Savits, T. H.
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THE EXPLOSION PROBLEM FOR BRANCHING MARKOV PROCESS

THOMAS H. SAVITS

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0. Introduction

Consider a single-type branching process. Then a well-known result of Dynkin is the following: explosion happens (i.e., the number of particles will be infinite in a finite time with positive probability) iff $\int_{1-\varepsilon}^{1}\frac{du}{u-h(u)}$ converges for every $\varepsilon>0$, where h is the generating function of new-born particles (see, e.g., [3, p. 106]). N. Ikeda [4] has also given an interesting proof of this fact using probabilistic techniques. Indeed he shows that the convergence of $\int_{1-\varepsilon}^{1}\frac{du}{u-h(u)}$ is equivalent to the finiteness of the expected value of e_Δ, the time of explosion (i.e., the first time when the number of particles is infinite).

The purpose of this paper is to investigate the explosion problem for a more general class of branching processes: branching Markov process[1] (see Ikeda, Nagasawa and Watanabe [5]). For a large class of bmp. we are able to show that a sufficient condition for explosion (non-explosion) is the convergence (divergence) of a particular integral. In many cases of interest, this condition is also necessary and sufficient.

In §1 we introduce the necessary terminology and notation; in §2 we generalize the methods of Ikeda and thus treat the problem from a probabilistic viewpoint; in §3, we consider the explosion problem from the analytical viewpoint. These results are of a more local character than those of §2 and hence give stronger results in some sense. Section 4 is devoted to applications. In particular, we consider branching diffusion processes with absorbing boundary. Another interesting application is that of branching Brownian motion whose splits occur only on a "fat" Cantor set.

It should be remarked that the explosion problem is intimately related to the uniqueness (or non-uniqueness) of solution of certain semi-linear parabolic equations. Such questions have been considered by Fujita and Watanabe [2].

1. We usually abbreviate this as bmp.

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1. Definitions and statement of problem

Let S be a locally compact, second-countable, Hausdorff topological space. Form the n-fold direct-product topological space $S^{(n)}$. Let $S^n=S^{(n)}/\sim$ be the quotient topological space induced by the equivalence relation $\sim$ of permutation: $(x_1,\cdots,x_n)\sim(y_1,\cdots,y_n)$ iff there exists a permutation π on $\{1,\cdots,n\}$ such that $x_i=y_{\pi i}$, all $i=1,\cdots,n$. The topological sum $\bigcup_{n=0}^{\infty} S^n$ is denoted by $\boldsymbol{S}$, where $S^0=\{\partial\}$, ∂ being an isolated point. Since $\boldsymbol{S}$ is locally compact (but not compact) we let $\hat{\boldsymbol{S}}=\boldsymbol{S}\cup\{\Delta\}$ be its one-point compactification.

In order to define a branching Markov process, it is convenient to introduce the mapping $\wedge : \boldsymbol{B}_1(S)\to\boldsymbol{B}(\hat{\boldsymbol{S}})$[2] defined by

$$\hat{f}(\boldsymbol{x})=\begin{cases}1 & \text{if } \boldsymbol{x}=\partial\,,\\ \prod_{i=1}^{n} f(x_i) & \text{if } \boldsymbol{x}=[x_1,\cdots,x_n]\in S^n\,,\\ 0 & \text{if } \boldsymbol{x}=\Delta\,.\end{cases}$$

Another mapping that we shall have occasion to use is the following: given $f, g\in\boldsymbol{B}_1(S)$, we define the $\mathscr{B}(\hat{\boldsymbol{S}})$-measurable function $\langle f|g\rangle$ by

$$\langle f|g\rangle(\boldsymbol{x})=\begin{cases}\sum_{i=1}^{n} g(x_i)\prod_{\substack{j\neq i\\ j=1}}^{n} f(x_j) & \text{if } \boldsymbol{x}=[x_1,\cdots,x_n]\in S^n\,,\\ 0 & \text{if } \boldsymbol{x}=\partial \text{ or } \Delta\,.\end{cases}$$

Now let $\boldsymbol{X}=(\Omega, \mathscr{B}_t, \boldsymbol{P}_x, \boldsymbol{X}_t, \theta_t)$ be a Markov process on $\boldsymbol{S}$[3], and let $\boldsymbol{T}_t$ be the semi-group on $\boldsymbol{B}(\hat{\boldsymbol{S}})$ induced by $\boldsymbol{X}$; i.e., $\boldsymbol{T}_t f(\boldsymbol{x})=\boldsymbol{E}_x[f(\boldsymbol{X}_t)]$. Following Ikeda, Nagasawa, and Watanabe, we say that $\boldsymbol{X}$ is a branching Markov process[4] (on $\boldsymbol{S}$) if

$$\boldsymbol{T}_t\hat{f}(\boldsymbol{x})=\widehat{(\boldsymbol{T}_t\hat{f})|_S}(x)^{5}$$

2. For any topological space E, $\mathscr{B}(E)$ is the Borel sets, $\boldsymbol{B}(E)$ the space of all (real-valued) bounded Borel-measurable functions, and $\boldsymbol{B}_1(E)=\{f\in\boldsymbol{B}(E)\colon \|f\|=\sup_{x\in E}|f(x)|\leq 1\}$.

3. We refer the reader to Dynkin [1] for the relevant definitions and properties concerning Markov processes.

4. For a clear and detailed exposition of such processes, see Ikeda, Nagasawa, and Watanabe [5].

5. For $f\in\boldsymbol{B}(\boldsymbol{S})$, $f|S$ means the restriction of f to S.

for all $t \geq 0, \boldsymbol{x} \in \hat{\boldsymbol{S}}$, and $f \in \boldsymbol{B}_1(S)$. We shall always assume that $\boldsymbol{X}$ is right-continuous, strong Markov, and $\bar{\mathscr{B}}_t = \mathscr{B}_t$, $\mathscr{B}_{t+} = \mathscr{B}_t$, all $t \geq 0$.

One easily sees that Δ is a trap,[6] and if e_Δ is the first hitting time of Δ, then $\boldsymbol{P}_x(e_\Delta > t) = \boldsymbol{T}_t \hat{1}(\boldsymbol{x})$. This representation will play an important role in §2. We shall call e_Δ the explosion time. Furthermore, letting $e_t(\boldsymbol{x}) = \boldsymbol{P}_x(e_\Delta > t)$ it follows that $e_t \downarrow e$ as $t \to \infty$, where $e(\boldsymbol{x}) = \boldsymbol{P}_x(e_\Delta = \infty)$.

