have the homotopy type of a CW complex. Given a map $f\colon X\to Y$, we denote by C_f and E_f the cofibre $Y\bigcup_f CX$ (with $(x, 1)$ and $f(x)$ identified) and the fibre $\{(x, \beta)\in X\times Y^I \mid \beta(0)=*,\ \beta(1)=f(x)\}$ respectively, where CX is the reduced cone over X. As usual, S and Ω denote the suspension and loop functors respectively.

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1. Whitney join

The Whitney join of two Hurewicz fibrations is, roughly speaking, defined to be a weak pushout of the pullback of them. Let $f\colon A\to Y$ and $g\colon B\to Y$ be fibrations with fibres F and G and let K denote the pullback of f and g; thus, $K=\{(a, b)\in A\times B \mid f(a)=g(b)\}$ with projections $p_1\colon K\to A$ and $p_2\colon K\to B$. The Whitney join

$$f\oplus g\colon A\oplus B\to Y$$

of f and g is defined (cf. [8]) by setting

$A \oplus B =$ the unreduced mapping cylinder (cf. [12]) $A \bigcup_{p_1} K \times I \bigcup_{p_2} B$ of p_1 and p_2 with strong topology

$$(f \oplus g)(a, b, t) = f(a) = g(b) .$$

Then $f \oplus g$ is a fibration with fibre $F * G$ (with strong topology). Since the passage to the reduced construction does not affect the homotopy type in our category, we shall replace $A \oplus B$ by the reduced one in the sequel.

Now consider the commutative diagram

$$\begin{tikzcd} & A \arrow[dr, "f"] & \\ X \arrow[ur, "h"] \arrow[dr, "k"'] & & Y \\ & B \arrow[ur, "g"'] & \end{tikzcd}$$

and let $T\begin{pmatrix} h & f \\ k & g \end{pmatrix}$ denote the space obtained from $Y \cup CA \cup CB \cup CCX$ by the identification

$$(a, 1) \sim f(a), \quad (b, 1) \sim g(b), \quad (x, s, 1) \sim (h(x), s), \quad (x, 1, t) \sim (k(x), t) .$$

Let us write $C_{h,k}$ for the double mapping cylinder $A \bigcup_h X \times I \bigcup_k B$ in the following. We shall define

$$\xi : C_{h,k} \to Y$$

by $\xi(x, t) = fh(x) = gk(x)$, $\xi(a) = f(a)$, $\xi(b) = g(b)$, $x \in X$, $0 \leqq t \leqq 1$, $a \in A$, $b \in B$. Note that $\xi = f \oplus g$ for $X = K$, $h = p_1$ and $k = p_2$.

Lemma 1.1. *The cofibre C_ξ of ξ is homeomorphic to $T\begin{pmatrix} h & f \\ k & g \end{pmatrix}$.*

Proof. The desired homeomorphism

$$\eta : C_\xi \to T\begin{pmatrix} h & f \\ k & g \end{pmatrix}$$

is obtained by setting

$$\eta(x, s, t) = \begin{cases} (x, t, 2st+1-2s) & \text{for } 0 \leqq 2s \leqq 1 \\ (x, (2-2s)t+2s-1, t) & \text{for } 1 \leqq 2s \leqq 2 . \end{cases}$$

$$\eta(a, t) = (a, t), \quad \eta(b, t) = (b, t), \quad \eta(y) = y .$$

Next consider the commutative diagram

$$\begin{tikzcd} X \arrow[r, "h"] \arrow[d, "k"] & A \arrow[r] \arrow[d, "f"] & C_h \arrow[d, "\chi_1"] \\ B \arrow[r, "g"] \arrow[d] & Y \arrow[r] \arrow[d] & C_k \\ C_h \arrow[r, "\chi_2"] & C_f & \end{tikzcd}$$

in which χ_1 and χ_2 are induced maps between cofibres resulting from $fh=gk$. The following is a direct consequence of the definitions.

Lemma 1.2. (3×3 lemma) *The cofibres C_{χ_1} and C_{χ_2} of χ_1 and χ_2 are both homeomorphic to $T\begin{pmatrix} h & f \\ k & g \end{pmatrix}$.*

The above two lemmas give rise to the following result for the situation in the beginning of this section (cf. Lemma 6 of [14]).

Theorem 1.3. *Suppose that the fibrations $f\colon A\to Y$ and $g\colon B\to Y$ over a path-connected Y are m- and n-connected respectively, where $m\geqq 0$, $n\geqq 0$. Then $f\oplus g$, $\chi_1\colon C_{p_1}\to C_g$ and $\chi_2\colon C_{p_2}\to C_f$ are $(m+n+1)$-connected.*

Given $f\colon A\to Y$ and $g\colon B\to Y$, let $E_{f,g}$ denote the mapping track $\{(a, \gamma, b)\in A\times Y^I\times B \mid f(a)=\gamma(0),\ g(b)=\gamma(1)\}$ with obvious projections $P_1\colon E_{f,g}\to A$ and $P_2\colon E_{f,g}\to B$. We see easily that the diagram

$$\begin{tikzcd} E_{f,g} \arrow[r, "P_1"] \arrow[d, "P_2"'] & A \arrow[d, "f"] \\ B \arrow[r, "g"] & Y \end{tikzcd}$$

is homotopically equivalent to the pullback

$$\begin{tikzcd} E_{f,g} \arrow[r, "p_1"] \arrow[d, "p_2"'] & E_{f,1_Y} \arrow[d, "p"] \\ E_{1_Y,g} \arrow[r, "q"] & Y \end{tikzcd}$$

of two Serre path fibrations p and q. Hence $\xi\colon C_{P_1,P_2}\to Y$ is homotopically equivalent to $p\oplus q$. Thus we have (cf. Theorem 3.10 of [12])

Corollary 1.4. *Let $f\colon A\to Y$ and $g\colon B\to Y$ be m- and n-connected maps, where $m\geqq 1$, $n\geqq 1$. Then the diagram*

$$\begin{tikzcd} & {[A, V]} \arrow[dl, "P_1^*"'] & \\ {[E_{f,g}, V]} & & {[Y, V]} \arrow[ul, "f^*"'] \arrow[dl, "g^*"] \\ & {[B, V]} \arrow[ul, "P_2^*"] & \end{tikzcd}$$

is exact for a connected space V with $\pi_i(V)=0$ for $i \geq m+n+1$.

Proof. Suppose given maps $a: A \to V$ and $b: B \to V$ such that $aP_1 \simeq bP_2$. Then we can construct an extension c: $C_{P_1,P_2} \to V$ of a and b. In order to show the existence of a map y: $Y \to V$ satisfying the conditions $a \simeq yf$ and $b \simeq yg$, it suffices to verify that c can be extended to the mapping cylinder of ξ: $C_{P_1,P_2} \to Y$. The obstruction for such an extension lies in $H^{i+1}(C_\xi; \pi_i(V))=0$ for $i \geqq 1$.

Corollary 1.5. *Let p: $E \to B$ be a fibration with fibre F and assume that B and F are m- and $(n-1)$-connected, where $m \geqq 0$, $n \geqq 0$. Then the maps*

$$\rho: E \cup CF \to B \quad \text{and} \quad \sigma: SF \to C_p,$$

defined by $\rho(e)=e$, $\rho(x, t)=*$, $\sigma(x, t)=(x, t)$ for $e \in E$, $x \in F$, $0 \leqq t \leqq 1$, are $(m+n+1)$-connected.

