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EXACT SEQUENCES INVOLVING COBORDISM GROUPS OF IMMERSIONS

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Introduction

The central problems of differential topology are the classification of differentiable manifolds and the classification of mappings between differentiable manifolds. Thom [6] has introduced the notion of cobordism to classify the differentiable manifolds. The bordism theory of Atiyah [1] provides a classification "up to cobordism" of mappings of differentiable manifolds into a fixed differentiable manifold. Watabe [7] and Wells [8] have considered the classification up to cobordism of immersions of differentiable manifolds into Euclidean spaces. These have useful applications but these do not allow the image manifold to vary within cobordism class. On the other hand, Stong [5] introduced a classification of maps which is "compatible" with the classification of manifolds. The object of this paper is to consider such a compatible classification of immersions.

Two immersions $f: M\to N$ and $f': M'\to N'$ will be said cobordant if there is an immersion $F: V\to W$ such that ∂V is the disjoint union of M and M', ∂W is the disjoint union of N and N', $F|M=f$, $F|M'=f'$ and F is transverse regular over ∂W. The relation of cobordism turns out to be an equivalence relation and the immersions of closed m-manifolds in closed $(m+k)$-manifolds form an abelian group $\boldsymbol{I}(m, k)$ modulo cobordism. In the above definition, if the term "immersion" is replaced by "embedding" and "generic immersion", one may define a cobordism group of embeddings $\boldsymbol{E}(m, k)$ and a cobordism group of generic immersions $\boldsymbol{G}(m, k)$ respectively. The group $\boldsymbol{I}(m, k)$ is complicated, so we will consider the group $\boldsymbol{G}(m, k)$ instead of $\boldsymbol{I}(m, k)$ since $\boldsymbol{G}(m, k)$ is isomorphic to $\boldsymbol{I}(m, k)$ if $2k>m+1$ from the theorem of Haefliger [4].

Next, in section 3, we will introduce a cobordism group $\boldsymbol{B}(m, k)$ of k-plane bundles over m-manifolds with involution. Then our main result is the existence of the following two exact sequences

$$\boldsymbol{A}: \cdots\to \boldsymbol{E}(m, k) \xrightarrow{\alpha_*} \boldsymbol{G}(m, k) \xrightarrow{\beta_*} \boldsymbol{B}(m-k, k) \xrightarrow{\partial_*} \boldsymbol{E}(m-1, k) \xrightarrow{\alpha_*}\cdots,$$

$$\boldsymbol{B}: \cdots\to \boldsymbol{B}(n, k) \xrightarrow{\rho_*} \mathfrak{N}_n(BO(k)\times BO(k)) \xrightarrow{\varphi_*} \boldsymbol{B}(n, k) \xrightarrow{\psi_*} \boldsymbol{B}(n-1, k) \xrightarrow{\rho_*}\cdots.$$

As a corollary of these two exact sequences, one may show that $\boldsymbol{G}(m, k)$ and $\boldsymbol{B}(m, k)$ are finitely generated.

By making use of the projective bundles, we will prove the existence of a homomorphism $P_*: \boldsymbol{B}(m-k, k) \to \boldsymbol{G}(m, k)$ such that $\beta_* P_* =$identity. Therefore $\boldsymbol{G}(m, k)$ is isomorphic to the direct sum $\boldsymbol{E}(m, k) \oplus \boldsymbol{B}(m-k, k)$.

In the last section we will consider the oriented cobordism groups of immersions and one may have exact sequences analogous to $\boldsymbol{A}$ and $\boldsymbol{B}$. But in the oriented case we could not find such a homomorphism as P_*.

1. Cobordism of immersions

An immersion of dimension (m, k) is a triple (f, M, N) consisting of two closed differentiable manifolds M and N of dimensions m and $m+k$ respectively and an immersion $f: M \to N$. We identify (f, M, N) with (f', M', N') if and only if there are diffeomorphisms $\varphi: M \to M'$ and $\psi: N \to N'$ for which $\psi f = f' \varphi$.

Two immersions (f, M, N) and (f', M', N') of dimension (m, k) will be said to be cobordant if there exists a triple (F, V, W) where:

(1) V and W are compact differentiable manifolds of dimensions $m+1$ and $m+k+1$ respectively, with $\partial V = M \cup M'$, $\partial W = N \cup N'$ where the symbol $\cup$ denotes disjoint union, and

(2) $F: V \to W$ is an immersion transverse regular over ∂W, whose restriction to M is f and whose restriction to M' is f'.

If (F, V, W) defines a cobordism of (f, M, N) and (f', M', N'), write $\partial(F, V, W) = (f, M, N) + (f', M', N')$. The symbol $+$ denotes disjoint union. It is immediate that this relation is reflexive and symmetric. It is also transitive, since F is transverse regular over ∂W.

The set of equivalence classes under this relation of immersions of dimension (m, k) will be denoted $\boldsymbol{I}(m, k)$. As usual, an abelian group structure is imposed on $\boldsymbol{I}(m, k)$ by disjoint union, which may be considered as the cobordism group of immersions of dimension (m, k). Given (f, M, N), one has

$$\partial(f \times i, M \times I, N \times I) = (f, M, N) + (f, M, N)$$

where i is the identity map on $I = [0, 1]$, showing that every element of $\boldsymbol{I}(m, k)$ is its own inverse.