Let ξ_t be the number of particles at time t; i.e., $\xi_t(\omega) = n$ if $\boldsymbol{X}_t(\omega) \in S^n$, $n = 0, ,\cdots, \infty$, where $S^\infty = \{\Delta\}$. Then the first splitting time τ is defined by

$$\tau(\omega) = \inf\{t \colon \xi_t(\omega) \neq \xi_0(\omega)\} \qquad (\inf \phi = \infty)\,.$$

The successive spliting times τ_n are defined inductively by $\tau_0 \equiv 0$ and $\tau_{n+1} = \tau_n + \tau \circ \theta_{\tau_n}$. Let $\tau_\infty = \lim_{n \to \infty} \tau_n$. We shall always assume that a bmp $\boldsymbol{X}$ satisfies the conditions

$$\text{(i)} \quad \boldsymbol{P}_x[\tau_\infty = e_\Delta\,;\, \tau_\infty < \infty] = \boldsymbol{P}_x[\tau_\infty < \infty],$$
$$\text{(ii)} \quad \boldsymbol{P}_x[\tau = s] = 0$$

for every $\boldsymbol{x} \in \boldsymbol{S}$ and $s \geq 0$.[7]

Given a bmp $\boldsymbol{X}$, we call X^0 the non-branching part, where

$$X_t^0(\omega) = \begin{cases} \boldsymbol{X}_t(\omega) & \text{if} \quad t < \tau(\omega) \\ \Delta & \text{otherwise.} \end{cases}$$

We have the following important property for a bmp $\boldsymbol{X}$. For every $f \in \boldsymbol{B}_1(S)$, $u(t, x) = \boldsymbol{T}_t \hat{f}(x)$[8] $(t \geq 0,\ x \in S)$ is a solution of the S-equation with initial value f:

$$u(t, x) = T_t^0 f(x) + \int_0^t \int_{\boldsymbol{S}} \Psi(x\colon ds\, d\boldsymbol{y})\, \widehat{u(t-s,\,\cdot)}(\boldsymbol{y})\,, \tag{1.1}$$

where $T_t^0 f(\boldsymbol{x}) = \boldsymbol{E}_x[f(\boldsymbol{X}_t)\colon t < \tau]$ and $\Psi(\boldsymbol{x}\colon ds\, d\boldsymbol{y}) = \boldsymbol{P}_x[\tau \in ds,\ \boldsymbol{X}_\tau \in d\boldsymbol{y}]$. Moreover, it is the minimal solution in the sense that when $0 \leq f \leq 1$ and if $0 \leq v \leq 1$ also satisfies (1.1), then $u \leq v$.

Two other properties enjoyed by a bmp which we shall have need of are

$$(1.2)\quad \begin{aligned} &\text{(i)} \quad T_t^0 \hat{f}(\boldsymbol{x}) = \widehat{(T_t^0 f)}|_S(\boldsymbol{x}) \\ &\text{(ii)} \quad \text{if} \quad \boldsymbol{x} \in S^n, \end{aligned}$$

$$\int_0^t \int_{S^m} \Psi(\boldsymbol{x};\, ds\, d\boldsymbol{y}) \hat{f}(\boldsymbol{y}) = \begin{cases} \displaystyle\int_0^t \langle T_s^0 f \,|\, \int_{S^{m-n+1}} \Psi(\cdot\,;\, ds\, d\boldsymbol{y}) \hat{f}(\boldsymbol{y}) \rangle(\boldsymbol{x}) \\ \qquad \text{provided } m \neq n,\ m \geq n-1 \\ 0 \quad \text{otherwise} \end{cases}$$

for $f \in \boldsymbol{B}_1(S)$.

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6. $\boldsymbol{P}_x[\boldsymbol{X}_t = \Delta \Rightarrow \boldsymbol{X}_s = \Delta,\ \forall s \geq t] = 1$, all $\boldsymbol{x} \in \hat{\boldsymbol{S}}$.

7. For most cases of interest, this constitutes no loss of generality. See [5] for more detail. There the conditions are labelled as (c. 1) and (c. 2) respectively.

8. When restricting our attention to $\boldsymbol{x} \in \boldsymbol{S}$, we often write x instead of $\boldsymbol{x}$.

A large class of bmp may be described in the following intuitive manner. Let $X^0=(X_t^0, P_x^0)$ be a Markov process on $S\cup B\cup\{\nabla\}$[9], ∇ an isolated point (B may be empty). Let ζ be the first hitting time of the set $B\cup\{\nabla\}$. Then, a particle moves on S according to X^0 up to time ζ. If at time ζ, $X_{\zeta-}^0\in B$, the particle is absorbed into ∂; otherwise, it splits into n-particles starting at $\boldsymbol{y}\in S^n$ with probability $\pi(X_{\zeta-}^0, d\boldsymbol{y})$, where π is a given stochastic kernel on $S\times\mathcal{B}(\hat{S})$[10] such that $\pi(x, S)=0$, all $x\in S$. Each newborn particle then exhibits the same motion as the original independent of one another. The S-equation then becomes $u(t, x)=T_t^0f(x)+h(t, x)+\int_0^t\int_S K(x; ds\,dy)F[y; u(t-s, \cdot)]$, where $T_t^0f(x)=E_x^0[f(X_t^0); t<\zeta]$, $h(t, x)=P_x^0[\zeta\leq t, X_{\zeta-}^0\in B]$, $K(x; ds\,dy)=P_x^0[\zeta\in ds, X_{\zeta-}^0\in dy\cap S]$, and $F[y; g]=\int_{\hat{S}}\pi(y; d\boldsymbol{z})\hat{g}(\boldsymbol{z})$; furthermore, we have the relation $h(t, x)=1-T_t^0 1(x)-K(x; [o, t]\times S)$. In this case we say that $\boldsymbol{X}$ possesses the fundamental system (T_t^0, K, π). In particular, if X^0 is obtained from a conservative Markov process $X=(X_t, P_x)$ by first absorbing it into δ (an isolated point) when it hits B and then killing this process with a non-negative measurable function k ($k=o$ on δ), we say that the fundamental system (T_t^0, K, π) is determined by $[X, k, \pi]$, or briefly, that $\boldsymbol{X}$ possesses the regular fundamental system $[X, k, \pi]$.
Here

$$T_t^0f(x) = E_x^X[e^{-\int_0^t k(X_s)\,ds}f(X_t); t<\eta]$$
$$K(x; ds\,dy) = T_s^0(kI_{\{dy\}})(x)\,ds\,,$$ [11]

where η is the first hitting time of the set B. This paper primarily concerns itself with discussing the explosion problem for such processes.[12]

Before moving on to the main results of this paper, we first make some general comments. The problem we are concerned with is the following; is it possible to produce an infinite number of particles in a finite amount of time? As we shall soon see (Lemma 2.1), it suffices to ask the question: starting from one particle, is it possible to produce an infinite number of particles in a finite amount of time? More precisely, is $\boldsymbol{P}_x(\xi_t=\infty$ for some $t\geq 0)>0$, or equivantly, is $e(x)=\boldsymbol{P}_x(e_\Delta=\infty)<1$? Recall that $e_t=\boldsymbol{T}_t\hat{1}\downarrow e$ and e_t is the minimal solution of the S-equation with initial value $f=1$:

$$(1.3)\qquad u_t(x) = T_t^0 1(x)+\int_0^t\int_S \Psi(x: ds\,d\boldsymbol{y})\hat{u}_{t-s}(\boldsymbol{y})\,.$$

9. Think of B as the boundary of a domain S in $\boldsymbol{R}^n$. We call B the absorbing set for $\boldsymbol{X}$.

10. For fixed $x\in S$, $\pi(x, \cdot)$ is a probability on $\mathcal{B}(\hat{S})$ and for fixed $\Lambda\in\mathcal{B}(\hat{S})$, $\pi(\cdot, \Lambda)$ is $\mathcal{B}(S)$-measurable.

