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# ON THE TRANSVERSAL IMMERSION

KAZUAKI KOBAYASHI

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Zeeman and Armstrong [1], [2], [4] introduced the notion of transversality for the intersection of two manifolds or two polyhedra and proved some results. This notion can be considered as a refinement of general position. In this paper we extend this to the self-intersection of an piecewise linear immersion between manifolds.

The main result of this paper (Theorem) says that any locally flat immersion of a closed manifold into a manifold without boundary can be approximated by a transversal immersion.

It will be assumed, without further mention, that all manifolds have a piecewise linear structure and that all maps are piecewise linear immersions of a closed (i.e. compact and without boundary) manifold into a manifold without boundary.

## Definitions and Theorem

Let $M$ be a closed $m$-manifold and $Q$ be a $q$-manifold without boundary. Let $f:M\to Q$ be an immersion. Let $\hat{A}$ be the barycenter of a simplex $A$. We denote $E^q$ the $q$-dimensional euclidean space.

DEFINITION 1. Let $J$, $K$ be triangulations of $M$, $Q$ such that $f:J\to K$ is simplicial. If $(Lk(fx, K), f(Lk(x, J)))$ is unknotted sphere pair for any $x\in M$, we say that $f$ is a *locally flat immersion*. This definition is independent of the triangulation of $M$, $Q$. Let $S_f=\{x\in M\,|\,f^{-1}f(x)\neq\{x\}\}$. Then $S_f=\bar{S}_f$ where $\bar{S}_f$ is the closure of $S_f$ since $f$ is an immersion. And let $S_f(r)=\{x_1\in M\,|\,f^{-1}f(x_1)=\{x_1,\cdots, x_n\}, n\geq r\}$. Hence $S_f=S_f(2)$.

DEFINITION 2. Let $f^{-1}f(x_1)=\{x_1,\cdots, x_n\}$ for some $x_1\in S_f$. If the following diagram commutes for any $i\,(1\leq i\leq n)$ except $j$, we call $f$ *transversal to $fM$ at $x_j$* where $\varphi_j$, $\varphi_i$, $\psi$ are homeomorphism onto some neighborhoods of $x_j$, $x_i$, $f(x_1)$ respectively.

$$\begin{array}{ccccc}
D^{2m-q}\times D^{q-m},\ 0\times 0 & \xrightarrow{1\times 1\times 0} & D^{2m-q}\times D^{q-m}\times D^{q-m},\ 0\times 0\times 0 & \xleftarrow{1\times 0\times 1} & D^{2m-q}\times D^{q-m},\ 0\times 0 \\
\varphi_j\downarrow & & \downarrow\psi & & \downarrow\varphi_i \\
M,\ x_j & \xrightarrow{\quad f\quad} & Q,\ f(x_1) & \xleftarrow{\quad f\quad} & M,\ x_i
\end{array}$$

And we simply call $f$ *transversal to* $fM$ *at* $f(x_1)$ if for any $j\,(1\leq j\leq n)$ $f$ is transversal to $fM$ at $x_j$.

Furthermore we define $f$ *transversal immersion* if $f$ is transversal to $fM$ at any $f(x)$, $x\in S_f$.

DEFINITION 3. An immersion $f$ of $M$ into $Q$ is *in general postition itself* if dim $S_f(r)\leq rm-(r-1)q$ for all $r$.

**Theorem.** *If $f$ is a locally flat immersion of $M^m$ into $Q^q$, then we can homotope $f$ into a locally flat transversal immersion $g$ by an arbitrarily small homotopy. Furthermore if $f$ is in general position itself, then $g$ is also in general position itself.*

REMARK 1. The above theorem still holds for an immersion $f$ such that $f$ is locally flat at any point of $St(S_f, M)$ although $g$ is a transversal immersion such that $g$ is locally flat at any point of $St(S_g, M)$.

REMARK 2. Conversely if $f$ is a transversal immersion of $M$ into $Q$, $f$ is locally flat at any point of $S_f$. For it is obvious by the left half or the right half of the above diagram.

REMARK 3. When $m=q$, theorem is obvious. So we shall prove it for $m<q$.