One may define a product $\boldsymbol{I}(m, k) \times \mathfrak{N}_n \to \boldsymbol{I}(m+n, k)$ by sending $([f, M, N], [L])$ into the class $[f \times id, M \times L, N \times L]$. This makes the direct sum $\sum_m \boldsymbol{I}(m, k)$ into a graded right $\mathfrak{N}_*$-module for any $k \geqq 0$ where $\mathfrak{N}_* = \sum_n \mathfrak{N}_n$ is the unoriented cobordism ring.

In the above definition, if the term "immersion" is replaced by "embedding" and "generic immersion", one may define the cobordism group of embeddings

$\boldsymbol{E}(m, k)$ and the cobordism group of generic immersions $\boldsymbol{G}(m, k)$, of dimension (m, k) respectively.

REMARK 1.1. Let V and W be compact differentiable manifolds and f: $V \to W$ be an immersion for which $f(\partial V) \subset \partial W$ and f is transverse regular over ∂W. Then f will be said to be generic, if

(a) $y=f(x)=f(x')$ and $x \neq x'$, then the images of tangent spaces of V at x and x' by df generate the tangent space of W at y, and

(b) f has no triple point (cf. [4], §2.5).

REMARK 1.2. From the theorem of Haefliger [4, Th. 2.5], if $2k > m+1$, then $\boldsymbol{G}(m, k)$ is isomorphic to $\boldsymbol{I}(m, k)$.

REMARK 1.3. $\boldsymbol{E}(m, k)$ is isomorphic to $\mathfrak{N}_{m+k}(MO(k))$ (cf. [5], p. 249, Remark (d)).

2. Bundles associated with generic immersion

Let f: $V \to W$ be a generic immersion, where V and W are compact differentiable manifolds of dimensions m and $m+k$ respectively. Then the set

$$D_f = \{x \in V \mid \exists x' \in V,\ x \neq x',\ f(x) = f(x')\}$$

is a compact submanifold of V of dimension $m-k$ for which $\partial(D_f) \subset \partial V$ and the inclusion $D_f \subset V$ is transversal over ∂V, and the set $\Delta_f = f(D_f)$ is a submanifold of W for which $\partial(\Delta_f) \subset \partial W$ and the inclusion $\Delta_f \subset W$ is transversal over ∂W (cf. [4], §2.5).

Moreover D_f has a canonical fixed point free differentiable involution $T = T_f$ defined by $T(x) \neq x$, $f(T(x)) = f(x)$, and the orbit manifold D_f/T is diffeomorphic to Δ_f.

Let $\nu = \nu_f$ and $\hat{\nu} = \hat{\nu}_f$ be the normal bundles of the embeddings $D_f \subset V$ and $\Delta_f \subset W$ respectively, and let $T^*\nu$ be the induced bundle of ν by the involution T. Since $(T^*\nu)_x = \nu_{T(x)}$, one may have a bundle map:

$$\begin{array}{ccc} \nu \oplus T^*\nu & \xrightarrow{\hat{T}} & \nu \oplus T^*\nu \\ \downarrow & & \downarrow \\ D_f & \xrightarrow{T} & D_f \end{array}$$

where $\hat{T}(u, v) = (v, u)$ and η_x is the fiber over x of bundle η. On the other hand by the condition (a) of Remark 1.1, one may have a bundle map:

$$\begin{array}{ccc} \nu \oplus T^*\nu & \xrightarrow{\hat{f}} & \hat{\nu} \\ \downarrow & & \downarrow \\ D_f & \xrightarrow{f} & \Delta_f \end{array}$$

such that $\hat{f}\hat{T}=\hat{f}$. Therefore the bundle $\hat{\nu}$ over Δ_f may be identified with the bundle $T_*\nu=(\nu\oplus T^*\nu)/\hat{T}$ over D_f/T. In the next section, we will consider the triple (D_f, T_f, ν_f).

3. Cobordism of bundles over manifolds with involution

The basic object in this section is a triple (W, T, ξ) where T is a fixed point free differentiable involution on a compact differentiable m-manifold W and ξ is a differentiable k-plane bundle over W.

We identify (W, T, ξ) with (W', T', ξ') if and only if there exists a bundle equivalence:

$$\begin{array}{ccc} \xi & \longrightarrow & \xi' \\ \downarrow & & \downarrow \\ W & \xrightarrow{\varphi} & W' \end{array}$$

for which φ is an equivariant diffeomorphism (i.e. $\varphi T=T'\varphi$).