11. I_A is the indicator function of the set A.

12. For a more rigorous treatment of these processes, see [5].

The only case in which the problem is interesting is when $P_x(X_\tau=\Delta;\ \tau<\infty)=0$ and so we shall always assume this.[13] Note then that $u_t\equiv 1$ is also a solution of (1.3). Hence we are interested in the uniqueness and non-uniqueness of certain integral equations; in fact, we have

(1.4) **Proposition.** $P_x[e_\Delta=+\infty]=1$ *for every* $x\in S$ *iff* $u(t, x)\equiv 1$ *is the unique solution of* (1.3) (*unique within the class of all solutions* v *such that* $0\leq v\leq 1$).

(1.5) **Corollary.** *Let* X *possess a regular fundamental system* $[X, k, \pi]$ *such that* $||k||<\infty$ *and suppose that* $\sup_{x\in S}\sum_{n=0}^{\infty} n\pi(x;\ S^n)<\infty$. *Then* $P_x(e_\Delta=+\infty)=1$ *for every* $x\in S$.

The proof of the corollary follows from the fact that F is Lipschitz continuous in this case.

We should also remark that in many cases, the S-equation has a differential analogue. For example, if X possesses a sufficiently "nice" regular fundamental system $[X, k, \pi]$, then the differential equation analogue of (1.3) is the non-linear evolution equation

$$\begin{aligned} \frac{d}{dt}u_t &= Au_t+k[F(\cdot\,;\ u_t)-u_t] \\ u(0+, x) &= 1 \\ u(t, x)|_{x\to B} &= 1\,, \end{aligned}$$

where A is the infinitesimal generator of the process X. H. Fujita and S. Watanabe [2] considered such problems of uniqueness and non-uniqueness.

2. A probabilistic approach

In this section we shall always assume that S *is compact.* So let X be a bmp on S. Recall the functions e_t and e defined in §1: $e_t(\boldsymbol{x})=T_t\hat{1}(\boldsymbol{x})=P_x(e_\Delta>t)\downarrow e(\boldsymbol{x})=P_x(e_\Delta=\infty)$. Thus, we can say that explosion happens starting from $\boldsymbol{x}$ iff $e(\boldsymbol{x})<1$. Our first aim will be to show that under suitable conditions $e\equiv 1$ or $e\equiv 0$ on $\boldsymbol{S}\backslash\{\partial\}$. Moreover, the former is true iff $E.[e_\Delta]$ is everywhere infinite there.

As a first step we observe

(2.1) **Lemma.**

$$\begin{aligned} &\text{(i)}\quad \widehat{e|_S} = e \\ &\text{(ii)}\quad T_t e = e \quad \text{for all} \quad t\geq 0\,. \end{aligned}$$

13. When X possesses the fundamental system (T_t^0, K, π), this amounts to assuming that $\pi(x;\ \{\Delta\})=0$, all $x\in S$.

Proof. Since $\widehat{e_t|_S}=e_t$ all $t\geq 0$, the first assertion is clear.
Also

$$\boldsymbol{T}_t e(\boldsymbol{x}) = \lim_{s\to\infty} \boldsymbol{T}_t \boldsymbol{T}_s \hat{1}(\boldsymbol{x}) = \lim_{s\to\infty} \boldsymbol{T}_{t+s}\hat{1}(\boldsymbol{x}) = e(\boldsymbol{x})\,.$$

We now impose the following set of assumptions $[A]$.

(2.2)
(A_1) $\boldsymbol{P}_x[\boldsymbol{X}_\tau = \partial;\, \tau<\infty] = 0$ for all $x\in S$.

(A_2) e_t and e are upper semi-continuous.

(A_3) For every $t>0$, all $x\in S$, and every non-empty open $U\subset S$, there exists a $V\in\mathcal{B}(\hat{\boldsymbol{S}})$ such that $\boldsymbol{P}_x[\boldsymbol{X}_t\in V]>0$ and for every $\boldsymbol{y}\in V$, say $\boldsymbol{y}=[y_1,\cdots,y_m]$, some $y_i\in U$.

(A_1) is the assumption of no death; (A_2) is a regularity condition on $\boldsymbol{X}$; (A_3) is some type of communication assumption. Roughly, (A_3) states that for every $t>0$ and open $U\subset S$, at least one particle is in U at time t with positive probability.

(2.3) **Theorem.** $\boldsymbol{P}_x[e_\Delta = \infty] \equiv 1$ *or* $\equiv 0$ *on* S.

Proof. Note that (A_1) implies $\boldsymbol{P}(t,\, x,\, \{\partial\})=0$ for all $t\geq 0$, $x\in S$, where $\boldsymbol{P}$ is the transition function for $\boldsymbol{X}$. Let $\beta=\sup_{x\in S} e(x)$. Then $0\leq\beta\leq 1$. From (A_2) and the assumption of compactness it follows that there exists some $x_0\in S$ with $e(x_0)=\beta$. If $\beta=0$ we are through. So suppose not. Then we claim that $\beta=1$. For otherwise $0<\beta<1$. By Lemma 2.1 and (A_1) we can write for any $t\geq 0$

$$\begin{aligned}(2.4)\qquad \beta = e(x_0) &= \boldsymbol{E}_{x_0}[\widehat{e|_S}(\boldsymbol{X}_t)] = \int_{\hat{S}} \widehat{e|_S}(\boldsymbol{y})\boldsymbol{P}(t,\, x_0,\, d\boldsymbol{y}) \\ &= \sum_{n=1}^{\infty}\int_{S^n} \widehat{e|_S}(\boldsymbol{y})\boldsymbol{P}(t,\, x_0,\, d\boldsymbol{y}) \leq \sum_{n=1}^{\infty}\beta^n \boldsymbol{P}(t,\, x_0,\, S^n)\,.\end{aligned}$$

Now if $\boldsymbol{P}(t,\, x_0,\, S)=1$ for all $t\geq 0$, it would imply by right-continuity that $\boldsymbol{P}_{x_0}[\boldsymbol{X}_t\in S,$ all $t\geq 0]=1$, contradicting the assumption that $\beta<1$. Thus, there exists some t_0 such that $\boldsymbol{P}(t_0,\, x_0,\, S)<1$. For this t_0 it would follow from (2.4) that $\beta<\beta$.

We will now show that $e|_S\equiv 1$ if $\beta=1$. Suppose not. Then there exists an $\varepsilon>o$ and open $U\subset S$ such that $e|_U\leq 1-\varepsilon$. Fix any $t>0$. Let V be a set corresponding to U in (A_3). Then

$$\begin{aligned}1 = e(x_0) &= \Big(\int_V + \int_{\hat{S}\setminus V}\Big)\widehat{e|_S}(\boldsymbol{y})\boldsymbol{P}(t,\, x_0,\, d\boldsymbol{y}) \\ &\leq (1-\varepsilon)\boldsymbol{P}(t,\, x_0,\, V) + \boldsymbol{P}(t,\, x_0,\, \hat{\boldsymbol{S}}\setminus V) < 1\,.\end{aligned}$$

Contradiction.

Theorem 2.3 states that $e\equiv 1$ or $\equiv 0$ on S. Clearly if $e|_S\equiv 1$ then $\boldsymbol{E}_x[e_\Delta]\equiv\infty$ on S. An interesting and useful fact, however, is that the converse is also true.