**Corollary 1.** *If $g$ is a transversal immersion and $S_g$ consists only on double points, then $S_g$ is a closed locally flat $(2m-q)$-submanifold of $M$.*

If the codimension of $M$ and $Q$ is greater than 3, any immersion is locally flat [4]. Hence we obtain the following Cor. 2.

**Corollary 2.** *If $m+3\leq q$, any piecewise linear immersion of $M^m$ into $Q^q$ is arbitrarily approximated by an transversal immersion. The approximation is made to be homotopic.*

**Corollary 3.** *If $f$ is any piecewise linear immersion of $M^2$ into $Q^4$, then $f$ is approximated by an transversal immersion. The approximation can be chosen so near and to be homotopic.*

DEFINITION 4. Let $f$ be an immersion of $M$ into $Q$ and $J$, $K$ be simplicial subdivisions of $M$, $Q$ respectively. We call $f$ *in general position with respect to $J$ and $K$ at $x$* if for any simplexes $\Delta^i\in J$ and $\Delta^k\in K$ such that $x\in\mathring{\Delta}^i$ and $f(x)\in\mathring{\Delta}^k$,

$$\dim\,(f\Delta^i\cap\Delta^k)\leq i+k-q\,.$$

For any $x\in M$ let $A$ be a simplex of $K$ such that $f(x)\in\mathring{A}$. Choose a vertex $v$ of $A$, let $L=Lk(A, K)$, and $s:AL\to vL$ the simplicial map defined as the join of $A\to v$ to the identity on $L$.

DEFINITION 5. (Armstrong, Zeeman [1], [2]) Let the map $f$ be an immersion of $M$ into $Q$ and be in general position with respect to $J$ and $K$ where $|J|=M$, $|K|=Q$. The map $f$ is *transimplicial to K at the point* $x\in M$ if there exists a neighborhood $N$ of $v$ in $vL$, and a commutative diagram

$$\begin{array}{ccccc} N\times D^{m+a-q} & \xrightarrow{1\times k} & N\times D^{a} & \xrightarrow{p} & N \\ \alpha\downarrow & & \beta\downarrow & & \downarrow\subset \\ St(x, J)\cap f^{-1}AL & \xrightarrow{f} & AL & \xrightarrow{s} & vL \end{array}$$

where a is the dimension of $A$, $k$ is a proper embedding $D^{m+a-q}\to D^{a}$, $p$ is a projection and $\alpha$, $\beta$ are embeddings onto neighborhoods of $x$, $f(x)$ respectively.

REMARK 4. The definition is independent of the choice of $v$.

REMARK 5. If $f$ is transimplicial to $K$ at $x$ and $x\in\mathring{\Delta}^i$ $(i<m)$, then $m+a>q$. For since $f$ is in general position and $f(x)\in\mathring{A}$,

$$\dim(f\Delta^i\cap A)\leq i+a-q<m+a-q\,.$$

Hence $f\Delta^i\cap A=\phi$ if $m+a-q\leq 0$. This contradicts $x\in\mathring{\Delta}^i$ and $f(x)\in\mathring{A}$.

If $f$ is transimplicial to $K$ at $x$ and $x\in\mathring{\Delta}^m$, then $m+a\geq q$. For $\dim(f\Delta^m\cap A)\leq m+a-q$. Hence $f\Delta^m\cap A=\phi$ if $m+a<q$. This is contradiction.

REMARK 6. Let $K'$ be a subdivision of $K$. If $f$ is transimplicial to $K'$ at $x\in M$ (hence $f$ is in general position with respect to $K'$), then $f$ is also transimplicial to $K$ at $x$ (see [2]).

DEFINITION 6. Let $K$ be a combinatorial manifold of dimension $q$. Then $K$ is called a *Brouwer manifold* if

(i) For each $v\in K$ there is a linear embedding
$St(v, K)\to E^q$

(ii) For each $v\in K$ there is a linear embedding
$St(v, K)$, $St(v, K)\to E^q_+$, $E^{q-1}$.

REMARK 7. Not all combinatorial manifolds are Brouwer, [3].

REMARK 8. Any subdivision of a Brouwer is Brouwer.

**Lemma 0** (Zeeman [2]). *Any combinatorial manifold has a Brouwer subdivision.*

**Lemmata**

When $m=q$, theorem is obvious. So we assume $m<q$ throughout this section.