A boundary operator may be defined as

$$\partial(W, T, \xi)=(\partial W, T\,|\,\partial W, \xi\,|\,\partial W)\,.$$

The cobordism group $\boldsymbol{B}(m, k)$ of k-plane bundles over m-manifolds with involution may be now defined. If M_1 and M_2 are closed m-manifolds then (M_1, T_1, ξ_1) is cobordant to (M_2, T_2, ξ_2) if and only if there is a triple (W, T, ξ) for which $\partial(W, T, \xi)=(M_1, T_1, \xi_1)+(M_2, T_2, \xi_2)$. The symbol $+$ denotes disjoint union. It is immediate that this relation is reflexive and symmetric. It is also transitive by the existence of the equivariant collared neighborhood (cf. [3], Th. 21.2). Denote a cobordism class by $[M, T, \xi]$ and the set of all such cobordism classes by $\boldsymbol{B}(m, k)$. As usual an abelian group structure is imposed on $\boldsymbol{B}(m, k)$ by disjoint union. And every element is its own inverse.

One may define a product $\boldsymbol{B}(m, k)\times\mathfrak{N}_n\to\boldsymbol{B}(m+n, k)$ by sending $([M, T, \xi], [N])$ into the class $[M\times N, T\times id, \xi\times 0]$ where 0 is the 0–plane bundle over N. This makes the direct sum $\sum_m \boldsymbol{B}(m, k)$ into a graded right $\mathfrak{N}_*$-module for any $k\geqq 0$.

REMARK 3.1. $\boldsymbol{B}(m, 0)$ is isomorphic to $\mathfrak{N}_m(Z_2)$ which is the bordism group of fixed point free involutions (cf. [3]). For any $k\geqq 0$, $\boldsymbol{B}(0, k)$ is isomorphic to $\mathfrak{N}_0(Z_2)\simeq Z_2$.

Let $c\in\boldsymbol{B}(m, k)$ be represented by a triple (M, T, ξ). One may have a bundle $T_*\xi$ over M/T and a projection π: $\xi\oplus T^*\xi\to T_*\xi$, similarly defined as in section 2. And there is a bundle monomorphism h: $\xi\to\xi\oplus T^*\xi$ defined by $h(u)=(u, 0)$.

If a Riemannian metric is given on ξ, then one may have a generic immersion

$$h_\xi\colon D(\xi) \to D(T_*\xi)$$

where $D(\eta)$ is the total space of the disk bundle associated with η and h_ξ is the restirction of πh. Moreover the restriction of h_ξ on $\partial D(\xi)$ is an embedding, and the class of $\partial(h_\xi, D(\xi), D(T_*\xi))$ in $\boldsymbol{E}(m+k-1, k)$ is independent of the choice of a representation (M, T, ξ) and the choice of a Riemannian metric on ξ.

4. Exact sequence A

In the above sections the cobordism groups $\boldsymbol{E}(m, k)$, $\boldsymbol{G}(m, k)$ and $\boldsymbol{B}(m, k)$ are defined. Now we define homomorphisms:

$$\begin{aligned} \alpha_*&\colon\ \boldsymbol{E}(m, k) \to \boldsymbol{G}(m, k)\\ \beta_*&\colon\ \boldsymbol{G}(m, k) \to \boldsymbol{B}(m-k, k)\\ \partial_*&\colon\ \boldsymbol{B}(m, k) \to \boldsymbol{E}(m+k-1, k)\,. \end{aligned}$$

The main result of this paper will be the existence of exact sequence involving these homomorphisms.

(4.1) Let $a\in\boldsymbol{E}(m, k)$ be represented by an embedding $f\colon M\to N$, then f is also a generic immersion and $\alpha_*(a)$ is represented by f.

(4.2) Let $b\in\boldsymbol{G}(m, k)$ be represented by a generic immersion $g\colon M\to N$, then $\beta_*(b)$ is represented by the triple (D_g, T_g, ν_g) defined in section 2.

(4.3) Let $c\in\boldsymbol{B}(m, k)$ be represented by a triple (M, T, ξ), then $\partial_*(c)$ is represented by the embedding $\partial(h_\xi, D(\xi), D(T_*\xi))$ defined in section 3.

Then these are the well-defined homomorphisms compatible with $\mathfrak{N}_*$-module structures and we can state the main result.

Theorem A. *For any $k\geqq 0$ the following sequence is exact*:

$$\cdots\xrightarrow{\partial_*}\boldsymbol{E}(m, k)\xrightarrow{\alpha_*}\boldsymbol{G}(m, k)\xrightarrow{\beta_*}\boldsymbol{B}(m-k, k)\xrightarrow{\partial_*}\boldsymbol{E}(m-1, k)\xrightarrow{\alpha_*}\cdots.$$

Theorem A′. *For any $k\geqq 0$ there exists an $\mathfrak{N}_*$-module homomorphism P_*: $\sum_m \boldsymbol{B}(m, k)\to\sum_m \boldsymbol{G}(m, k)$ such that $\beta_* P_*=identity$.*