(2.5) **Lemma.** *If $e\equiv 0$ on S, then for all $t>0$, $\|e_t|_S\|<1$.*

Proof. Suppose there exists some $t_0>0$ such that $\|e_{t_0}|_S\|=1$. Let $y_0\in S$ be such that $e_{t_0}(y_0)=1$ and choose $h>0$ such that $t_1=t_0-h>0$. Then

$$1=e_{t_0}(y_0)=\boldsymbol{T}_h\boldsymbol{T}_{t_1}\hat{1}(y_0)=\boldsymbol{T}_h\widehat{(e_{t_1}|_S)}(y_0)\,.$$

By the same reasoning as in Theorem 2.3, we conclude that $e_{t_1}|_S\equiv 1$. Hence for every n,

$$\begin{aligned} e_{nt_1}(y_0) &= \boldsymbol{T}_{nt_1}\hat{1}(y_0)=\boldsymbol{T}_{(n-1)t_1}\boldsymbol{T}_{t_1}\hat{1}(y_0)=\boldsymbol{T}_{(n-1)t_1}\widehat{e_{t_1}|_S}(y_0)\\ &= \boldsymbol{T}_{(n-1)t_1}\hat{1}(y_0)=\cdots=\boldsymbol{T}_{t_1}\hat{1}(y_0)=1\,, \end{aligned}$$

and so $e(y_0)=\lim_{n\to\infty}e_{nt_1}(y_0)=1$. Contradiction.

(2.6) **Theorem.** $\boldsymbol{P}_x[e_\Delta=+\infty]=1$ *iff* $\boldsymbol{E}_x[e_\Delta]=\infty$.

Proof. We need only prove sufficiency as necessity is clear. Applying Dynkin's formula to $g=\boldsymbol{R}_1\hat{1}=\int_0^\infty e^{-t}\boldsymbol{T}_t\hat{1}dt$,

$$\boldsymbol{E}_x[g(\boldsymbol{X}_{e_\Delta\wedge M})]-g(x)=\boldsymbol{E}_x\Big[\int_0^{e_\Delta\wedge M}(g-\hat{1})(\boldsymbol{X}_t)\,dt\Big]\quad\text{for every } M>0\,.$$

So suppose $\boldsymbol{P}_x(e_\Delta=\infty)=0$. Applying Lemma 2.5 we conclude that there exists some $\alpha>0$ such that $0\le g(\boldsymbol{y})\le 1-\alpha$ for all $\boldsymbol{y}\neq\partial$. But from the right-continuity of the process and the assumption of no dying we have $\boldsymbol{P}_x[\forall t\ge 0, \boldsymbol{X}_t\neq\partial]=1$. Consequently,

$$\alpha\boldsymbol{E}_x[e_\Delta\wedge M]\le 2\|g\|_{\hat{S}}\le 2\,.$$

Letting $M\uparrow\infty, \boldsymbol{E}_x[e_\Delta]\le\dfrac{2}{\alpha}$ (independent of x).

Combining Theorems 2.3 and 2.6 we have

(2.7) **Theorem.** *Let $\boldsymbol{X}$ be a bmp on a compact space S satisfying $[A]$. Then $\boldsymbol{P}_x(e_\Delta=+\infty)\equiv 1$ or $\equiv 0$ accordingly as $\boldsymbol{E}_x[e_\Delta]\equiv\infty$ or uniformly bounded on S.*

(2.8) **Corollary.** *Let $\boldsymbol{X}$ possess a regular fundamental system $[X, k, \pi]$ with no absorbing set (i.e., $B=\phi$) and such that*

(i) $\pi(x;\{\partial\})=0$ *all* $x\in S$,

(ii) $\|k\|<\infty$,

(iii) *T_t^0 strongly Feller*[14], *and*

(iv) *for every $t>0$, $x\in S$, and non-empty open $U\subset S$,*

$$P^0(t, x, U)\equiv T_t^0 I_U(x)>0 .$$

Then the conclusions of Theorem 2.7 are valid.

Proof. (A_1) follows from (i). Since $e_t(x)$ is a solution of the S-equation

$$u(t, x) = T_t^0 1(x)+\int_0^t T_{t-s}^0[k(\cdot)F(\cdot\,; u_s)](x)\,ds\,,$$

(ii) and (iii) imply that e_t is continuous for all t. Thus e is upper semi-continuous. (A_3) follows easily from (iv). Now apply Theorem 2.7.

In the remainder of this section we assume that X possesses a fundamental system (T_t^0, K, π) with no absorbing set such that $\pi(x, \{\partial\})=\pi(x, \{\Delta\})=0$ on S. Our aim here is to derive a condition for explosion similar to that of E.B. Dynkin. We shall only sketch the details. In section 3 we are able to derive essentially much stronger results.[15]

Consider

$$\begin{aligned}(2.9)\qquad \boldsymbol{E}_x[e_\Delta] &= \boldsymbol{E}_x[\tau_\infty] = \sum_{n=0}^{\infty} \boldsymbol{E}_x[\tau\circ\theta_{\tau_n};\ \tau_n<\infty]\\ &= \sum_{n=0}^{\infty} \boldsymbol{E}_x[\boldsymbol{E}_{X_{\tau_n}}[\tau];\ \tau_n<\infty]\\ &= \sum_{n=0}^{\infty}\sum_{\nu_1=2}^{\infty}\sum_{\nu_2=\nu_1+1}^{\infty}\cdots\sum_{\nu_n=\nu_{n-1}+1}^{\infty} \boldsymbol{E}_x[\boldsymbol{E}_{X_{\tau_n}}[\tau];\ X_{\tau_1}\in S^{\nu_1},\cdots,\\ &\qquad\qquad X_{\tau_n}\in S^{\nu_n};\ \tau_n<\infty]\end{aligned}$$

by the S.M.P.. Again by the repeated use of the S.M.P., we can write

$$\begin{aligned}&\boldsymbol{E}_x[\boldsymbol{E}_{X_{\tau_n}}[\tau];\ X_{\tau_1}\in S^{\nu_1},\cdots, X_{\tau_n}\in S^{\nu_n};\ \tau_n<\infty]\\ &\quad= \boldsymbol{E}_x[\boldsymbol{E}_{X_\tau}[\cdots[\boldsymbol{E}_{X_\tau}[\tau];\ X_\tau\in S^{\nu_n},\ \tau<\infty]\cdots];\ X_\tau\in S^{\nu_1};\ \tau<\infty]\\ &\quad= \int_{S^{\nu_1}}\cdots\int_{S^{\nu_n}} \boldsymbol{E}_{y_n}[\tau]\boldsymbol{P}_{y_{n-1}}(X_\tau\in dy_n)\cdots \boldsymbol{P}_{y_1}(X_\tau\in dy_2)\boldsymbol{P}_x(X_\tau\in dy_1)\,.\end{aligned}$$

Furthermore, for $z\neq\Delta$

$$\begin{aligned}\boldsymbol{E}_z[\tau] &= \boldsymbol{E}_z[\int_0^\tau dt] = \int_0^\infty \boldsymbol{E}_z[1;\ t<\tau]\,dt\\ &= \int_0^\infty T_t^0\hat{1}(z)dt = \int_0^\infty \widehat{(T_t^0\hat{1})|_S}(z)\,dt\end{aligned}$$