Now let $f:M\to Q$ be an immersion and $J$, $K$ be subdivision of $M$, $Q$ such that $f$ is simplicial with respect to $J$, $K$ and such that $K$ is Brouwer subdivision. And we may suppose $f$ is in general position itself by [4, Chap. 6]. Let $x_1\in S_f$, and $f^{-1}f(x_1)=\{x_1,\cdots, x_n\}$.

We suppose $x_i\in B_i^b$, $f(x_1)\in \mathring{A}$ where $B_i$ $(i=1, 2,\cdots, n)$ and $A$ are simplexes of $J$ and $K$ respectively.

Let $X_i=St(\hat{B}_i, J')$, $W=St(\hat{A}, K')$.

We construct approximating maps $g_i^b(i=1, 2,\cdots, n-1)$ of $f$ inductively as follows. First take $v_1$ in $\mathring{W}$ such that $v_1\notin f(M)$ and $v_1$ is in general position with respect to the vertices of $W$. Then we define $g_1^b$ as follows

$$g_1^b\,|\,M-\mathring{S}t(\hat{B}_1, J')=f\,|\,M-\mathring{S}t(\hat{B}_1, J')$$
$$g_1^b(\hat{B}_1)=v_1$$

and for any $y\in St(\hat{B}_1, J')$ if $y=(1-\lambda)d+\lambda\hat{B}_1$ $(d\in\dot{S}t(\hat{B}_1, J'))$, $g_1(y)=(1-\lambda)f(d)+\lambda v_1$.

Obviously $g_1^b$ is not simplicial with respect to $J'$ and $K'$. We take subdivisions $J_1$, $K_1$ of $J'$, $K'$ such that they are subdividing the parts $f^{-1}(\mathring{W})$ and $\mathring{W}$ and such that $g_1^b:J_1\to K_1$ is simplicial.

Next take $v_2$ in $\mathring{S}t(\hat{A}, K'_1)$ satisfying $v_2\notin g_1^b(M)$ and $v_2$ is in general position with respect to the vertices of the subcomplex $W_1$ of $K'_1$ covering $W$. And we define $g_2^b$ satisfying

$$g_2^b\,|\,M-\mathring{S}t(\hat{B}_2, J')=g_2^b\,|\,M-\mathring{S}t(\hat{B}_2, J')$$
$$g_2^b(\hat{B}_2)=v_2 \qquad \text{and}$$

for any $y\in\mathring{S}t(\hat{B}_2, J')$ it maps linearly same as $g_1^b$. We take subdivisions $J_2$, $K_2$ of $J'_1$, $K'_1$ such that they are subdividing the parts $(g_1^b)^{-1}\mathring{W}_1$ and $\mathring{W}_1$ and such that $g_2^b:J_2\to K_2$ is simplicial. We construct $g_i^b$ $(3\leq i\leq n-1)$ in the same way. Furthermore we similarly construct approximating maps of $f$ for any other point $x\in S_f$ such that $x\in\mathring{B}^b$.

And we again put the maps $g_j^b$.

Obviously $g_i^b\,|\,X_i$ is in general position with respect to $J'_{i-1}$ and $K'_{i-1}$ $(i=1, 2,\cdots, n-1)$ at any point $x\in\mathring{X}_i$ where $J_0=J$, $K_0=K$ and $g_i^b$ is in general position itself.

**Lemma 1.** *For all $i\,(i=1, 2,\cdots, n-1)$ the map $g_i^b$ constructed above is transimplicial to $K'_{i-1}$ at any point of $\mathring{X}_i$ where $K_0=K$.*