Proof of Theorem A′. Let $[M, T, \xi]$ be an element of $\boldsymbol{B}(m, k)$. Let $P(\xi\oplus\theta^1)$ be the associated projective bundle where θ^1 is the trivial line bundle, then the total space $E(\xi)$ of the bundle ξ is canonically embedded in $P(\xi\oplus\theta^1)$ as an open set and its complement is $P(\xi)$. Therefore one may have the following commutative diagram:

$$\begin{array}{ccccc} M & \longrightarrow & E(\xi) & \longrightarrow & P(\xi\oplus\theta^1)\\ \Big\downarrow{\scriptstyle \bar\pi} & & \Big\downarrow{\scriptstyle \pi h} & & \Big\downarrow{\scriptstyle P(\pi h\oplus 1)}\\ M/T & \longrightarrow & E(T_*\xi) & \longrightarrow & P(T_*\xi\oplus\theta^1) \end{array}$$

where $T_*\xi$, π and h are defined in section 3, $\bar{\pi}$ is the orbit map, and the horizontal lines are embeddings. Moreover $P(\pi h\oplus 1)$ is a generic immersion such that $D_{P(\pi h\oplus 1)}=M$ and ξ is the normal bundle of the embedding $M\subset P(\xi\oplus\theta^1)$. Thus the assignment of $[M, T, \xi]$ to $[P(\pi h\oplus 1), P(\xi\oplus\theta^1), P(T_*\xi\oplus\theta^1)]$ is a desired homomorphism. q.e.d.

Corollary 4.4. *For any m, k $\boldsymbol{G}(m, k)$ is isomorphic to the direct sum $\boldsymbol{E}(m, k)\oplus\boldsymbol{B}(m-k, k)$. If $m<k$, then $\boldsymbol{G}(m, k)$ is isomorphic to $\boldsymbol{E}(m, k)$.*

Corollary 4.5. *For any $m\geqq 0$ $\boldsymbol{G}(m, 0)$ is isomorphic to the direct sum $\mathfrak{N}_m\oplus\mathfrak{N}_m\oplus\mathfrak{N}_m(Z_2)$.*

Proof. $\boldsymbol{E}(m, 0)\cong\mathfrak{N}_m(S^0)\cong\mathfrak{N}_m\oplus\mathfrak{N}_m$ (Remark 1.3) and $\boldsymbol{B}(m, 0)\cong\mathfrak{N}_m(Z_2)$ (Remark 3.1). q.e.d.

5. Proof of Theorem A

It clearly suffices to prove the following statements:

(a) $\beta_*\alpha_*=0$ (b) $\alpha_*\partial_*=0$ (c) $\partial_*\beta_*=0$

(d) $\ker\partial_*\subset\text{image}\,\beta_*$ (e) $\ker\alpha_*\subset\text{image}\,\partial_*$ (f) $\ker\beta_*\subset\text{image}\,\alpha_*$.

Since the set D_f is empty for any embedding f, (a) is trivial. And (b) is trivial by the definition (4.3) of ∂_*.

We prove (c). Let $f\colon M\to N$ be a generic immersion. Given suitable Riemannian metrics on M and N, there are embeddings

$$\varphi\colon D(\nu)\to M\,,\quad \psi\colon D(\hat{\nu})\to N$$

such that $\varphi|D_f$ is the embedding $D_f\subset M$, $\psi|\Delta_f$ is the embedding $\Delta_f\subset N$ and $dfd\varphi=d\psi dh_\nu$ on D_f, where ν and $\hat{\nu}$ are the normal bundles of the embeddings $D_f\subset M$, $\Delta_f\subset N$ respectively, $\hat{\nu}$ is identified with $T_*\nu$ where T is the canonical involution on D_f, and $h_\nu\colon D(\nu)\to D(T_*\nu)$ is defined in section 3. There exists a regular homotopy f_t such that $f_0=f$, $f_t|D_f=f|D_f$, $D_{f_t}=D_f$, and $f_1\varphi=\psi h_\nu$ on D_ε for sufficiently small $\varepsilon>0$ where $D_\varepsilon=D_\varepsilon(\nu)$ is the set of all vectors of ν whose length is smaller than or equal to ε. Now let $V=M-\operatorname{Int}\varphi(D_\varepsilon)$, $W=N-\operatorname{Int}\psi(D_\varepsilon')$ and $g\colon V\to W$ be the restriction of f_1, where $D_\varepsilon'=D_\varepsilon(\hat{\nu})$. Then g is an embedding and $\partial(g, V, W)=\partial(h_\nu|D_\varepsilon, D_\varepsilon, D_\varepsilon')=\partial(h_\nu, D(\nu), D(T_*\nu))$. Thus $\partial_*\beta_*=0$.

Next we prove (d). Let $\partial_*([M, T, \xi])=0$. Then there exists an embedding $g\colon V\to W$ such that

$$\partial(g, V, W)=\partial(h_\xi, D(\xi), D(T_*\xi))\,.$$

Let X be a closed manifold obtained from the disjoint union $V\cup D(\xi)$ by the identification $\partial V=\partial D(\xi)$ and let Y be a closed manifold obtained from the disjoint union $W\cup D(T_*\xi)$ by the identification $\partial W=\partial D(T_*\xi)$. Define $f\colon X\to Y$ by $f=g$ on V and $f=h_\xi$ on $D(\xi)$, then f is a generic immersion and $D_f=M$, $T_f=T$ and the normal bundle ν_f of the embedding $D_f\subset X$ is ξ. Therefore $\beta_*([f, X, Y])=[M, T, \xi]$.