14. That is, T_t^0: $\mathbf{B}(S)\longrightarrow\mathbf{C}(S)=\{$bounded continuous functions on $S\}$, all $t\geq 0$.

15. We need assume there, however, that $K(x$: dsdy$)=J(x, s;$ dy$)$ds.

making use of (1.2). Now define the following for $t \geq 0$, $0 \leq \xi \leq 1$:

$$\alpha(t) = \inf_{x \in S} T_t^0 \hat{1}(x) \qquad F_*(\xi) = \inf_{z \in S} \sum_{\nu=2}^{\infty} q_\nu(z) \xi^\nu$$

$$\beta(t) = \sup_{x \in S} T_t^0 \hat{1}(x) \qquad F^*(\xi) = \sup_{z \in S} \sum_{\nu=2}^{\infty} q_\nu(z) \xi^\nu ,$$

where $q_\nu(z) = \pi(z;\, S^\nu)$.

Observe that if $q_\nu(z)$ is independent of z, all ν, then $F_* = F^*$. Continuing, we estimate for $\boldsymbol{y} \in S^\mu$

$$\begin{aligned}
&\sum_{\nu=\mu+1}^{\infty} \int_{S^\nu} \boldsymbol{E}_z[\tau] \boldsymbol{P}_y[X_\tau \in d\boldsymbol{z}] \\
&= \int_0^\infty \int_0^\infty \sum_{\nu=\mu+1}^{\infty} \int_{S^\nu} \widehat{(T_t^0 1) \mid S}(\boldsymbol{z}) \Psi(\boldsymbol{y};\, ds\, d\boldsymbol{z})\, dt \\
&\geq \int_0^\infty \sum_{\nu=\mu+1}^{\infty} \alpha^\nu(t) \int_0^\infty \int_{S^\nu} \hat{1}(\boldsymbol{z}) \Psi(\boldsymbol{y};\, ds\, d\boldsymbol{z})\, dt \\
&= \int_0^\infty \sum_{\nu=\mu+1}^{\infty} \alpha^\nu(t) \int_0^\infty \langle T_s^0 1 \mid \int_{S^{\nu-\mu+1}} \hat{1}(\boldsymbol{z}) \Psi(\cdot\,;\, ds\, d\boldsymbol{z}) \rangle (\boldsymbol{y})\, dt \\
&= \int_0^\infty \int_0^\infty \langle T_s^0 1 \mid \int_S K(\cdot\,;\, ds\, dz) \sum_{\nu=\mu+1}^{\infty} \alpha^\nu(t) \pi(z,\, S^{\nu-\mu+1}) \rangle (\boldsymbol{y})\, dt \\
&\geq \int_0^\infty \alpha^\mu(t) \left(\frac{F_*[\alpha(t)]}{\alpha(t)} \right) \int_0^\infty \langle T_s^0 1 \mid \int_S K(\cdot\,;\, ds\, dz)\, 1(z) \rangle\, (\boldsymbol{y})\, dt
\end{aligned}$$

using (1.2) and the fact that $\Psi(x;\, ds\, d\boldsymbol{z}) = \int_S K(x;\, ds\, dy) \pi(y,\, d\boldsymbol{z})$. If we assume that $\lim_{t \uparrow \infty} T_t^0 \hat{1}(x) = 0$ for all $x \in S$, then

$$\begin{aligned}
&\int_0^\infty \langle T_s^0 1 \mid \int_S K(\cdot\,;\, ds\, dz) 1(z) \rangle (\boldsymbol{y}) \\
&\qquad = 1 - \lim_{t \uparrow \infty} \widehat{(T_t^0 1) \mid S}(\boldsymbol{y}) = 1 .
\end{aligned}$$

Hence

$$\sum_{\nu=\mu+1}^{\infty} \int_{S^\nu} \boldsymbol{E}_z[\tau] \boldsymbol{P}_y[X_\tau \in d\boldsymbol{z}] \geq \int_0^\infty \alpha^\mu(t) \left(\frac{F_*[\alpha(t)]}{\alpha(t)} \right) dt .$$

Iterating this in (2.9) one obtains the estimate

$$\begin{aligned}
\boldsymbol{E}_x[e_\Delta] &\geq \int_0^\infty \alpha(t) \sum_{n=0}^{\infty} \left(\frac{F_*[\alpha(t)]}{\alpha(t)} \right)^n dt \\
&= \int_0^\infty \frac{\alpha^2(t)\, dt}{\alpha(t) - F_*[\alpha(t)]} .
\end{aligned}$$

Although the intermediate calculations in the case $\alpha(t) = 0$ are not valid, the end result is provided we interpret the integrand to be zero for such t. A similar calculation yields

$$E_x[e_\Delta] \leq \int_0^\infty \frac{\beta^2(t)\,dt}{\beta(t)-F^*[\beta(t)]}.$$

So under the assumptions $[B]$,

$$(2.11)\quad \begin{array}{ll} (B_1) & \boldsymbol{X} \text{ possesses a fundamental system } (T_t^0, K, \pi) \text{ with no absorbing set,} \\ (B_2) & \pi(x, \{\partial\}) = \pi(x, \{\Delta\}) = 0 \quad \text{on} \quad S\,, \\ (B_3) & \lim_{t \uparrow \infty} T_t^0 \hat{1}(x) = 0 \quad \text{on} \quad S\,, \end{array}$$

we have the following

(2.12) **Proposition.**

$$\int_0^\infty \frac{\alpha^2(t)\,dt}{\alpha(t)-F_*[\alpha(t)]} \leq E_x[e_\Delta] \leq \int_0^\infty \frac{\beta^2(t)\,dt}{\beta(t)-F^*[\beta(t)]}$$

for every $x \in S$.

(2.13) REMARK. If α is integrable (on $[o, \infty)$), then $\dfrac{\alpha^2(t)}{\alpha(t)-F_*[\alpha(t)]}$ is integrable iff it is locally integrable at 0. Similarly for β. In particular, if $(T_t^0\hat{1}, K, \pi)$ is determined by $[X, k, \pi]$ such that $0<k_1\leq k\leq k_2$ for some constants k_i then

(i) $\displaystyle\int_{1-\varepsilon}^1 \frac{d\xi}{\xi-F^*[\xi]} < \infty$ implies $E_x[e_\Delta] < \infty$ for all $x \in S$,

(ii) $\displaystyle\int_{1-\varepsilon}^1 \frac{d\xi}{\xi-F_*[\xi]} = \infty$ implies $E_x[e_\Delta] = \infty$ for all $x \in S$.

By combining Theorem 2.7 and Proposition 2.12 we obtain

(2.14) **Theorem.** *Let* $\boldsymbol{X}$ *be a bmp on compact* S *satisfying* $[A]$ *and* $[B]$. *Then*

(i) $\displaystyle\int_0^\infty \frac{\beta^2(t)\,dt}{\beta(t)-F^*[\beta(t)]} < \infty$ *implies* $P_x(e_\Delta=\infty) = 0$ *on* S.