Proof. First we show $g_1^b$ transimplicial to $K'$ at any point of $\mathring{X}_1$. For any $y\in\mathring{S}t(\hat{B}_1, J')$ let $C$, $D$ be simplexes of $J'$, $K'$ respectively such that $y\in\mathring{C}$, $g_1^b y\in\mathring{D}$. Since $K$ is Brouwer triangulation, we may suppose $St(A, K)$ embedded linearly in $E^q$. If $D$ is a principal simplex of $K'$, $g_1^b$ is obviously transimplicial to $K'$ at $y$. Hence we may suppose dim $D<q$. $g_1^b C$ is the linear join in $E^q$ of $v_1$ to some simplex of $\mathring{W}$ by the construction of $g_1^b$. And since $v_1$ is in general position with respect to the vertices of $W$, $g_1^b C$ and $D$ span $E^q$. $g_1^b C\cap D$ is a $(c+d-q)$-linear convex cell. Let $(g_1^b)^{-1}(g_1^b C\cap D)\cap C=E$ and $F$ a $(q-d)$-cell through $y$ that is perpendicular to $E$ in $C$. Let $C^*$ be a simplex of $J'$ having $C$ as a face and $E^*=(g_1^b)^{-1}(g_1^b C^*\cap D)\cap C^*$. Then although $E^*$ is not necessarily perpendicular to $F$ in $C^*$, it has the property that any $(q-d)$-cell parallel to $F$ in $C^*$, and sufficiently close to $F$, meets it in exactly one point. Therefore for some sufficiently small neighborhood $U$ of $y$ in $St(\hat{B}_1, J')$ we can define a map $\rho_1: U\to(g_1^b)^{-1}D$ by projecting $U\cap C^*$ parallel to $F$ onto the corresponding $E^*$. Now return to $E^q$. Since we defined $F$ perpendicular to $E$ in $C$, we know that the linear subspace $[g_1^b F]$ and $[D]$, spanned by $g_1^b F$ and $D$, are complementary in $E^q$. Hence we define the map

$$\rho_2\colon E^q\to[D] \text{ parallel to } [g_1^b F]\,.$$

Then $\rho_2 g_1^b=g_1^b\rho_1$ on $U$.

Choose a vertex $v$ of $D$ and let $L=lk(D, K')$, $s: DL\to vL$ and define
$\alpha=sg_1^b\times\rho_1: U\to vL\times(g_1^b)^{-1}D$

$$\beta = s\times\rho_2\colon St(D, K')\to vL\times[D]\,.$$

We can check that $\alpha$ and $\beta$ are both piecewise linear embeddings onto neighborhoods of $(v, y)$, $(v, g_1^b y)$ respectively. Choose ball neighborhoods $N$ of $v$ in $vL$,

$D^{m+d-q}$ of $y$ in $(g_1^b)^{-1}D$ and

$D^d$ of $g_1^b y$ in $D$

such that $N\times D^{m+d-q}$ image of $\alpha$

$N\times D^d$ image of $\beta$, and

$g_1^b D^{m+d-q}\subset D^d$

Hence the following diagram commutes

$$\begin{array}{ccccc}
N\times D^{m+d-q} & \xrightarrow{1\times(g_1^b\mid D^{m+d-q})} & N\times D^d & \xrightarrow{p} & N \\
\Big\downarrow{\scriptstyle\alpha^{-1}} & & \Big\downarrow{\scriptstyle\beta^{-1}} & & \Big\downarrow{\scriptstyle\subset} \\
g^{-1}DL\cap St(x_1, J') & \xrightarrow{g_1^b} & DL & \xrightarrow{s} & vL
\end{array}$$

This complete the proof for $g_1^b$.

Next we show $g_2^b$ transimplicial to $K_1'$ at any point of $\mathring{X}_2$. The main part of the proof is equally for $g_1^b$ and we shall not repeat all, but give the difference part of $g_2^b$. Since $g_2^b$ is simplicial with respect to $J'$ and $v_2 * \dot{W}$ on $St(\hat{A}_2, J')$, if $C_2$ is the simplex of $J'$ such that $y_2\in\mathring{C}_2$ where $y_2$ is any point of $\mathring{X}_2$, $g_2^b C_2$ is the linear join of $v_2$ and some simplex of $\mathring{W}$. Let $D_2$ be the simplex of $K_1'$ such that $g_2^b y_2 \in \mathring{D}_2$. Since $v_2$ is in general position with respect to the vertices of $W_1$, $g_2^b C_2$ and $D_2$ span $E^q$. Hence $(g_2^b)^{-1}(g_2^b C_2 \cap D_2) \cap C_2$ is the $(c+d-q)$-linear convex cell $E_2$. We define $F_2$ the $(q-d)$-cell through $y_2$ that is perpendicular to $E_2$. Let $C_2^*$ be a simplex of $J'$ having $C_2$ as a face. In the same way as $g_1^b$, $g_2^b$ is transimplicial to $K_1'$ at any point of $\mathring{X}_2$. We can prove the lemma similarly for $g_i^b$ $(3 \leq i \leq n-1)$.