One may prove (e) by similar argument to (c), so we omit the proof.

Finally we prove (f). Let $\beta_*([f, M, N])=0$. Then there exists a triple (V, T, ξ) such that $\partial(V, T, \xi)=(D_f, T_f, \nu_f)$. As the proof of (c), there exists a regular homotopy f_t such that $f_0=f$, $f_t|D_f=f_t|D_f$, $D_{f_t}=D_f$ and $f_1\varphi=\psi h_\nu$ on $D_\varepsilon(\nu_f)$ for some $\varepsilon>0$. One may assume $\varepsilon=1$. Let X be a manifold obtained from the disjoint union of $M\times[0, 1]$ and $D(\xi)$ by identifying $D(\nu_f)\subset D(\xi)$ with $\varphi(D(\nu_f))\times\{1\}$ by φ and straightening the angle at $\partial D(\nu_f)$, and let Y be a manifold obtained from the disjoint union of $N\times[0, 1]$ and $D(T_*\xi)$ by identifying $D(T_*(\xi|\partial V))$ with $\psi(D(\hat{\nu}_f))\times\{1\}$ by ψ. Define $F\colon X\to Y$ by $F(x, t)=f_t(x)$ on $M\times[0, 1]$ and $F=h_\xi$ on $D(\xi)$, then F is a generic immersion and $\partial(F, X, Y)=(f, M, N)+(f', M', N')$ where $f'\colon M'\to N'$ is an embedding. Therefore $[f, M, N]$ is in the image of α_*.

These complete the proof of Theorem A.

6. The Smith homomorphism

Let (M, T, ξ) be a triple where T is a fixed point free differentiable involution on a closed differentiable n-manifold M and ξ is a differentiable k-plane bundle over M. For $N\geqq n$ there exists a difierentiable equivariant map $g\colon (M, T)\to(S^N, A)$ which is transverse regular over $S^{N-1}\subset S^N$ where A is the antipodal map on S^N. Let $V=g^{-1}(S^{N-1})$. The function

$$\psi_*\colon \boldsymbol{B}(n, k)\to\boldsymbol{B}(n-1, k)$$

defined by $\psi_*([M, T, \xi])=[V, T|V, \xi|V]$ is a well-defined homomorphism for $N>n$ independent of N which we call the Smith homomorphism (cf. [3], §26).

Next we consider the bordism group $\mathfrak{N}_n(BO(k)\times BO(k))$ whose element is represented by a triple (M, ξ, η) where ξ and η are differentiable k-plane bundles over a closed differentiable n-manifold M. Let $(\tilde{M}, T, \zeta)$ be defined by $\tilde{M}=M\times\{0\}\cup M\times\{1\}$, $T(m, i)=(m, 1-i)$, $\zeta|M\times\{0\}=\xi\times\{0\}$ and $\zeta|M\times\{1\}=\eta\times\{1\}$. The function

$$\varphi_*\colon \mathfrak{N}_n(BO(k)\times BO(k))\to\boldsymbol{B}(n, k)$$

defined by $\varphi_*([M, \xi, \eta])=[\tilde{M}, T, \zeta]$ is a well-defined homomorphism. The function

$$\rho_*\colon \boldsymbol{B}(n, k)\to\mathfrak{N}_n(BO(k)\times BO(k))$$

defined by $\rho_*([M, T, \xi])=[M, \xi, T^*\xi]$ is also a well-defined homomorphism.

Theorem B. *For any $k\geqq 0$ the following sequence is exact:*

$$\cdots \dashrightarrow \boldsymbol{B}(n, k) \xrightarrow{\rho_*} \mathfrak{N}_n(BO(k)\times BO(k)) \xrightarrow{\varphi_*} \boldsymbol{B}(n, k) \xrightarrow{\psi_*} \boldsymbol{B}(n-1, k) \xrightarrow{\rho_*} \cdots.$$

Corollary 6.1. *For any n, k the cobordism group $\boldsymbol{B}(n, k)$ is finitely generated.*

Proof. $\boldsymbol{B}(0, k)=Z_2$ and $\mathfrak{N}_n(BO(k)\times BO(k))$ is finitely generated. Therefore $\boldsymbol{B}(n, k)$ is finitely generated by induction on n. q.e.d.

Corollary 6.2. *For any n, k the cobordism group of generic immersions $\boldsymbol{G}(n, k)$ is finitely generated.*

Proof. Since $\boldsymbol{E}(n, k)$ and $\boldsymbol{B}(n, k)$ are finitely generated, $\boldsymbol{G}(n, k)$ is also finitely generated from the exact sequence $\boldsymbol{A}$. q.e.d.

REMARK. Let τ: $BO(k)\times BO(k)\rightarrow BO(k)\times BO(k)$ be the map defined by $\tau(a, b)=(b, a)$. Then $\rho_*\varphi_*(x)=x+\tau_*(x)$ for any $x\in\mathfrak{N}_*(BO(k)\times BO(k))$.