(ii) $\displaystyle\int_0^\infty \frac{\alpha^2(t)\,dt}{\alpha(t)-F_*[\alpha(t)]} = \infty$ *implies* $P_x(e_\Delta=\infty) = 1$ *on* S.

We conclude this section with the following theorem. These results were first obtained by N. Ikeda [4] for the single-type branching process.

(2.15) **Theorem.** *Let* $\boldsymbol{X}$ *be a bmp on compact* S. *Suppose it possesses a regular fundamental system* $[X, k, \pi]$ *with no absorbing set such that*

(i) $\pi(\cdot, S^n) = q_n(constant)$, $n = 0, 1, \cdots, +\infty$ *and* $q_0 = q_1 = q_\infty = 0$,

(ii) *there exist constants* k_1, k_2 *with* $0 < k_1 \leq k \leq k_2$,

(iii) T_t^0 *strongly Feller and*

(iv) *for every non-empty open* $U \subset S$, $T_t^0 I_U(x) > 0$, *for all* $t > 0, x \in S$.

Then the following statements are equivalent:

$$\begin{array}{ll} (1)\ \ \boldsymbol{P}_x[e_\Delta = \infty] = 1 \ \ on\ \ S & (1)'\ \ \boldsymbol{P}_x[e_\Delta < \infty] = 1 \ \ \text{on}\ \ S \\ (2)\ \ \boldsymbol{E}_x[e_\Delta] = \infty \ \ on\ \ S & (2)'\ \ \boldsymbol{E}_x[e_\Delta] < \infty \ \ \text{on}\ \ S \\ (3)\ \ \displaystyle\int_{1-\varepsilon}^{1} \frac{d\xi}{\xi - F[\xi]} = \infty & (3)'\ \ \displaystyle\int_{1-\varepsilon}^{1} \frac{d\xi}{\xi - F[\xi]} < \infty \end{array}$$

where $F[\xi] = \sum_{n=2}^{\infty} q_n \xi^n$.

3. An analytic approach

Recall that in §1 we pointed out that $e_t(x) = \boldsymbol{T}_t \hat{1}(x)$ is the minimal solution of the S-equation with initial value $f=1$. We shall exploit this fact here.

We shall suppose that $\boldsymbol{X}$ *possesses a fundamental system* (T_t^0, K, π) *such that*

$$(3.1) \qquad K(x; ds\,dy) = J(x, s; dy)\,ds.$$

In particular this is true if $\boldsymbol{X}$ possesses a regular fundamental system. Then $\bar{v}(t, x) = 1 - e_t(x)$ is the maximal solution[16] of

$$(3.2) \qquad u(t, x) = \int_0^t ds \int_S J(x, t-s; dy)\, G[y; u(s, \cdot)],$$

where $G[x; f] = 1 - F[x; 1-f]$. The idea now is to compare $\bar{v}$ with a solution of a related integral equation. A key lemma in this direction is the following:

(3.3) **Lemma.** *Let* g *be a non-negative non-decreasing function on* $[0, 1]$, *and let* τ *be a non-negative integrable function on* $[0, \delta]$, *some* $\delta > 0$. *Consider the integral equation*

$$(3.4) \qquad v(t) = \int_0^t \tau(s) g[v(s)]\, ds.$$

Then

16. Maximal in the sense that if v is also a solution, $0 \leq v \leq 1$, then $v \leq \bar{v}$.

(i) *if* $\int_0 \frac{d\xi}{g(\xi)}=\infty$,[17] *any solution* v *of* (3.4) *defined on* $[o, \eta]$ *such that* $0\leq v\leq 1$ *is identically zero on* $[0, \delta\wedge\eta]$.

(ii) *if* $\int_0 \frac{d\xi}{g(\xi)}<\infty$ *and* τ *is (essentially) locally positive at* 0,[18] *then there exists an increasing solution* v *of* (3.4) *on* $[0, \eta]$, *some* $\eta>0$, *such that* $0\leq v\leq 1$; *moreover* $v(t)>0$ *for* $t>0$.

Proof.

(i) Let $0\leq v\leq 1$ be a solution of (3.4) on $[0, \eta]$. Without loss of generality, we may assume $\eta\leq\delta$. Clearly v is absolutely continuous and increasing. Set

$$\mu = \sup\{t: 0\leq t\leq\eta \text{ and } g[v(t)] = 0\} \quad (\sup\phi = 0)\,.$$

If $\mu=\eta$, then $g\circ v=0$ on $[0, \eta)$. Consequently $v=0$ on $[0, \eta]$ and we are done. So suppose $\mu<\eta$. Then $g[v(t)]>0$ for $\mu<t\leq\eta$. Now from (3.4) it follows that

$$v'(t) = \tau(t)g[v(t)] \quad \text{a.e.}$$

Consequently for every $\varepsilon>0$,

$$\int_{\mu+\varepsilon}^{\eta} \frac{v'(s)}{g[v(s)]}\,ds = \int_{\mu+\varepsilon}^{\eta} \tau(s)\,ds\,,$$

or, by a change of variables,

$$\int_{v(\mu+\varepsilon)}^{v(\eta)} \frac{d\xi}{g(\xi)} = \int_{\mu+\varepsilon}^{\eta} \tau(s)\,ds\,.$$

Letting $\varepsilon\downarrow 0$ we obtain

$$\int_0^{v(\eta)} \frac{d\xi}{g(\xi)} = \int_{\mu}^{\eta}\tau(s)\,ds<\infty\,.$$

Contradiction.

(ii) Define on $[0, 1]\times[0, \delta]$ the function

$$A(v, t) = \int_0^{v}\frac{d\xi}{g(\xi)}-\int_0^{t}\tau(s)\,ds\,.$$

Note that for fixed t, A is strictly increasing and continuous in v, and that $A(0, 0)=0$. Set

---

17. By $\int_0 \frac{d\xi}{g(\xi)}=+\infty$, we mean that $\int_0^{\varepsilon} \frac{d\xi}{g(\xi)}=+\infty$ for every sufficiently small $\varepsilon>0$; i.e., $\left(\frac{1}{g}\right)$ is not locally integrable at 0.

18. That is, for every sufficiently small $r>0$, $\int_0^r\tau>0$.

$$\eta = \sup_{0 \le t \le \delta} \{t: A(1, t) \ge 0\} .$$

Clearly $\eta > 0$. So, for every t with $0 \le t \le \eta$, there exists a unique v such that $0 \le v \le 1$ and $A(v, t) = 0$. Denote it by $v = v(t)$. It is not hard to show that v has all the required properties.

In order to apply this lemma it is convenient to make the following set up. Let $\mathfrak{S} = \boldsymbol{B}_1([0, \infty) \times S)$ and define the operator Φ by

$$(3.5) \qquad (\Phi v)(t, x) = \int_0^t ds \int_S J(x, t-s; dy) G[y: v(s, \cdot)] .$$

It is clear that Φ has the properties

$$(3.6) \qquad \begin{array}{ll} \text{(i)} & \Phi\mathfrak{S} \subset \mathfrak{S} \\ \text{(ii)} & \Phi u \le \Phi v \quad \text{if} \quad u \le v \end{array}$$

(3.7) DEFINITION. v is a solution of $\Phi u = u$ if $v \in \mathfrak{S}$ and $\Phi v = v$; $\bar{v}$ is a maximal solution if $\bar{v}$ is a solution and if v is also a solution, then $\bar{v} \ge v$.