Proof. By definition, E_p is the pullback of p: $E \to B$ and the Serre path fibration q: $E_{*,1_B} \to B$. So we have the commutative diagram:

$$\begin{tikzcd} E_p \arrow[r, "p_1"] \arrow[d, "p_2"'] & E \arrow[d, "p"] \\ E_{*,1_B} \arrow[r, "q"] & B \end{tikzcd}$$

Clearly the deformation retraction r: $E_{*,1_B} \to *$ can be lifted to the deformation retraction $\tilde{r}$: $E_p \to F$. Hence the above diagram is homotopy equivalent to the following one:

$$\begin{tikzcd} F \arrow[r, "i"] \arrow[d] & E \arrow[d, "p"] \\ * \arrow[r] & B. \end{tikzcd}$$

These considerations reveal that ρ and σ are homotopically equivalent to χ_1: $E \cup CE_p \to C_q$ and χ_2: $C_{p_2} \to C_p$ respectively. Since p and q are n- and m-connected respectively, it follows from 1.3 that $p \oplus q$, χ_1 and χ_2 are $(m+n+1)$ connected. This proves 1.5.

Finally, we consider the commutative diagram

$$\begin{tikzcd} X \arrow[r, "h"] \arrow[d, "k"'] & A \arrow[d, "f"] \\ B \arrow[r, "g"] & Y \end{tikzcd}$$

which leads to the diagram

$$
\begin{tikzcd}
A\vee B \arrow[r, "e"] \arrow[d, equal] & C_{h,k} \arrow[r] \arrow[d, "\xi"] & C_e \arrow[r] \arrow[d, "\chi'"] & SX=C_{h,k}/(A\vee B) \arrow[dl, "w"] \\
A\vee B \arrow[r, "\{f,g\}"] \arrow[d] & Y \arrow[r] \arrow[d] & C_{\{f,g\}} & \\
C(A\vee B) \arrow[r, "\chi''"] & C_\xi & &
\end{tikzcd}
$$

where e denotes the natural injection, $\{f, g\}$ is the map determined by f and g, χ' and χ'' are the maps defined in such a way that all the squares are commutative and w is given by

$$
w(x, t) = \begin{cases} (h(x), 2t) & \text{for} \quad 0 \leqq 2t \leqq 1 \\ (k(x), 2-2t) & \text{for} \quad 1 \leqq 2t \leqq 2\,. \end{cases}
$$

Note that the triangle in the right corner is homotopy-commutative.

Lemma 1.6. *With the above notation, the cofibre C_w of w is of the same homotopy type as C_ξ, i.e., $T\begin{pmatrix} h & f \\ k & g \end{pmatrix}$.*

This follows from 1.2, by observing that $C(A\vee B)$ is contractible and the map $C_e \to SX$ is a homotopy equivalence.

2. Certain cofibres

In this section we examine the cofibre of a map which admits a right inverse. The verification of lemmas is straightforward and is left to the reader.

Lemma 2.1. *Let the composite $Y \xrightarrow{g} X \xrightarrow{f} Y$ be the identity map of Y. Then*

(i) *the maps*

$$
\Phi\colon SX \to C_f \vee SY \quad \text{and} \quad \Psi\colon C_f \vee SY \to SX\,,
$$

given by

$$
\Phi(x, t) = \begin{cases} (x, 3t) & \text{for} \quad 0 \leqq 3t \leqq 1 \\ (gf(x), 2-3t) & \text{for} \quad 1 \leqq 3t \leqq 2 \\ (f(x), 3t-2) & \text{for} \quad 2 \leqq 3t \leqq 3 \end{cases}
$$

$$
\Psi(y, t) = (g(y), t), \quad \Psi(x, t) = (x, t), \quad \Psi(y) = *\,,
$$

are mutually inverse homotopy equivalences with the homotopy commutative diagram

$$
\begin{tikzcd}
 & SX \arrow[dr, "Sf"] \arrow[d, shift right, "\Phi"'] & \\
C_f \arrow[ur] \arrow[dr] & C_f \vee SY \arrow[r] \arrow[u, shift right, "\Psi"'] & SY \\
\end{tikzcd}
$$

(ii) *the maps*

$$F\colon C_f \to SC_g \quad \textit{and} \quad G\colon SC_g \to C_f$$

given by

$$F(x, s) = (x, s), \quad F(y) = *$$

$$G(x, s) = \begin{cases} (x, 2s) & \textit{for} \quad 0 \leqq 2s \leqq 1 \\ (gf(x), 2-2s) & \textit{for} \quad 1 \leqq 2s \leqq 2 \end{cases}$$

$$G(y, t; s) = \begin{cases} (g(y), 2st) & \textit{for} \quad 0 \leqq 2s \leqq 1 \\ (g(y), 2t-2st) & \textit{for} \quad 1 \leqq 2s \leqq 2 \end{cases}$$

are mutually inverse homotopy equivalences.

Lemma 2.2. *Let p: $A \times B \to B$ denote the projection on the second factor and let Θ: $C_p \to SA \vee A*B$ be defined by*

$$\Theta(a, b; t) = \begin{cases} (a, 3t) & \textit{for} \quad 0 \leqq 3t \leqq 1 \\ (3t-1)a \oplus (2-3t)* & \textit{for} \quad 1 \leqq 3t \leqq 2 \\ (3-3t)a \oplus (3t-2)b & \textit{for} \quad 2 \leqq 3t \leqq 3 \end{cases}$$

$$\Theta(b) = b = 0* \oplus 1b$$

where $(1-s)a \oplus sb$ denotes the point of $A*B$ represented by $(a, b, s) \in A \times B \times I$. Then Θ is a homotopy equivalence with inverse T: $SA \vee A*B \to C_p$ gievn by

$$T(a, t) = (a, *; t), \quad T((1-t)a \oplus tb) = (a, b; t)\,.$$

REMARK. Lemma 2.2 can also be proved by observing that, in the week pushout diagram

$$\begin{tikzcd} A \times B \arrow[r, "p"] \arrow[d] & B \arrow[r] \arrow[d] & C_p \arrow[d, "\chi"] \\ A \arrow[r, "i"] & A*B \arrow[r] & C_i, \end{tikzcd}$$

χ is a homotopy equivalence (cf. Lemma 1.2′ of [12]) and i is null-homotopic, hence C_i is homotopy-equivalent to $SA \vee A*B$.

Corollary 2.3. *The Thom space of the trivial n-dimensional real vector bundle over X is homotopy-equivalent to $S^n \vee S^n X$.*

3. Homology of group extensions

In this section we try to generalize an exact sequence due to T. Ganea (Theorem 1.1 of [7]) to an arbitrary extension of groups.

Consider an exact sequence

$$1 \to N \xrightarrow{\hat{\imath}} G \xrightarrow{\hat{p}} Q \to 1 \tag{1}$$

of groups which operate trivially on an abelian group A and the additive group of integers. We shall identify N with its image $\hat{\imath}(N)$ in G. Since N is normal in G, each element $g \in G$ determines an automorphism of N, as $n \to gng^{-1}$, $n \in N$. Let

$$N \tilde{\times} G$$

denote the *semi-direct product* of N and G with respect to this operation of G on N.