**Lemma 2.** *If $f$ is a locally flat immersion, $g_i^b$ $(i=1, 2, \cdots, n-1)$ are also a locally flat immersion.*

Proof. By the construction of $g_i^b$, it is sufficiently to show that $g_1^b$ is locally flat. Since $g_1^b$ is different from $f$ only on $\mathring{S}t(\hat{B}_1, J')$ and since $f$ is locally flat, $(Lk(f\hat{B}_1, K'), f(Lk(\hat{B}_1, J'))$ is the unknotted $(q-1, m-1)$-sphere pair. Hence

$$\begin{aligned}&(St(g_1^b B, K_1), g_1^b(St(B_1, J)))\\&\quad = v_1 * (Lk(fB_1, K), f(Lk(B_1, J)))\end{aligned}$$

is an unknotted $(q, m)$-ball pair.

Therefore $g_1^b$ is locally flat at $x \in \mathring{S}t(\hat{B}_1, J')$. Obviously $g_1^b$ is locally flat at the point $x \in M - St(\hat{B}_1, J')$. Let $u$ be a vertex in $Lk(\hat{B}_1, J')$. Since $f$ is locally flat, $(Lk(f(u), K'), f(Lk(u, J'))$ is an unknotted sphere pair.

Case 1. If $u$ is a vertex of $J$ in $Lk(\hat{B}_1, J')$,

$$(Lk(f(u), K'), f(Lk(u, J')) \cong \partial(\square, f\triangledown)$$

where $\triangledown$ is the dual cell of $u$ in $J$ and where $\square$ is the dual cell of $f(u)$ in $K$. And $\partial(\square, f\triangledown) \cap St(f(\hat{B}_1), K') = (\widetilde{\square}, f(\widetilde{\triangledown}))$ where $\widetilde{\triangledown} = St(\hat{B}_1, \partial\triangledown)$ and where $\widetilde{\square} = St(f(\hat{B}_1), \partial\square)$. Since $f$ is locally flat, $(\widetilde{\square}, f(\widetilde{\triangledown}))$ is the unknotted $(q-1, m-1)$-ball pair. Hence $\partial(\square, f\triangledown)$-Int$(\widetilde{\square}, f(\widetilde{\triangledown}))$ is the unknotted ball pair $(D, C)$ by [5. Cor. 8] and

$$(Lk(g_1^b(u), K'), g_1^b(Lk(u, J')) \cong (D, C) \cup v_1 * \partial(D, C)$$

is the unknotted sphere pair. Therefore $g_1^b$ is locally flat at $u$.

Case 2. If $u$ is a vertex of $J'$ not $J$ in $Lk(\hat{B}_1, J')$,

$$\begin{aligned}&(Lk(f(u), K'), f(Lk(u, J'))) \cap St(f\hat{B}_1, K')\\&\quad = (St(f\hat{B}_1), Lk(f(u)), f(St(\hat{B}_1, Lk(u, J')))) .\end{aligned}$$

And since $f$ is locally flat, it is the unknotted $(q-1, m-1)$-ball pair $(E, F)$. Hence $(Lk(f(u), K'), f(Lk(u, J')))$-Int$(E, F)$ is the unknotted $(q-1, m-1)$-ball pair $(D, C)$. Therefore $g_1^b$ is locally flat at $u$ as Case 1.

**Lemma 3.** *If a locally flat immersion $f: M \to Q$ is transimplicial to $K$ at*

*$x_j \in S_f$ where $f^{-1}f(x_j)=\{x_1,\cdots, x_j,\cdots, x_n\}$ and $|K|=Q$ and if $f|M-\mathring{S}t(x_j, J)$ is simplicial with respect to $J$ and $K$ where $|J|=M$, then $f$ is transversal to $f(M)$ at $x_j$.*