We already know, of course, the maximal solution v in terms of the semi-group $\boldsymbol{T}_t$ induced from the bmp $\boldsymbol{X}$. This appears to be difficult to work with directly, however. It is more convenient to use the subterfuge of an approximating sequence.

(3.8) **Proposition.** *There exists a sequence v_n, $1 \le n \le \infty$, with $0 \le v_n \le 1$ such that $v_0 = 1$, $v_\infty = \bar{v}$, and $v_n \downarrow v_\infty$.*

Proof. Set $v_0 \equiv 1$ and define inductively for $n \ge 1$, $v_n = \Phi v_{n-1}$. Since $v_0 \in \mathfrak{S}$ it follows from (3.6) that $v_n \in \mathfrak{S}$ and $v_n \downarrow$. Set $v_\infty = \lim v_n$, which clearly exists. By the dominated convergence theorem, $v_\infty = \Phi v_\infty$.

Now suppose u is any other solution. But $u \le 1 = v_0$. So suppose $u \le v_n$. Then

$$u = \Phi u \le \Phi v_n = v_{n+1} .$$

Hence $u \le v_\infty$. By the uniqueness of the maximal solution, we have then that $v_\infty = \bar{v}$.

(3.9) DEFINITION. The sequence $v_0 = 1$, $v_n = \Phi v_{n-1}$ for $n \ge 1$ is called the defining sequence for the maximal solution $\bar{v}$.

We are now ready to reap the main results of this section.

(3.10) **Theorem.** *Let $\delta > 0$ be fixed and set*

$$(3.11) \qquad \tau^*(s) = \sup_{\substack{x \in S \\ t \in [s, \delta]}} J(x, t-s; S) , \quad 0 \le s \le \delta .$$

Suppose $\int_0^\delta \tau^* < \infty$. *Define* $G^*(\xi) = \sup_{x \in S} G[x; \xi 1]$, $0 \leq \xi \leq 1$. *Then if* $\int_0 \frac{d\xi}{G^*(\xi)} = \infty$, $\bar{v} \equiv 0$ (i.e., *no explosion*).

Proof. Let G_+^* be the right-continuous version of G^*; i.e., $G_+^*(\xi) = \lim_{\eta \downarrow \xi} G^*(\eta)$, $0 \leq \xi < 1$, and $G_+^*(1) = G^*(1)$. Then G_+^* is monotone increasing, $G_+^* \geq G^*$ and $G_+^* = G^*$ a.e. Let $\langle v_n \rangle$ be the defining sequence for $\bar{v}$. Take $u_0 \equiv 1$ and define u_n iteratively by

$$u_n(t) = \int_0^t \tau^*(s) G_+^*[u_{n-1}(s)]\, ds\,, \quad n \geq 1\,.$$

Set $\eta = \sup_{0 \leq t \leq \delta} \{t\colon G_+^*(1) \int_0^t \tau^*(s)\, ds \leq 1\}$. Then $\eta > 0$. Also, since $0 \leq u_{n+1} \leq u_n \leq 1$, we have $u_n \downarrow u_\infty$ exists on $[0, \eta]$ with $0 \leq u_\infty \leq 1$.

Now $v_0 = 1 \leq u_0$; so suppose $v_n(t, x) \leq u_n(t) 1(x)$ on $[0, \eta] \times S$. Then for $(t, x) \in [0, \eta] \times S$

$$\begin{aligned} v_{n+1}(t, x) &= \int_0^t ds \int_S J(x, t-s; dy) G[y; v_n(s, \cdot)] \\ &\leq \int_0^t ds \int_S J(x, t-s; dy) G[y; u_n(s) 1(\cdot)] \\ &\leq \int_0^t G^*[u_n(s)] J(x, t-s; S)\, ds \\ &\leq \int_0^t \tau^*(s) G_+^*[u_n(s)]\, ds = u_{n+1}(t)\,. \end{aligned}$$

Consequently, $\bar{v} \leq u_\infty$. But u_∞ satisfies

$$u_\infty(t) = \int_0^t \tau^*(s) G_+^*[u_\infty(s)]\, ds\,.$$

From Lemma 3.3 we conclude that $u_\infty \equiv 0$ on $[0, \eta]$; hence $\bar{v} \equiv 0$ on $[0, \eta] \times S$.

Now set

$$\sigma = \sup\{t\colon \bar{v}(s, x) = 0 \quad \text{on} \quad [0, t] \times S\}\,.$$

If $\sigma = \infty$, we are done; so suppose not. Then $\sigma \geq \eta > 0$. Now set $u(t, x) = \bar{v}(t+\sigma, x)$. Then u satisfies the equation

$$\begin{aligned} u(t, x) = \bar{v}(t+\sigma, x) &= \int_0^{t+\sigma} ds \int_S J(x, t+\sigma-s; dy) G[y; \bar{v}(s, \cdot)] \\ &= \int_\sigma^{t+\sigma} ds \int_S J(x, t+\sigma-s; dy) G[y; \bar{v}(s, \cdot)] \end{aligned}$$

since from the condition $\int_0 \frac{d\xi}{G^*(\xi)} = \infty$, we must have $G(0) = 0$.

Then

$$u(t, x) = \int_0^t ds \int_S J(x, t-s: dy) G[y; u(s, \cdot)],$$

and so u is a solution of (3.2). Consequently, $\bar{v} \geq u$. But $\bar{v}=0$ on $[0, \sigma] \times S$ which implies that $\bar{v}=0$ on $[0, 2\sigma] \times S$. Contradiction.

(3.12) **Theorem.** *Let* $\Gamma \in \mathscr{B}(S)$ *and* $\delta > 0$. *Set*

(3.13) $$\tau_*(s) = \inf_{\substack{x \in \Gamma \\ t \in [s, \delta]}} J(t-s, x; \Gamma), \quad 0 \leq s \leq \delta.$$

Suppose τ_* *is locally positive at* 0 (*cf. footnote* 18). *Define* $G_*(\xi) = \inf_{x \in \Gamma} G[x: \xi I_\Gamma]$, $0 \leq \xi \leq 1$ *and suppose* $\int_0 \frac{d\xi}{G_*(\xi)} < \infty$. *Then* $\bar{v} > 0$ *on* $(0, \infty) \times \Gamma$ (*i.e., explosion happens starting from* Γ).

Proof. Since τ_* is integrable, it follows from Lemma 3.3 that there exists a function u defined on $[0, \eta]$, some $\eta > 0$, such that $0 < u \leq 1$ on $(0, \eta]$ and satisfies the integral equation

$$u(t) = \int_0^t \tau_*(s) G_*[u(s)] ds, \quad 0 \leq t \leq \eta.$$

Let v_n be the defining sequence for $\bar{v}$. Then $v_0(t, x) \equiv 1 \geq u(t) I_\Gamma(x)$ on $[0, \eta] \times S$. Suppose $v_n \geq u I_\Gamma$ on $[0, \eta] \times S$. Then for $(t, x) \in [0, \eta] \times \Gamma$, we have

$$\begin{aligned} v_{n+1}(t, x) &= \int_0^t ds \int_S J(x, t-s; dy) G[y; v_n(s, \cdot)] \\ &\geq \int_0^t ds \int_S J(x, t-s; dy) G[y; u(s) I_\Gamma] \\ &\geq \int_0^t ds \int_\Gamma J(x, t-s; dy) G[y; u(s) I_\Gamma] \\ &\geq \int_0^t \tau_*(s) G_*[u(s)] = u(t). \end{aligned}$$

Consequently $\bar{v} \geq u I_\Gamma$ on $[0, \eta] \times S$. But $\bar{v}(t, x)$ is an increasing function of t and so $\bar{v} > 0$ on $(0, \infty) \times \Gamma$.