We see that the mappings

$$\hat{p}_2 \colon N \tilde{\times} G \to G \quad \text{and} \quad \hat{p}_1 \colon N \tilde{\times} G \to G\,,$$

which are given by

$$\hat{p}_2(n, g) = ng, \quad \hat{p}_1(n, g) = g\,,$$

are homomorphisms. We have $\hat{p}_{2*} \colon H_2(N \tilde{\times} G) \to H_2(G)$ and $\hat{p}_{1*} \colon H_2(N \tilde{\times} G) \to H_2(G)$, and $\hat{p}_{2*}$ induces

$$\operatorname{Ker} \hat{p}_{1*} \to H_2(G)$$

which we denote also by $\hat{p}_{2*}$.

Theorem 3.1. *The sequence*

$$\operatorname{Ker} \hat{p}_{1*} \xrightarrow{\hat{p}_{2*}} H_2(G) \xrightarrow{\hat{p}_*} H_2(Q) \to N/[N, G] \to H_1(G) \xrightarrow{\hat{p}_*} H_1(Q) \to 0$$

is exact, where $[N, G]$ is the normal subgroup of N generated by the elements $ngn^{-1}g^{-1}$, $n \in N$, $g \in G$.

Proof. Take a Hurewicz fibration $p \colon E \to B$ between aspherical spaces which induces $\hat{p} \colon \pi_1(E) \cong G \to \pi_1(B) \cong Q$, and let $i \colon F \to E$ denote the fiber inclusion. Then F is aspherical and i induces $\hat{\imath} \colon \pi_1(F) \cong N \to G$.

Let K denote the pullback of p by p, that is, $K = \{(e, e') \mid p(e) = p(e')\}$, with projections $p_1, p_2 \colon K \to E$. We form the Puppe sequence for p_1 and p to obtain the commutative ladder:

$$\begin{tikzcd}
K \arrow[r, "p_1"] \arrow[d, "p_2"'] & E \arrow[r] \arrow[d, "p"] & C_{p_1} \arrow[r, "r"] \arrow[d, "\chi"] & SK \arrow[r, "Sp_1"] \arrow[d, "Sp_2"] & SE \arrow[d, "Sp"] \\
E \arrow[r, "p"] & B \arrow[r] & C_p \arrow[r] & SE \arrow[r, "Sp"] & SB
\end{tikzcd} \tag{2}$$

where χ is induced by the commutative square in the left corner and r denotes the map shrinking E to a point. Since p is 1-connected, it follows from 1.3 that χ is 3-connected.

We shall show that K is an aspherical space such that $\pi_1(K)$ is isomorphic to the semi-direct product $N \tilde{\times} G$. Take $\kappa \in \pi_1(K)$ and let κ_k denote $p_{k*}(\kappa) \in \pi_1(E)$, $k=1, 2$. Since $p_*(\kappa_1)=p_*(\kappa_2)$, there exists a unique $\nu \in \pi_1(F)$ with $i_*(\nu)=\kappa_2\kappa_1^{-1}$. Define

$$\varphi: \pi_1(K) \to N \tilde{\times} G$$

by $\varphi(\kappa)=(\nu, \kappa_1)$, which is easily seen to be a homomorphism. Let us consider the diagram of homotopy groups induced from the diagram (2). Then $\pi_2(B)=0$ implies $\mathrm{Ker}\, p_{1*} \cap \mathrm{Ker}\, p_{2*}=\{1\}$ and hence φ is monic. Suppose given an element $(n, g) \in N \tilde{\times} G$. Clearly we have $p_*(i_*(n)g)=p_*(g)$, so the homotopy lifting property assures the existence of an element $\kappa \in \pi_1(K)$ such that $p_{1*}(\kappa)=g$ and $p_{2*}(\kappa)=i_*(n)g$. This proves that φ is epic.

We show next that $H_2(C_p) \cong H_2(C_{p_1})$ is isomorphic to $N/[N, G]$. Let M be the mapping cylinder of p_1 and consider the Hurewicz homomorphism

$$h: \pi_2(M, K) \to H_2(M, K) \cong H_2(C_{p_1}).$$

According to the Hurewicz theorem (see [15, p. 397]), h is epic with kernel generated by the elements $\omega \cdot \tau - \tau$, $\omega \in \pi_1(K)$, $\tau \in \pi_2(M, K)$. Since the boundary operator $\partial: \pi_2(M, K) \to \pi_1(K)$ induces an isomorphism $\pi_2(M, K) \cong \mathrm{Ker}\, [\pi_1(K) \to \pi_1(E)]$ we obtain $\pi_2(M, K) \cong \pi_1(F)$. These facts lead to the conclusion that $\partial(\mathrm{Ker}\, h)$ is the subgroup of $\pi_1(K)$ generated by $\partial(\omega \cdot \tau - \tau) = \omega\partial(\tau)\omega^{-1}\partial(\tau)^{-1}$. Now let us write $\omega=(n, g)$ and $\partial(\tau)=(\nu, 1)$; then a simple calculation shows that $\omega\partial(\tau)\omega^{-1}\partial(\tau)^{-1}=(ngvg^{-1}n^{-1}\nu^{-1}, 1)$, from which we see that $\mathrm{Ker}\, h$ coincides with $[N, G]$.

Finally, consider the following commutative diagram

$$\begin{array}{ccccc}
H_3(C_{p_1}) & \xrightarrow{r_*} & H_2(K) & \xrightarrow{p_{1*}} & H_2(E) \\
\downarrow{\scriptstyle \chi_*} & & \downarrow{\scriptstyle p_{2*}} & & \downarrow{\scriptstyle p_*} \\
H_3(C_p) & \longrightarrow & H_2(E) & \longrightarrow & H_2(B)
\end{array}$$

As shown at the beginning of the proof, χ_* is epic, hence the image of $H_3(C_p) \to H_2(E)$ coincides with $\mathrm{Im}\,(p_{2*}r_*)=p_{2*}(\mathrm{Ker}\, p_{1*})$. This concludes the proof of 3.1.

REMARK. We can infer from 2.1, (i) that, in 3.1, $\mathrm{Ker}\, \hat{p}_{1*} \xrightarrow{\hat{p}_{2*}} H_2(G)$ may be replaced by $\mathrm{Coker}\, s_* \xrightarrow{\hat{p}_{2*} - \hat{p}_{1*}} H_2(G)$, where $s: G \to N \tilde{\times} G$ is given by $s(g)=(1, g)$.