Proof. Since $f$ is a locally flat immersion, $g_i^b$ is also by Lemma 2. Let $A$ be the simplex of $K$ satisfying $f(x_j)\in\mathring{A}$ and $L=Lk(A, K)$. Choose a vertex $v$ of $A$, let $s:AL\to vL$ the simplicial map as defined before. Since $f$ is transimplicial to $K$ at $x_j$, $f|St(x_j, J)$ is in general position with respect to $K$ and there exists a neighborhood $N$ of $v$ in $vL$ such that the following diagram commutes (see Remark 5),

$$(0)\qquad \begin{array}{ccccc} N\times D^{m+a-q} & \xrightarrow{1\times k} & N\times D^a & \xrightarrow{p} & N \\ \alpha\downarrow & & \beta\downarrow & & \downarrow\subset \\ f^{-1}AL\cap St(x_j, J) & \xrightarrow{f} & AL & \xrightarrow{s} & vL \end{array}$$

where $k:D^{m+a-q}\to D^a$ is a proper embedding. Furthermore as $f$ is locally flat, the pair $(N\times D^a,\ N\times k(D^{m+a-q}))$ is unknotted $(q, m)$-ball pair. Hence we can write the diagram (0) as follows,

$$\begin{array}{ccc} N\times D^{m+a-q},\ v\times 0 & \xrightarrow{1\times 1\times 0} & N\times D^{m+a-q}\times D^{q-m},\ v\times 0\times 0 \\ \alpha\downarrow & & \downarrow\beta \\ f^{-1}AL\cap St(x_j, J),\ x_j & \xrightarrow{f} & AL,\ f(x_j) \end{array}$$

On the other hand $f|M-St(x_j, J)$ is simplicial, there exist simplexes $B_i$ of $J$ such that $\mathring{B}_i\ni x_i$ and $fB_i=A$ $(i=1, 2,\cdots, \overset{*}{j},\cdots, n)$. Let $L_1=f(Lk(B_i, J))$ and $N_1$ be a neighborhood of $v$ in $vL_1$, then $N_1=N\cap vL_1$. And since $f$ is locally flat, $(N, N_1)$ is an unknotted $(q-a,\ m-a)$-ball pair. Thus there exists an unknotting homeomorphism

$$h:D^{q-m}\times D^{m-a},\ 0\times D^{m-a},\ 0\times 0\to N,\ N_1,\ v\,.$$

Hence the following diagram commutes.

$$\begin{array}{ccccc} D^{a-m}\times D^{m-a}\times D^{m+a-q},\ 0\times 0\times 0 & \xrightarrow{1\times 1\times 1\times 0} & D^{q-m}\times D^{m-a}\times D^{m+a-q}\times D^{q-m},\ 0\times 0\times 0\times 0 & \xleftarrow{0\times 1\times 1\times 1} & D^{m-a}\times D^{m+a-q}\times D^{q-m},\ 0\times 0\times 0 \\ h\times 1\downarrow & & \downarrow h\times 1\times 1 & & (h|D^{m-a})\times 1\times 1\downarrow \\ N\times D^{m+a-q},\ v\times 0 & \xrightarrow{1\times 1\times 0} & N\times D^{m+a-q}\times D^{q-m},\ v\times 0\times 0 & \xleftarrow{\subset\times 1\times 1} & N_1\times D^{m+a-q}\times D^{q-m}, v\times 0\times 0 \\ \alpha\downarrow & & \beta\downarrow & & f^{-1}\alpha\downarrow \\ f^{-1}AL\ St(x_j, J),\ x_j & \xrightarrow{f} & AL,\ f(x_j) & \xleftarrow{f} & f^{-1}AL\cap St(x_i, J),\ x_i. \end{array}$$

From the top and the bottom of the diagram, $f$ is transversal to $f(M)$ at $x_j$.

**Lemma 4.** *If* $y_k \in Sg_k^b - Sg_{k-1}^b$ *and* $(g_k^b)^{-1}(g_k^b)(y_k)=\{\cdots y_k \cdots y_m\}$ *where* $y_i \in \mathring{S}t(\hat{B}_i, J')$ *and* $\mathring{B}_i \ni x_i$, *then* $g_k^b$ *is transversal to* $g_k^b M$ *at* $y_i$ $(1 \leq i \leq k)$.