7. Proof of Theorem B

It clearly suffices to prove the following statements:

(a) $\psi_*\varphi_*=0$ (b) $\rho_*\psi_*=0$ (c) $\varphi_*\rho_*=0$

(d) $\ker\varphi_*\subset\text{image}\,\rho_*$ (e) $\ker\rho_*\subset\text{image}\,\psi_*$ (f) $\ker\psi_*\subset\text{image}\,\varphi_*$.

(a) is trivial by the definitions of φ_* and ψ_*.

We prove (b). Let (M, T, ξ) be a triple and let $(V, T|V, \xi|V)$ be a triple defined by making use of an equivariant map g: $(M, T)\rightarrow(S^N, A)$ in section 6. Let E_+ and E_- be the upper and the lower hemispheres in S^N respectively such that $S^{N-1}=E_+\cap E_-$ and let $W_\pm=g^{-1}(E_\pm)$. Then

$$\partial(W_+, \xi|W_+, T^*\xi|W_+)=(V, \xi|V, (T|V)^*(\xi|V))$$

and this shows $\rho_*\psi_*=0$.

We prove (c). $\varphi_*\rho_*([M, T, \xi])$ is represented by a triple (M', T', ξ') where $M'=M\times\{0\}\cup M\times\{1\}$, $T'(m, i)=(m, 1-i)$, $\xi'|M\times\{0\}=\xi\times\{0\}$ and $\xi'|M\times\{1\}=T^*\xi\times\{1\}$. Let W be a manifold with the boundary M' obtained from the disjoint union of $M\times[0, 1/2]$ and $M\times[1/2, 1]$ by identifying a point $(m, 1/2)$ in $M\times[0, 1/2]$ with $(T(m), 1/2)$ in $M\times[1/2, 1]$. Then W has a canonical fixed point free involution $\tilde{T}$ defined by $\tilde{T}(m, t)=(m, 1-t)$ whose restriction on M' is T'. Let ζ be a bundle over W defined by the clutching construction from $\xi\times[0, 1/2]$ and $T^*\xi\times[1/2, 1]$ (cf. [2]). Then $\partial(W, \tilde{T}, \zeta)=(M', T', \xi')$ and this shows $\varphi_*\rho_*=0$.

Next we prove (d). Let $\varphi_*([M, \xi, \eta])=0$. Then there exists a triple

(W, T, ζ) such that $\partial W = M\times\{0\}\cup M\times\{1\}$, $(T\,|\,\partial W)\ (m, i)=(m, 1-i)$, $\zeta\,|\,M\times\{0\}=\xi\times\{0\}$ and $\zeta\,|\,M\times\{1\}=\eta\times\{1\}$. For sufficiently large N there exists a differentiable equivariant map $g\colon (W, T)\to(S^N, A)$ which is transverse regular over S^{N-1} and $g(\partial W)$ does not meet with S^{N-1}. Let $V=g^{-1}(S^{N-1})$, then $\rho_*([V, T\,|\,V, \zeta\,|\,V])=[M, \xi, \eta]$. This is similarly proved as (b), so we omit the proof.

We prove (e). Let $\rho_*([M, T, \xi])=0$. Then there exists a triple (W, η, η') such that $\partial(W, \eta, \eta')=(M, \xi, T^*\xi)$. Let X be a twisted double of W obtained from the disjoint union of $W\times\{0\}$ and $W\times\{1\}$ by identifying a point $(m, 0)$ in $M\times\{0\}=\partial W\times\{0\}$ with $(T(m), 1)$ in $M\times\{1\}=\partial W\times\{1\}$. Then X is a closed manifold and X has a canonical fixed point free involution $\tilde{T}$ whose restriction on $M=M\times\{0\}$ is the involution T. Let ζ be a bundle over X defined by the clutching construction from $\eta\times\{0\}$ and $\eta'\times\{1\}$. Then $\psi_*([X, \tilde{T}, \zeta]) =[M, T, \xi]$ by the definition of ψ_*.

Finally we prove (f). Let $\psi_*([M, T, \xi])=0$. Since the class $\psi_*([M, T, \xi])$ is represented by a triple $(V, T\,|\,V, \xi\,|\,V)$ by making use of a differentiable equivariant map $g\colon(M, T)\to(S^N, A)$ such that g is transverse regular over S^{N-1} and $V=g^{-1}(S^{N-1})$, there exists a triple (W, S, η) such that $\partial(W, S, \eta)=(V, T\,|\,V, \xi\,|\,V)$. On the other hand there exists an equivariant embedding h of $(V\times[-1, 1], (T\,|\,V)\times(-id))$ into (M, T) such that $h(v, 0)=v$ and $h^*\xi\cong(\xi\,|\,V)\times[-1, 1]$. Let X be a manifold obtained from the disjoint union of $M\times[0, 1]$ and $W\times[-1, 1]$ by identifying a point (v, t) in $V\times[-1, 1]=\partial W\times[-1, 1]$ with a point $(h(v, t), 1)$ in $M\times[0, 1]$ and straightening the angle at $V\times\{-1, 1\}$. Then X has a fixed point free involution $\tilde{T}$ such that $\tilde{T}(m, t)=(T(m), t)$ on $M\times[0, 1]$ and $\tilde{T}(w, t)=(S(w), -t)$ on $W\times[-1, 1]$. Moreover one may have a bundle ζ over X by the clutching construction from $\xi\times[0, 1]$ and $\eta\times[-1, 1]$ by the isomorphism $h^*\xi\cong(\xi\,|\,V)\times[-1, 1]=(\eta\,|\,V)\times[-1, 1]$.