Corollary 3.2. *The sequence*

$$\mathrm{Coker}\, \hat{p}_1{}^* \xleftarrow{\hat{p}_2{}^*} H^2(G, A) \xleftarrow{\hat{p}^*} H^2(Q, A) \leftarrow \mathrm{Hom}\,(N/[N, G], A)$$
$$\leftarrow H^1(G, A) \xleftarrow{\hat{p}^*} H^1(Q, A) \leftarrow 0$$

is exact. If $\hat{p}_*: H_3(G) \to H_3(Q)$ is epic, then there is an exact sequence

$$H_1(N)\otimes H_1(N) \to \operatorname{Ker}\hat{p}_{1*} \xrightarrow{\hat{p}_{2*}} H_2(G) \to \cdots \to H_1(Q) \to 0\,.$$

The second assertion follows by noting that one has a 4-connected map $S(F*F) \to C_\chi$ by 1.1, 1.2 and 1.5

Let ι: $N \to N \tilde{\times} G$ denote the homomorphism defined by $\iota(n)=(n, 1)$. The following theorem is an extension of Theorem 2.1 in [7] to an arbitrary extension.

Theorem 3.3. *There exist homomorphisms*

$$\rho\colon H_2(N)\oplus H_1(N)\otimes H_1(N) \to \operatorname{Ker}\hat{p}_{1*}\cap H_2(N\tilde{\times} G)$$

and

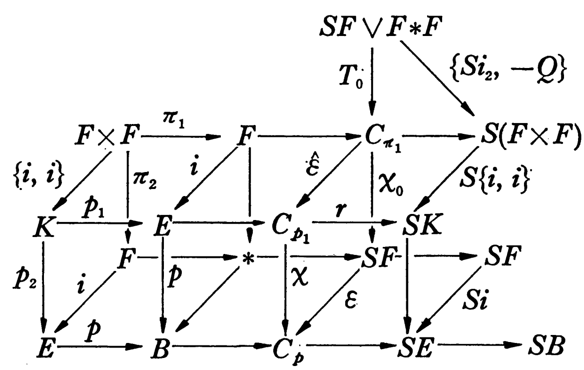
$$\sigma\colon \operatorname{Coker}\hat{p}_1{}^* \to H^2(N, A)\oplus \operatorname{Hom}(H_1(N)\otimes H_1(N), A)$$

which make the following sequences exact:

$$\begin{aligned}
&H_3(Q) \to \operatorname{Coker}\rho \to H_2(G)/\hat{\imath}_*H_2(N) \to H_2(Q) \to N/[N, G]\\
&\to H_1(G) \to H_1(Q) \to 0\,,\\
&0 \to H^1(Q, A) \to H^1(G, A) \to \operatorname{Hom}(N/[N, G], A) \to H^2(Q, A)\\
&\to H^2(G, A)\cap \operatorname{Ker}\hat{\imath}^* \to \operatorname{Ker}\sigma \to H^3(Q, A)\,,
\end{aligned}$$

where $\rho\,|\,H_2(N)=\iota_*$ and the first component of σ is induced by ι^*.

Proof. Let π_k: $F\times F \to F$ denote the projection onto the k th factor ($k=1, 2$) and let T_0: $SF\vee F*F \to C_{\pi_1}$ denote the homotopy equivalence given by $T_0(x, t)=(*, x; t)$, $T_0((1-t)x\oplus tx')=(x, x'; 1-t)$ (see 2.2). We shall refer to the diagram



where ε, $\hat{\varepsilon}$ and χ_0 are induced by the left-corner commutative cube, i_2: $F \to F\times F$ is the injection into the second factor, and Q: $F*F \to S(F\times F)$ denotes the quotient map.

Now the Puppe sequences resulting from the fibrations p and $p|F$ yield the following exact sequences

$$H_j(B)\to \mathrm{Coker}\,\mathcal{E}_*\to \mathrm{Coker}\,(Si)_*\to H_j(SB)$$
$$H^j(SB)\to \mathrm{Ker}\,(Si)^*\to \mathrm{Ker}\,\mathcal{E}^*\to H^j(B)\,.$$

Next we show that $\mathrm{Coker}\,\mathcal{E}_*$ and $\mathrm{Ker}\,\mathcal{E}^*$ have the more comprehensive expression in case $j=3$ as follows.

First we observe by 1.1 and 1.2 that C_χ and C_{χ_0} are homeomorphic to the cofibres of the fibrations $p\oplus p$: $C_{p_1,p_2}\to B$ and $C_{\pi_1,\pi_2}\to *$ respectively. Here applying 1.5 to these fibrations we see that the map $S(F*F)\to C_\chi$ is 4-connected and the map $S(F*F)\to C_{\chi_0}$ gives a homotopy equivalence. Thus we obtain the commutative diagram:

$$\begin{tikzcd}
H_3(F*F) \arrow[r] \arrow[d, equal] & H_3(C_{\pi_1}) \arrow[r] \arrow[d, "\hat{\mathcal{E}}_*"] & H_3(SF) \arrow[r] \arrow[d, "\mathcal{E}_*"] & 0 \\
H_3(F*F) \arrow[r] & H_3(C_{p_1}) \arrow[r] & H_3(C_p) \arrow[r] & 0
\end{tikzcd}$$

where rows are exact. This implies the isomorphisms

$$\mathrm{Coker}\,\mathcal{E}_*\cong \mathrm{Coker}\,\hat{\mathcal{E}}_* \quad\text{and}\quad \mathrm{Ker}\,\mathcal{E}^*\cong \mathrm{Ker}\,\hat{\mathcal{E}}^*\,.$$

Secondly we observe that r_* is monic, since p_1 has a cross-section. Therefore we have $\mathrm{Coker}\,\hat{\mathcal{E}}_*\cong r_*(\mathrm{Coker}\,\hat{\mathcal{E}}_*)$. We know that T_0 is a homotopy equivalence and therefore we reach the conclusion:

$$r_*(\mathrm{Coker}\,\hat{\mathcal{E}}_*)\cong \mathrm{Ker}\,(Sp_1)_*/\mathrm{Im}\,(S\{i,\,i\}_*\{Si_2,\,-Q\}_*)\,.$$

Taking $S\{i,\,i\}_*\{Si_2,\,-Q\}_*$ as ρ, we have the required result.

Corollary 3.4. *If $\hat{p}_*\colon H_3(G)\to H_3(Q)$ is epic, then there is an exact sequence*

$$H_2(N)+H_1(N)\otimes H_1(N)\xrightarrow{\rho}\mathrm{Ker}\,\hat{p}_*\cap H_2(N\,\widetilde{\times}\,G)\to H_2(G)/\hat{\imath}_*H_2(N)\to$$
$$\cdots\to H_1(Q)\to 0\,.$$

If $\hat{p}^*\colon H^3(Q,\,A)\to H^3(G,\,A)$ is monic, then there is an exact sequence

$$0\to H^1(Q,\,A)\to\cdots\to H^2(G,\,A)\cap \mathrm{Ker}\,\hat{\imath}^*\to \mathrm{Coker}\,\hat{p}_1{}^*\to$$
$$H^2(N,\,A)+\mathrm{Hom}\,(H_1(N)\otimes H_1(N),\,A)\,.$$

This follows from 3.3 by observing that $H_3(B)\to\mathrm{Coker}\,\mathcal{E}_*$ and $\mathrm{Ker}\,\mathcal{E}^*\to H^3(B;\,A)$ are trivial.

We shall next construct an exact sequence which is slightly "larger" than the previous ones. For this purpose consider the Puppe sequence for $\{p,\,p\}$: $E\vee E\to B$

$$E \vee E \xrightarrow{\{p,\, p\}} B \to C_{\{p,p\}} \to SE \vee SE \to SB \to \cdots .$$

The map $w\colon SK \to C_{\{p,p\}}$ is 3–connected from 1.6 and 1.3. Therefore we obtain the following result.