Proof. We shall prove the lemma by induction on $k$. Since $g_k^b$ is transimplical to $K'_{k-1}$ at $y_k$ by Lemma 1, $g_k^b$ is transversal to $g_k^b M$ at $y_k$ by Lemma 3. We suppose $g_k^b$ transversal to $g_k^b M$ at $y_j$ $(1 \leq j \leq k)$. Next we shall show $g_k^b$ transversal to $g_k^b M$ at $y_l$ where $l'$ is the largest number satisfying $l'<l$. Since $g_k^b | \mathring{S}t(\hat{B}_l, J')=g_{l'}^b | St(\hat{B}_{l'}, J')$ and $g_k^b$ is transimplicial to $K'_i$ at $y_{l'}$ $(i<l')$, $g_k^b | \mathring{S}t(\hat{B}_{l'}, J')$ is transversal to $g_k^b(\mathring{S}t(\hat{B}_i, J'))$ at $y_{l'}$, $(i<l)$ by Lemma 3. And $g_k^b$ is transversal to $g_k^b(\mathring{S}t(\hat{B}_j, J'))$ $(1 \leq j \leq k)$ at $y_{l'}$ by inductive hypothesis. For $j<k$, since $g_k^b$ does not effect on $\mathring{S}t(\hat{B}_j, J')$, $g_k^b(\mathring{S}t(\hat{B}_j, J'))$ is embedded as a subcomplex of $K_i'$ for any $i$. Hence $g_k^b$ is transversal to $g_k^b(St(\hat{B}_j, J'))$ $(j<k)$ at $y_{l'}$ because $g_k^b$ is transimplicial to $K_i'$ $(i<l')$ at $y_l$. Therefore $g_k^b$ is transversal to $g_k^b M$ at $y_{l'}$.

We denote the last map like $g_{n-1}^b$ for all other point $x \in S_f$ such that $x \in B^b$ as $g^b$.

Proof of Theorem.

We order $k$-simplexes of $J$ containing the point of $S_f$ such $\Delta_{ij}^k$ that if $f(\Delta_{ij}^k)=f(\Delta_{mn}^k)$, $i=m$ and that the set of $k$-simplexes of $f^{-1}f(\Delta_{ij}^k)$ is properly numbered for the second suffix $j$. We number point contained $|\Delta_{ij}^k| \cap S_f$ as $x_{ij}^k$.

We perform the shift $g_{ij}^b$ for every $x_{ij}^b \in S_f$ such that $x_{ij}^b \in B_{ij}^b$ and in order to decreasing dimension of $b$. Since $J$ is a finite dimensional finite complex, the time of shifts is finite.

First perform on $m$-shift $g_{ij}^m$ for $x_{ij}^m \in S_f$ $(i=1, 2,\cdots, l; j=1, 2,\cdots, q)$ where $l$ is the number of $m$-simplexes such that $|\Delta_{ij}^m| \cap S_f \neq \phi$, $|\Delta_{st}^m| \cap S_f \neq \phi$ and $f(\mathring{\Delta}_{ij}^m) \cap f(\mathring{\Delta}_{st}^m)=\phi$ and where $f^{-1}f(\Delta_{ij}^k)=\{\Delta_{i1}^k,\cdots, \Delta_{iq}^k\}$. We denote $g_{lq}^m=g^m$ then $g^m | J-J^{m-1}$ is non-singular. Next perform on $(m-1)$-shift $g_{ij}^{m-1}$ for $x_{ij}^m \in S_f$ $(i=1, 2, \cdots, p; j=1, 2,\cdots, r)$ where $p$ and $r$ are same as above. We denote $g_{pr}^{m-1}=g^{m-1}$. Then $x_{ij}^{m-1} \notin S_{g^{m-1}}$ for $i=1, 2,\cdots, p; j=1, 2,\cdots, r$. For any point $y \in S_{gm-1}-S_f$ such that $y \in \mathring{S}t(\hat{B}_{ij}^{m-1}, J')$, $g^{m-1}$ is transversal to $g^{m-1}M$ at $g^{m-1}(y)$ by Lemma 4.