Then

$$\partial(X, \tilde{T}, \zeta) = (M, T, \xi)+(M', T', \xi')$$

and $[M, T, \xi]=[M', T', \xi']$ is clearly in the image of φ_* from the above construction.

These complete the proof of Theorem B.

8. Stability of $\boldsymbol{B}(n, k)$

Let θ^1 be a trivial line bundle. Then the function

$$\theta_*\colon \boldsymbol{B}(n, k) \to \boldsymbol{B}(n, k+1)$$

defined by $\theta_*([M, T, \xi])=[M, T, \xi\oplus\theta^1]$ is a well-defined homomorphism.

Theorem C. *If $n\leq k$, then the homomorphism*

$$\theta_*:\ \boldsymbol{B}(n,\ k) \to \boldsymbol{B}(n,\ k+1)$$

is an isomorphism.

Proof. Let i: $BO(k) \to BO(k+1)$ be a canonical inclusion map. Then the following diagram is commutative:

$$\begin{array}{ccccccccc}
\boldsymbol{B}(n, k) & \xrightarrow{\rho_*} & \mathfrak{N}_n(BO(k)\times BO(k)) & \xrightarrow{\varphi_*} & \boldsymbol{B}(n, k) & \xrightarrow{\psi_*} & \boldsymbol{B}(n-1, k) & \xrightarrow{\rho_*} & \\
\downarrow{\scriptstyle \theta_*} & & \downarrow{\scriptstyle (i\times i)_*} & & \downarrow{\scriptstyle \theta_*} & & \downarrow{\scriptstyle \theta_*} & & \\
\boldsymbol{B}(n, k+1) & \xrightarrow{\rho_*} & \mathfrak{N}_n(BO(k+1)\times BO(k+1)) & \xrightarrow{\varphi_*} & \boldsymbol{B}(n, k+1) & \xrightarrow{\psi_*} & \boldsymbol{B}(n-1, k+1) & \xrightarrow{\rho_*} & .
\end{array}$$

Since $(i\times i)_*$: $\mathfrak{N}_n(BO(k)\times BO(k)) \to \mathfrak{N}_n(BO(k+1)\times BO(k+1))$ is an isomorphism for $n \leqq k$ and $\theta_*: \boldsymbol{B}(0,\ k) \to \boldsymbol{B}(0,\ k+1)$ is an isomorphism for any $k \geqq 0$, $\theta_*: \boldsymbol{B}(n,\ k) \to \boldsymbol{B}(n,\ k+1)$ is also an isomorphism for $n \leqq k$ by induction on n. q.e.d.

Theorem D. *$\boldsymbol{B}(n,\ k)$ contains a direct summand isomorphic to $\mathfrak{N}_n(Z_2)$.*

Proof. Let $\pi: \boldsymbol{B}(n,\ k) \to \mathfrak{N}_n(Z_2)$ be a homomorphism defined by $\pi([M,\ T,\ \xi])=[M,\ T]$, and let ι: $\mathfrak{N}_n(Z_2) \to \boldsymbol{B}(n,\ k)$ be a homomorphism defined by $\iota([M,\ T])=[M,\ T,\ \theta^k]$ where θ^k is a trivial k-plane bundle. Then $\pi\iota=$ identity. Therefore $\boldsymbol{B}(n,\ k)$ contains a direct summand isomorphic to $\mathfrak{N}_n(Z_2)$. q.e.d.

9. A direct summand of $\boldsymbol{I}(m,\ k)$

Let s be a point of k-sphere S^k. Let M, N be closed differentiable manifolds of dimensions m, $m+k$ respectively, and N' be the disjoint union $M\times S^k \cup N$. Define a function

$$f:\ M \to N'$$

by $f(m)=(m,\ s)$, then f is an embedding. The function

$$\iota:\ \mathfrak{N}_m \oplus \mathfrak{N}_{m+k} \to \boldsymbol{E}(m,\ k)$$

defined by $\iota([M],\ [N])=[f,\ M,\ N']$ is a well-defined homomorphism and $\boldsymbol{E}(m,\ k)$ may be replaced by $\boldsymbol{G}(m,\ k)$ and $\boldsymbol{I}(m,\ k)$.

Theorem E. *$\boldsymbol{E}(m, k)$, $\boldsymbol{G}(m, k)$ and $\boldsymbol{I}(m, k)$ contain a direct summand isomorphic to the direct sum $\mathfrak{N}_m \oplus \mathfrak{N}_{m+k}$ respectively for any $k \geqq 0$.*

Proof. Let π: $\boldsymbol{E}(m,\ k) \to \mathfrak{N}_m \oplus \mathfrak{N}_{m+k}$ be a homomorphism defined by $\pi([f,\ M,\ N])=([M],\ [N])$, then $\pi\iota=$ identity. Therefore $\boldsymbol{E}(m,\ k)$ contains a direct summand isomorphic to $\mathfrak{N}_m \oplus \mathfrak{N}_{m+k}$. Similarly $\boldsymbol{G}(m, k)$ and $\boldsymbol{I}(m, k)$ contain a direct summand isomorphic to $\mathfrak{N}_m \oplus \mathfrak{N}_{m+k}$. q.e.d.