Theorem 3.5. *There is an exact sequence*

$$H_2(N \tilde{\times} G) \xrightarrow{\{\hat{p}_{1*},\, -\hat{p}_{2*}\}} H_2(G)+H_2(G) \xrightarrow{p_*+p_*} H_2(Q) \to H_1(N \tilde{\times} G)$$
$$\xrightarrow{\{\hat{p}_{1*},\, -\hat{p}_{2*}\}} H_1(G)+H_1(G) \to H_1(Q) \to 0 .$$

Corollary 3.6. *If $\hat{p}_*\colon H_3(G) \to H_3(Q)$ is epic, then there is an exact sequence*

$$H_1(N) \otimes H_1(N) \to H_2(N \tilde{\times} G) \to H_2(G)+H_2(G) \to \cdots \to H_1(Q) \to 0 .$$

If $H_3(Q)$ is free abelian, then there is a unnatural homomorphism $H_3(Q)+H_1(N)\otimes H_1(N) \to H_2(N \tilde{\times} G)$ which makes the sequence

$$H_3(Q)+H_1(N)\otimes H_1(N) \to H_2(N \tilde{\times} G) \to H_2(G)+H_2(G) \to \cdots \to H_1(Q) \to 0$$

exact.

The first exact sequence can be derived from the preceding Puppe sequence by combining the fact that $S(F*F) \to C_\xi$ is 4–connected owing to 1.5 and that C_w and C_ξ have the same homotopy type by 1.6.

In order to prove the second one, we consider the following commutative diagram (cf. Lemma 1.6)

$$\begin{tikzcd}
H_3(F*F) \arrow[r, equal] \arrow[d] & H_3(F*F) \arrow[d] & & \\
H_3(E \oplus E) \arrow[r] \arrow[d, "(p\oplus p)_*"] & H_3(SK) \arrow[d, "w_*"] & & \\
H_3(B) \arrow[r] \arrow[d] & H_3(C_{\{p,p\}}) \arrow[r] \arrow[d] & H_3(SE)\oplus H_3(SE) \arrow[r] & \cdots \\
0 & 0 & &
\end{tikzcd}$$

where the row and columns are exact. Since the left column splits, we obtain an exact sequence

$$H_3(B) \oplus H_3(F*F) \to H_2(K) \to H_2(E) \oplus H_2(E) \to H_2(B) \to \cdots$$

which concludes the proof.

From now on assume that (1) is a central extension. It follows from the proof of 3.1 that φ is an isomorphism of $\pi_1(K)$ onto the direct product $N \times G$.

This is realized by a homotopy equivalence $\bar{\varphi}: K \to F \times E$ such that $\bar{\varphi}=(\tau, p_1)$ for a map $\tau: K \to F$ with $\tau_*(\kappa)=\nu$. Let $\bar{\psi}: F\times E \to K$ denote a homotopy inverse of $\bar{\varphi}$ and let $q: F\times E \to E$ be the projection. Since p_1 is a fibration, we see from $q \simeq q\bar{\varphi}\bar{\psi}=p_1\bar{\psi}$ that there exists a map $\psi: F\times E \to K$ such that $\psi\simeq\bar{\psi}$ and $p_1\psi=q$. Put

$$\mu = p_2\psi\colon\ F\times E \to E\,.$$

Then (2) can be replaced by the following commutative ladder:

$$\begin{tikzcd}
 & & SF\vee F*E \arrow[d, shift right, "T"'] & & & \\
F\times E \arrow[r, "q"] \arrow[d, "\mu"] & E \arrow[r] \arrow[d, "p"] & C_q \arrow[r] \arrow[d, "\Delta"] \arrow[u, shift right, "\Theta"'] & S(F\times E) \arrow[r, "Sq"] \arrow[d, "S\mu"] & SE \arrow[r] \arrow[d, "Sp"] & \cdots \\
E \arrow[r, "p"] & B \arrow[r] & C_p \arrow[r, "\partial"] & SE \arrow[r, "Sp"] & SB \arrow[r] & \cdots
\end{tikzcd} \tag{3}$$

in which Δ is a 3-connected map induced by $pq=p\mu$ and T and Θ are homotopy equivalences as given in 2.2. Hence we have an exact sequence

$$H_3(SF)\oplus H_3(F*E) \xrightarrow{\partial_*\Delta_*T_*} H_2(E) \to H_2(B) \to \cdots\,.$$

Since $p\mu(1\times i)=pq(1\times i)=*$. there is a map

$$\mu_0\colon\ F\times F \to F$$

with $\mu(1\times i)\simeq i\mu_0$, which induces an H structure compatible with the group structure of N. Thus we obtain a homotopy-commutative diagram

$$\begin{tikzcd}
F*F \arrow[r, "Q"] \arrow[d, "1*i"'] & S(F\times F) \arrow[r, "S\mu_0"] \arrow[d, "S(1\times i)"'] & SF \arrow[d, "Si"] \\
F*E \arrow[r, "Q'"] & S(F\times E) \arrow[r, "S\mu"] & SE
\end{tikzcd}$$

On the other hand, there is an exact sequence

$$H_3(F*F) \xrightarrow{(S\mu_0)_*Q_*} H_3(SF) \to H_3(N, 2; Z) = 0\,,$$

which results from the canonical map $SF \to K(N, 2)$. Thus, in dimension 3,

$$\mathrm{Im}\,(S\mu)_*Q'_* \supset \mathrm{Im}\,(Si)_*\,.$$

Now we see from 2.2 that

$$\partial\Delta T\,|\,SF \simeq Si \quad\text{and}\quad \partial\Delta T\,|\,F*E = (S\mu)Q'\,.$$

From these data and the fact that $F*E$ is 2-connected we recover the exact sequences

$$N\otimes H_1(G)\xrightarrow{(S\mu)_*Q'_*}H_2(G)\to H_2(Q)\to N\to H_1(G)\to H_1(Q)\to 0\,,$$
$$0\to H^1(Q,A)\to H^1(G,A)\to \mathrm{Hom}\,(N,A)\to H^2(Q,A)\to H^2(G,A)$$
$$\to \mathrm{Hom}\,(N\otimes H_1(G),A)\oplus \mathrm{Ext}\,(N,A)\,,$$

which have been obtained by T. Ganea [7, Theorem 1.1].