Furthermore since for any $k$-simplex $\Delta^k \in J'$ whose interior is contained in $\mathring{S}t(\hat{B}^{m-1}, J') \cap \mathring{S}t(\hat{B}^m, J')$, $\Delta^k=\hat{B}^{m-1} * \tilde{\Delta}^{k-1}$ where $\tilde{\Delta}^{k-1}$ is the opposite face of $\hat{B}^{m-1}$ in $\Delta^k$ and since $g^{m-1}=g^m$ at $M-\bigcup_{B \in J} \mathring{S}t(\hat{B}^{m-1}, J')$, $g^{m-1}(\Delta^k)=v * g^m(\tilde{\Delta}^{k-1})$ where $v$ is a point of $\mathring{S}t(f(\hat{B}^{m-1}), K')$ satisfying the conditions 1) $v \notin f(M)$ and 2) $v$ is in general position with respect to the vertices of the complex covered by $St(f(\hat{B}^{m-1}), K')$. Hence it is obviously that if $g^m$ is transversal to $g^m M$ at $St(\hat{B}^m, J') \cap St(\hat{B}^m, J')$, $g^{m-1}$ is also transversal to $g^{m-1}M$ at there. Therefore

$g^{m-1}$ is transversal to $g^{m-1}M$ at every point of $|J-J^{m-2}|$. In this way any singular point of $S_f$ become non-singular point one time, and if $y\in \mathring{S}t(\hat{B}^b, J')$ become again a singular point by $g^b$, then, $g^b$ is transversal to $g^bM$ at $y$ by Lemma 4.

Furthermore for any point $y\in \mathring{S}t(\hat{B}^b, J')\cap \mathring{S}t(\hat{B}^{b+1}, J')$ $g^b$ is transversal to $g^bM$ at $y$ as above. Hence $g^b$ is transversal to $g^bM$ at every point of $|J-J^{b-1}|$. Therefore, in final, $g^0$ is transversal to $g^0M$ at all points of $M$ particularly of $S_{g^0}$. Then $g^0$ is the required $g$.

And if $f$ is in general position itself, $g$ is obviously in general position itself by construction. Complete the proof.

Proof of Corollary 1.

By the hypothesis if $x\in S_g$, $g^{-1}g(x)=x\cup y$ and the following diagram commutes

$$\begin{array}{ccccc} D^{2m-q}\times D^{q-m},\ 0\times 0 & \xrightarrow{k_x} & D^{2m-q}\times D^{q-m}\times D^{q-m},\ 0\times 0\times 0 & \xleftarrow{k_y} & D^{2m-q}\times D^{q-m},\ 0\times 0 \\ \varphi_x \downarrow & & \downarrow \psi & & \downarrow \varphi_y \\ M,\ x & \xrightarrow{\quad g \quad} & Q,\ g(x) & \xleftarrow{\quad g \quad} & M,\ y \end{array}$$

where $k_x=1\times 1\times 0$, $k_y=1\times 0\times 1$ and where $\varphi_x$, $\varphi_y$, $\psi$ are homeomorphism onto some neighborhoods of $x$, $y$, $g(x)$ respectively.

Since $St(x, S_g)\cong St(k_x(0\times 0), k_x(D^{2m-q}\times D^{q-m})\cap k_y(D^{2m-q}\times D^{q-m}))= St(0\times 0\times 0, D^{2m-q})\cong D^{2m-q}$, $S_g$ is a closed $(2m-q)$-manifold. And in the left side of the above diagram $(St(x, M), St(x, S_g))\cong(D^{2m-q}\times D^{q-m}, D^{2m-q})$ since $\varphi_x$: $(D^{2m-q}\times D^{q-m}, D^{2m-q}\times 0, 0\times 0)\to(M, S_g, x)$.

Hence $S_g$ is a closed locally flat $(2m-q)$-submanifold of $M$.

Proof of Cor. 3.

Let $J_0$ and $K_0$ be the subdivisions of $M$, $Q$ and $f$ be in general position with respect to $J_0$, $K_0$. Let $J$, $K$ be the subdivisions of $J_0$, $K_0$ such that $f: J\to K$ is simplicial.

Then $S_f$ consists only of the vertices of $J$ and the local knotness rises only on the vertices of $J$. Hence if $y_k\in S_{g^k}-S_{g^{k-1}}$, $y_k\notin J^0$ where $y_k$, $g_k$ are same as Lemma 3.

Then $g_k$ is locally flat at $y_k$ and $g_k$ is transversal to $g_kM$ at $y_k$ as the proof of theorem.

KOBE UNIVERSITY

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