10. Oriented cobordism groups of immersions

Let $f: M^n \to N^{n+k}$ be a generic immersion where M and N are oriented closed manifolds. We will use the notations $D=D_f$, $\Delta=\Delta_f$, $\nu=\nu_f$, $\hat{\nu}=\hat{\nu}_f$ and $T: D \to D$ defined in section 2. Let $\tau(M)$ be the tangent bundle of M, then

$$\tau(D) \oplus \nu = \tau(M) \mid D\,, \quad \tau(\Delta) \oplus \hat{\nu} = \tau(N) \mid \Delta$$

and there exists a bundle map

$$\begin{array}{ccc} \tau(D)\oplus\nu\oplus T^*\nu & \to & \tau(\Delta)\oplus\hat{\nu} \\ \downarrow & & \downarrow \\ D & \xrightarrow{\quad f\mid D \quad} & \Delta\,. \end{array}$$

Since $\tau(D)\oplus\nu$ and $\tau(\Delta)\oplus\hat{\nu}$ are oriented, one may define an orientation of $T^*\nu$ so that the above bundle map may be orientation preserving. Then $\nu = T^*(T^*\nu)$ is naturally oriented and therefore $\tau(D)$ may be oriented in order that the bundle isomorphism $\tau(D)\oplus\nu = \tau(M) \mid D$ becomes orientation preserving. Then the bundle map

$$\begin{array}{ccc} \tau(D)\oplus\nu\oplus T^*\nu & \xrightarrow{\quad dT\oplus\hat{T} \quad} & \tau(D)\oplus\nu\oplus T^*\nu \\ \downarrow & & \downarrow \\ D & \xrightarrow{\qquad T \qquad} & D \end{array}$$

becomes orientation preserving, since $\hat{\nu} \simeq T_*\nu = (\nu\oplus T^*\nu)/\hat{T}$ where the bundle map $\hat{T}$: $\nu\oplus T^*\nu \to \nu\oplus T^*\nu$ is defined by $\hat{T}(u, v)=(v, u)$. On the other hand $\hat{T}$ is orientation preserving if k is even and orientation reversing if k is odd.

Consequently the involution $T: D \to D$ is orientation preserving if k is even and orientation reversing if k is odd.

By similar argument to the unoriented case one may have the following exact sequences for any $k \geqq 0$

$$\boldsymbol{A}^+:\ \cdots \to \boldsymbol{B}^+(n-2k+1, 2k) \xrightarrow{\partial_*} \boldsymbol{E}^0(n, 2k) \xrightarrow{\alpha_*} \boldsymbol{G}^0(n, 2k) \xrightarrow{\beta_*} \boldsymbol{B}^+(n-2k, 2k) \to \cdots,$$

$$\boldsymbol{A}^-:\ \cdots \to \boldsymbol{B}^-(n-2k, 2k+1) \xrightarrow{\partial_*} \boldsymbol{E}^0(n, 2k+1) \xrightarrow{\alpha_*} \boldsymbol{G}^0(n, 2k+1) \xrightarrow{\beta_*} \boldsymbol{B}^-(n-2k-1, 2k+1) \to \cdots,$$

$$\boldsymbol{B}^+:\ \cdots \to \boldsymbol{B}^-(n, k) \xrightarrow{\rho_*} \Omega_n(BSO(k)\times BSO(k)) \xrightarrow{\varphi_*^+} \boldsymbol{B}^+(n, k) \xrightarrow{\psi_*} \boldsymbol{B}^-(n-1, k) \to \cdots,$$

$$\boldsymbol{B}^-:\ \cdots \to \boldsymbol{B}^+(n, k) \xrightarrow{\rho_*} \Omega_n(BSO(k)\times BSO(k)) \xrightarrow{\varphi_*^-} \boldsymbol{B}^-(n, k) \xrightarrow{\psi_*} \boldsymbol{B}^+(n-1, k) \to \cdots$$

where $\boldsymbol{E}^0(n, k)$ and $\boldsymbol{G}^0(n, k)$ are the oriented cobordism groups of embeddings and of generic immersions respectively, $\boldsymbol{B}^+(n, k)$ and $\boldsymbol{B}^-(n, k)$ are the cobordism groups of oriented k-plane bundles over oriented n-manifolds with orientation preserving involution and with orientation reversing involution respectively, and the homomorphisms $\varphi_*^\pm: \Omega_n(BSO(k)\times BSO(k))\to \boldsymbol{B}^\pm(n, k)$ are defined by sending $[M, \xi, \eta]$ into the class $[M\times 0\cup(\pm M)\times 1, T, \xi\times 0\cup\eta\times 1]$ as in section 6.

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