We can also see that the morphisms ρ and σ in 3.3 are equivalent to

$$1\oplus(1*i)_*:\ H_2(N)\oplus H_3(F*F)\to H_2(N)\oplus H_3(F*E)$$

and

$$1\oplus(1*i)^*:\ H^2(N,A)\oplus \mathrm{Hom}\,(N\otimes H_1(G),A)\to H^2(N,A)\otimes \mathrm{Hom}\,(N\otimes N,A)$$

respectively. Theorem 3.3 allows us to conclude Theorem 2.1 of T. Ganea [7]: there exist exact sequences

$$H_3(Q)\to N\otimes H_1(Q)\to H_2(G)/\hat{\imath}_*H_2(N)\to H_2(Q)\to N\to H_1(G)\to H_1(Q)\to 0$$
$$0\to H^1(Q,A)\to H^1(G,A)\to \mathrm{Hom}\,(N,A)\to H^2(Q,A)\to \mathrm{Ker}\,\hat{\imath}^*\to$$
$$\mathrm{Hom}\,(N\otimes H_1(Q),A)\to H^3(Q,A)\,.$$

4. Monomorphisms

A map $f\colon X\to Y$ is called a *monomorphism* if the induced function $f_*\colon[V,X]\to[V,Y]$ is injective for every V (see [5], [9, p. 168]). We shall prove

Theorem 4.1. *Let $f\colon X\to Y$ be an n-connected monomorphism and let X be a connected space such that $\pi_i(X)=0$ for $i\geqq 2n+1$ with $n\geqq 1$. Then f has a left homotopy inverse.*

Proof. By the assumtion, 1.4 implies that the following square is exact:

$$\begin{tikzcd}
 & {[X,X]} \arrow[dl, "P_1^*"'] & \\
{[E_{f,f},X]} & & {[Y,X]} \arrow[ul, "f^*"'] \arrow[dl, "f^*"] \\
 & {[X,X]} \arrow[ul, "P_2^*"] &
\end{tikzcd}$$

On the other hand, the definition of $E_{f,f}$ implies $f_*(P_1)=f_*(P_2)$. Hence we have $P_1\simeq P_2$, since f is a monomorphism. Now the exactness of the diagram proves the existence of a map $g\colon Y\to X$ satisfying $1_X=f^*(g)$.

According to T. Ganea [5], the Hopf fibrations

$$h\colon S^n\to RP^n\quad\text{and}\quad h\colon S^{2n+1}\to CP^n$$

are monomorphisms for odd $n>1$, and these do not admit left homotopy inverses, as shown by the inspection of cohomology.

Finally, we remark that the condition in 4.1 is indispensable for the conclusion. Consider a map $\theta: K(Z, 2)\to K(Z, 2m+2)$ which represents the $(m+1)$-fold cup product ι^{m+1} of the basic class $\iota\in H^2(Z, 2; Z)$ and let

$$K(Z, 2m+1) \xrightarrow{i} E_\theta \xrightarrow{p} K(Z, 2)$$

be the principal fibration induced by θ. Since $\Omega\theta\simeq 0$, we see that i is a monomorphism by Theorem 15.11′ of [9]. But it is known that p is the first stage for a Postnikov system of CP^m, so that there is a $(2m+2)$-connected map $g: CP^m\to E_\theta$. Thus $H_{2m+1}(E_\theta; Z)=0$, which shows that i does not admit any left homotopy inverse. Taking $m=1$, the inclusion i provides an example.

5. Thomas exact sequence

Let $p: E\to B$ be a principal fibration in the restricted sense, as defined in [13], with fibre inclusion $i: F\to E$ and with action $\mu: F\times E\to E$. Let K denote the square of p by p with projections $p_1, p_2: K\to E$, and let $q: F\times E\to E$ be the projection onto the second factor. Then, by Lemma 2.1 of [13], $\{q, \mu\}: F\times E\to K$ is a homotopy equivalence making the diagram

$$\begin{tikzcd}
 & & E \arrow[dr, "p"] & \\
F\times E \arrow[urr, "q"] \arrow[r, "\{q,\mu\}"] \arrow[drr, "\mu"'] & K \arrow[ur, "p_1"] \arrow[dr, "p_2"'] & & B \\
 & & E \arrow[ur, "p"'] &
\end{tikzcd}$$

commutative. Hence, Theorem 1.3 implies that, if p is n-connected, $n\geqq 1$, then the sequence

$$[F\times E, V] \underset{q^*}{\overset{\mu^*}{\leftleftarrows}} [E, V] \xleftarrow{p^*} [B, V]$$

is exact for a connected space V with $\pi_i(V)=0$ for $i\geqq 2n+1$, that is, $\mu^*(x)=q^*(x)$ for $x\in[E, V]$ if and only if there is a $y\in[B, V]$ such that $x=p^*(y)$.

Henceforth we assume that p is n-connected, $n\geqq 1$. Then one has the commutative ladder (3) with Δ being $(2n+1)$-connected.

Let $i_2: E\to F\times E$ denote the inclusion. In virtue of Lemma 2.1 (ii), we can replace C_q by SC_{i_2} and the mutually inverse homotopy equivalences

$$C_q \underset{G}{\overset{F}{\rightleftarrows}} SC_{i_2}$$

are given by

$$
\begin{array}{ll}
F(x, y; t) = (x, y; t),\ F(y) = * & \text{for } x \in F,\ y \in E,\ 0 \leq t \leqq 1 \\
G(x, y; s) = \begin{cases} (x, y; 2s) \\ (*, y; 2-2s) \end{cases} & \begin{array}{l} \text{for } 0 \leqq 2s \leqq 1 \\ \text{for } 1 \leqq 2s \leqq 2 \end{array} \\
G(y, t; s) = \begin{cases} (*, y; 2st) \\ (*, y; 2t-2st) \end{cases} & \begin{array}{l} \text{for } 0 \leqq 2s \leqq 1 \\ \text{for } 1 \leqq 2s \leqq 2\,. \end{array}
\end{array}
$$

It follows from these expressions that the diagram

$$
\begin{tikzcd}
S(F \times E) \arrow[r] \arrow[dd, "(S\mu)-(Sq)"'] & SC_{i_2} \arrow[d, "G"] \\
& C_q \arrow[d, "\Delta"] \\
SE & C_p \arrow[l, "\partial"']
\end{tikzcd}
$$

is commutative.

Let

$$J(\mu):\ F*E \to SE$$

denote the map obtained from μ by the Hopf construction, i.e. the composite $F*E \to S(F \times E) \xrightarrow{S\mu} SE$. Then the composite $\partial \Delta T$ in (3) is homotopic to Si on SF and $J(\mu)$ on $F*E$.

Summarizing the above discussion we have

Theorem 5.1. *The following sequences are exact for a connected space V with $\pi_k(V)=0$ for $k \geqq 2n+1$:*

$$[F \times E, V] \underset{q^*}{\overset{\mu^*}{\leftleftarrows}} [E, V] \xleftarrow{p^*} [B, V] \leftarrow [SC_{i_2}, V] \leftarrow [SE, V] \xrightarrow{(Sp)^*} \cdots$$

$$[F \times E, V] \underset{q^*}{\overset{\mu^*}{\leftleftarrows}} [E, V] \xleftarrow{p^*} [B, V] \leftarrow [SF, V] \oplus [F*E, V]$$

$$\xleftarrow{\{(Si)^*,\, J(\mu)^*\}} [SE, V] \leftarrow \cdots$$

The first sequence is an extension of an exact sequence due to E. Thomas [19].

6. Duality

6.1. It seems difficult to define the dual of the Whitney join in an effective way. However we can define the homotopy analogue of the dual of the Whitney join as shown in [12] in the following way.

Let us given a pushout diagram

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "g"'] & A \arrow[d, "i_1"] \\ B \arrow[r, "i_2"] & L \end{tikzcd}$$

where f and g are cofibrations. Then the dual of the Whitney join

$$f\widehat{\oplus}g\colon X \to E_{i_1,i_2}$$

is defined by $[(f\widehat{\oplus}g)(x)](t)=x$ for $x\in X$, $0\leqq t\leqq 1$. Note that, if we denote by T' the fibre of $E_f\to E_{i_2}$, $\pi_n(T')$ is isomorphic to the homotopy group $\pi_{n+2}(L;A,B)$ of a triad $(L;A,B)$.

Now, given a system of maps $A\xleftarrow{f}X\xrightarrow{g}B$, let

$$I_1\colon A\to C_{f,g} \quad \text{and} \quad I_2\colon B\to C_{f,g}$$

denote the canonical injections. Then the homotopy-commutative diagram

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "g"'] & A \arrow[d, "I_1"] \\ B \arrow[r, "I_2"] & C_{f,g} \end{tikzcd}$$

is homotopically equivalent to the pushout

$$\begin{tikzcd} X \arrow[r, "i"] \arrow[d, "j"'] & M_f \arrow[d, "i_1"] \\ M_g \arrow[r, "i_2"] & C_{f,g} \end{tikzcd}$$

of two injections into mapping cylinders M_f and M_g. Now we consider the map $\hat{\xi}\colon X\to E_{I_1,I_2}$ defined by $\hat{\xi}(x)=(f(x),\hat{x},g(x))$, where $\hat{x}$ denotes the path given by $\hat{x}(t)=(x,t)$. Clearly $\hat{\xi}$ is homotopically equivalent to $i\widehat{\oplus}j$.

Now, applying a theorem of Blakers-Massey to the triad $(C_{f,g};M_f,M_g)$, we we can prove the following theorem in the similar manner as in 1.3 and 1.4 (cf. [12, Theorem 4.3]).

Theorem 6.1. *Suppose f and g are m- and n-connected respectively, $m\geqq 1$, $n\geqq 1$, $m+n\geqq 3$, and let X be 1-connected. Then $\hat{\xi}\colon X\to E_{I_1,I_2}$ is $(m+n-1)$-connected and the diagram*

$$\begin{tikzcd} & {[V,A]} \arrow[dr, "I_{1*}"] & \\ {[V,X]} \arrow[ur, "f_*"] \arrow[dr, "g_*"'] & & {[V,C_{f,g}]} \\ & {[V,B]} \arrow[ur, "I_{2*}"'] & \end{tikzcd}$$

is exact for a CW complex V *with* $\dim V \leqq m+n-1$. *Further, if*

$$\chi_1\colon E_f \to E_{I_2} \quad \textit{and} \quad \chi_2\colon E_g \to E_{I_1}$$

denote the maps induced by a canonical homotopy $I_1 f \simeq I_2 g$, *then* χ_1 *and* χ_2 *are* $(m+n-1)$*-connected.*

We remark that, as shown in [14], the restrictions on m and n can be removed under appropriate assumption.

6.2. Following [9, p. 168], we say that $f\colon X \to Y$ is an *epimorphism* if $f^*\colon [Y, V] \to [X, V]$ is injective for every V. The following theorem can be deduced, in a similar way to 4.1, from 6.1.

Theorem 6.2. *Suppose* $f\colon X \to Y$ *is an* n*-connected epimorphism with* X 1*-connected,* $n \geqq 2$, *and let* Y *be a CW complex with* $\dim Y \leqq 2n-1$. *Then* f *has a right homotopy inverse.*

The condition of the above theorem cannot be removed with the conclusion unchanged, because the projection $f\colon S^n \times S^n \to S^{2n}$ $(n \geqq 2)$ shrinking $S^n \vee S^n$ to a point is an epimorphism with no right homotopy inverse, as mentioned in [9, p. 181].

6.3. Consider a principal cofibration

$$A \xrightarrow{i} B \xrightarrow{q} C \tag{4}$$

in the restricted sense [13], where C, the cofibre of i, is an H cogroup. Let K denote the pushout of $B \xleftarrow{i} A \xrightarrow{i} B$, i.e. the space formed from $B \vee B$ by identifying $(i(a), *)$ with $(*, i(a))$, $a \in A$, and let $j_k\colon B \to K$ $(k=1, 2)$ denote the canonical inclusions. There is defined a coaction

$$\mu\colon B \to C \vee B$$

such that $\{i_2, \mu\}\colon K \to C \vee B$ is a homotopy equivalence where $i_2\colon B \to C \vee B$ denotes the injection. Let $r_2\colon C \vee B \to B$ be the projection shrinking C to a point.

Henceforth we shall assume that i is n-connected, $n \geqq 2$. Then it follows from 6.1 that the sequence

$$[V, A] \xrightarrow{i_*} [V, B] \underset{i_{2*}}{\overset{\mu_*}{\rightrightarrows}} [V, C \vee B]$$

is exact for a CW complex with $\dim V \leqq 2n-1$.

Introduce the commutative ladder

$$\begin{array}{ccccccccc} \cdots \longrightarrow & \Omega B & \xrightarrow{\varepsilon} & E_i & \xrightarrow{p} & A & \xrightarrow{i} & B & \\ & \Omega\mu \downarrow & & \nabla \downarrow & & i \downarrow & & \downarrow \mu & \qquad (5) \\ \cdots \longrightarrow & \Omega(C \vee B) & \longrightarrow & E_{i_2} & \longrightarrow & B & \xrightarrow{i_2} & C \vee B & \end{array}$$

in which ∇ is a $(2n-1)$-connected map induced by the right-hand commutative square. We set

$$\bar{\mu} = \nabla_* \mathcal{E}_* : \pi_k(\Omega B) = \pi_{k+1}(B) \to \pi_k(E_{i_2}) = \pi_{k+1}(C \vee B, B)$$
$$\tau = p_*(\nabla_*)^{-1} : \pi_k(E_{i_2}) = \pi_{k+1}(C \vee B, B) \to \pi_k(A)$$

for $k \leqq 2n-2$.

With the above notation we can state an exact sequence which is dual to the cohomology exact sequence obtained by E. Thomas as follows:

Theorem 6.3. *The sequence*

$$\pi_{2n-1}(A) \xrightarrow{i_*} \pi_{2n-1}(B) \xrightarrow{\bar{\mu}} \pi_{2n-1}(C \vee B, B) \xrightarrow{\tau} \pi_{2n-2}(A)$$
$$\cdots \to \pi_k(A) \xrightarrow{i_*} \pi_k(B) \xrightarrow{\bar{\mu}} \pi_k(C \vee B, B) \xrightarrow{\tau} \pi_{k-1}(A) \to \cdots$$

is exact with the following additional properties:

(i) *For* $\alpha \in \pi_r(A)$, $\gamma \in \pi_s(C \vee B, B)$ *with* $r+s \leqq 2n$, $r \geqq 1$, $s \geqq 1$,

$$\tau[\gamma, i_*\alpha] = -[\tau(\gamma), \alpha]$$

where the bracket in the left-hand side denotes the relative Whitehead product.

(ii) *Let* i_1: $C \to (C \vee B, B)$ *be the inclusion and let* τ_0 *denote the homomorphism determined by the composite*

$$p_*(q_*)^{-1} : \pi_k(C) \leftarrow \pi_k(B, A) \to \pi_{k-1}(A)\,.$$

Then, for $\beta \in \pi_k(C) \cap \operatorname{Im} q_*$,

$$\tau(i_{1*}\beta) = \tau_0(\beta) \mod p_* (\operatorname{Ker} q_*)\,.$$

(iii) $\tau(\beta \circ \partial^{-1}\kappa) = \tau(\beta) \circ \kappa$ *for* $\kappa \in \pi_{m-1}(S^{k-1})$, $\beta \in \pi_k(C \vee B, B)$, *where* ∂ *is the boundary isomorphism*

$$\partial : \pi_m(CS^{k-1}, S^{k-1}) \to \pi_{m-1}(S^{k-1})\,.$$

Proof. The exactness follows immediately from (5). Since p_* is the boundary operator, we have, by (3.5) and (3.4) of [3],

$$\nabla_*[\nabla_*^{-1}\gamma, \alpha] = [\nabla_* \nabla_*^{-1}\gamma, i_*\alpha] = [\gamma, i_*\alpha]\,.$$

Hence

$$\begin{aligned}\tau[\gamma, i_*\alpha] &= p_*\nabla_*^{-1}[\gamma, i_*\alpha] = p_*[\nabla_*^{-1}\gamma, \alpha] \\ &= -[p_*\nabla_*^{-1}\gamma, \alpha] = -[\tau\gamma, \alpha]\,.\end{aligned}$$

The second property (ii) is a direct consequence of the following commutative diagram:

$$\begin{tikzcd}
\pi_k(C) \arrow[d, equal] & \pi_k(B, A) \arrow[l, "q_*"'] \arrow[r, "\partial"] \arrow[d, "\nabla_*"] \arrow[dl, "i_{1*}"] & \pi_{k-1}(A) \arrow[d, equal] \\
\pi_k(C) & \pi_k(C \vee B, B) \arrow[l, "r_{1*}"] \arrow[r, "\tau"] & \pi_{k-1}(A)
\end{tikzcd}$$

where r_1: $C \vee B \to C$ denotes the projection pinching B to a point and the commutativity follows from $r_1\mu \simeq q$ and $r_1 i_1 = 1$.

The last property (iii) is obvious from the definition of τ.

REMARK. Note that the exact sequence in 6.3 is not exactly dual to the one due to E. Thomas [19], and the precise dual will be the one obtained by replacing E_{i_2} by ΩE_{r_2}, where r_2: $C \vee B \to B$ is the retraction.

6.4. We consider here a situation as an illustration of Theorem 6.3. First we prove

Lemma 6.4. *Let i: $A \to B$ be the principal cofibration induced by θ: $S^{n-1} \to A$, i.e. $B = C_\theta$ where A is 1-connected and $n \geqq 3$. Then*

$$\tau(i_{1*}\iota_n) = \theta \qquad \textit{for the identity class} \quad \iota_n \in \pi_n(C)\,.$$

Proof. We see from the Blakers-Massey theorem that

$$q_*:\ \pi_n(B, A) \to \pi_n(S^n)$$

is bijective. Since the characteristic map $\hat{\theta}$: $(CS^{n-1}, S^{n-1}) \to (B, A)$ satisfies $p_*(\hat{\theta}) = \theta$ and $q_*(\hat{\theta}) = \iota_n$, and since i is $(n-1)$-connected, it follows from (ii) of 6.3 that $\tau(i_{1*}\iota_n) = \theta$.

Theorem 6.5. *Let i: $A \to B$ be as in Lemma 6.4 and let a relation*

$$\theta\circ\alpha - [\theta, w] = 0, \qquad w \in \pi_q(A),\ \alpha \in \pi_{n+q-2}(S^{n-1})$$

be given, where $1 < q < n-1$. Then there exists an element $\eta \in \pi_{n+q-1}(B)$ such that

$$\mu_*\eta = i_{1*}(S\alpha) + i_{2*}\eta + [i_{1*}\iota_n, i_{2*}i_*w],\ q_*\eta = S\alpha\ ,$$

where μ: $B \to S^n \vee B$ denotes the coaction and i_1 is the inclusion $S^n \to S^n \vee B$.

REMARK. Taking S^q and $m\iota_q$ (m: an integer) for A and w, we obtain a result due to I.M. James [10, Lemma (5.4)].

Proof. By (i), (iii) of 6.3 and Lemma 6.4, we have

$$\tau[i_{1*}\iota_n, i_*w] = -[\tau(i_{1*}\iota_n), w] = -[\theta, w]$$
$$\tau(i_{1*}S\alpha) = \tau(i_{1*}\iota_n)\circ\alpha = \theta\circ\alpha .$$

These imply that $i_{1*}S\alpha+[i_{1*}\iota_n, i_*w]$ lies in the kernel of τ and, by the exactness of 6.3, there exists $\eta\in\pi_{n+q-1}(B)$ such that

$$\bar{\mu}(\eta) = i_{1*}S\alpha+[i_{1*}\iota_n, i_*w] .$$

Now consider the following commutative diagram

$$\begin{array}{ccccc}
 & & \pi_{n+q-1}(B) & \xrightarrow{\varepsilon_*} & \pi_{n+q-1}(B, A) \\
 & & \downarrow \mu_* & & \downarrow \nabla_* \\
\pi_{n+q-1}(B) & \xrightarrow{i_{2*}} & \pi_{n+q-1}(S^n\vee B) & \xrightarrow{k_*} & \pi_{n+q-1}(S\vee B, B)
\end{array}$$

where k denotes the inclusion, which leads to

$$k_*\{\mu_*\eta-i_{1*}S\alpha-[i_{1*}\iota_n, i_{2*}i_*w]\} = 0$$

by (3.10) of [3]. Hence there exists a unique $\hat{\eta}\in\pi_{n+q-1}(B)$ with

$$i_{2*}\hat{\eta} = \mu_*\eta-i_{1*}S\alpha-[i_{1*}\iota_n, i_{2*}i_*w] .$$

Moreover, it follows from $r_2\mu\simeq1$ and $r_1\mu\simeq q$ that $\hat{\eta}=\eta$ and $S\alpha=q_*\eta$.

6.5. Take A and B to be the complex projective spaces $P^m(C)$ and $P^{m+1}(C)$ and let θ: $S^{2m+1}\to P^m(C)$ be the Hopf map. Then, by [2],

$$[\theta, w] = \begin{cases} \theta\circ\eta & \text{if } m \text{ is even} \\ 0 & \text{if } m \text{ is odd} \end{cases},$$

for the generator $w\in\pi_2(P^m(C))$ which comes from $\pi_1(S^1)$, where η is the generator of $\pi_{2m+2}(S^{2m+1})$. Thus there is an element $\rho\in\pi_{2m+3}(P^{m+1}(C))$ such that

$$\mu_*\rho = \begin{cases} i_{1*}S\eta+i_{2*}\rho+[i_{1*}\iota_{2m-2}, i_{2*}i_*w] & \text{for } m \text{ even} \\ i_{2*}\rho+[i_{1*}\iota_{2m-2}, i_{2*}i_*w] & \text{for } m \text{ odd} \end{cases}$$

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