(3.14) **Corollary.** *Let* Γ *be as in Theorem* 3.12. *If there exists a* $\Lambda \in \mathscr{B}(S)$ *sucht that for every* $x \in \Lambda$ *and for every sufficiently small* $r > 0$, $\int_0^r J(x, r-s; \Gamma)$ $ds > 0$, *then under the assumptions of the above theorem,* $\bar{v} > 0$ *on* $(0, \infty) \times \Lambda$.
(3.15) REMARK. Let (T_t^0, K, π) be determined by $[X, k, \pi]$. Then $J(x, s; dy) = P^0(s, x, dy) k(y)$, where P^0 is the transition function corresponding to T_t^0. Thus

(i) if $||k||<\infty$, then τ^* is integrable on $[0, \delta]$.

(ii) if $k|\Gamma\geq k_1>0$ for some $\Gamma\in\mathcal{B}(S)$, then τ_* is locally positive at 0 if

$$\inf_{\substack{x\in\Gamma\\0\leq s\leq\delta}} P^0(s, x, \Gamma)>0\,.$$

(iii) if $||k||<\infty$ and $k|\Gamma\geq k_1>0$ for some $\Gamma\in\mathcal{B}(S)$,

then τ_* is locally positive at 0 if

$$\inf_{\substack{x\in\Gamma\\0\leq s\leq\delta}} P^X_x(X_s\in\Gamma;\, s>\eta)>0\,,$$

where η is the first hitting time of B.

4. Applications

EXAMPLE 1 (multi-type bmp).

Let $S=\{a_1,\cdots,a_N\}$. Then a bmp $\boldsymbol{X}$ on S is called an N-type bmp. In particular, let X be a (π_{ij}, b_i)-Markov chain on S, where $0<b_i<\infty$ and $0\leq\pi_{ij}\leq 1$, $\pi_{ii}=0$, $\sum_{j=1}^{N}\pi_{ij}=1$, $i, j=1,\cdots, N$; i.e., X is the Markov chain on S such that

$$b_i=(E^X_{a_i}[\sigma])^{-1} \quad\text{and}\quad \pi_{ij}=P^X_{a_i}[X_\sigma=a_j]\,,$$

where σ is the first jump time. Let k be defined on S such that $k(a_i)=k_i>0$ and $q_n(n\geq 2)$ non-negative constants such that $\sum_{n=2}^{\infty}q_n=1$. Define the stochastic kernel π on $S\times\mathcal{B}(\hat{S})$ by

$$\pi(x, d\boldsymbol{y}\cap S^n)=q_n\delta_{\{\underbrace{[x,\cdots,x]}_{n}\}}(d\boldsymbol{y})\,, \tag{4.1}$$

where we set $q_0=q_1=q_\infty=0$. Then there exists a bmp $\boldsymbol{X}$ on S with $[X, k, \pi]$ as its regular fundamental system. Theorem 2.15 says that explosion happens with probability one independent of the starting point iff $\int^1\frac{d\xi}{\xi-F[\xi]}<\infty$, $F[\xi]=\sum_{n=2}^{\infty}q_n\xi^n$.

EXAMPLE 2. (Branching diffusion with reflecting boundary)

Let D be a bounded domain in $E=\boldsymbol{R}^l$ and set $S=\bar{D}$. We assume that D has a sufficiently smooth boundary, say $C^{(2)}$. Consider the operator

$$Af(x)=\sum_{i,j=1}^{l}a_{ij}(x)\frac{\partial^2 f}{\partial x_i\partial x_j}+\sum_{i=1}^{l}b(x)\frac{\partial f}{\partial x_i} \tag{4.2}$$

where the a_{ij}, b_i are bounded and satisfy a Hölder condition on S. We also assume that A is uniformly elliptic. Then it is known (cf., Itô [6]) that there exists a conservative diffusion process X on S such that for f sufficiently smooth, $u(t, x)=E_x^X[f(X_t)]$ satisfies

$$\frac{\partial u}{\partial t}=Au \qquad \frac{\partial u}{\partial n}\Big|_{\partial D}=0\,. \tag{4.3}$$

Furthermore, X is strongly Feller and if $p(t, x, y)$ is the fundamental solution of (4.3), it is strictly positive for $t>0$ and $x, y\in S$.

Let k be a non-negative measurable function on S such that there exists constants k_i with $0<k_1\leq k\leq k_2$ and let π be as in (4.1). The bmp $\boldsymbol{X}$ on S which has $[X, k, \pi]$ as its regular fundamental system will be called a branching diffusion process with reflecting boundary. We again conclude from Theorem 2.15 that explosion happens with probability one (independent of the starting position) iff $\int^1 \frac{d\xi}{\xi-F[\xi]}<\infty$.

EXAMPLE 3.

Let X be Brownian motion on $\boldsymbol{R}$, and let X^0 be the $e^{-\varphi_t}$-subprocess, where φ_t is local time at the origin. Given a kernel π on $S\times\mathscr{B}(\hat{S})$, let $\boldsymbol{X}$ be the (T_t^0, K, π) bmp on $S=\boldsymbol{R}$. We assume, of course, that $\pi(x, S)\equiv 0$. Here

$$\begin{aligned} K(x;\, ds\,dy) &= (-d_s E_x^X[e^{-\varphi_s}])\,\delta_{\{0\}}(dy)\\ &=J(x, s;\, dy)\,ds\,. \end{aligned}$$

In particular,

$$J(0, s;\, dy)=\frac{1}{\sqrt{\pi}}\left\{\frac{1}{\sqrt{2s}}-e^{s/2}\int_{\sqrt{s/2}}^{\infty}e^{-z^2}dz\right\}\delta_{\{0\}}(dy)\,.$$

It is easy to see that for $\Gamma=\{0\}$ and sufficiently small $\delta>0$, τ_* is locally positive at zero. So by Theorem 3.12 we conclude that if $\int_0 \frac{d\xi}{G[0;\, \xi I_{\{0\}}]}<\infty$, then explosion happens starting from zero. But since $J(x, s;\, \{0\})>0$ for every $x\in S$, $s>0$ we can conclude from Corollary 3.14 that explosion happens starting from any $x\in S$.

EXAMPLE 4. (Branching diffusion with absorbing boundary).

Let A be as in (4.2) except that we assume it to be defined on all of $\boldsymbol{R}^l$ for simplicity. Let $X=(X_t, P_x)$ be the corresponding conservative diffusion on E. Let S be a bounded domain in E with sufficiently smooth boundary $B=\partial S$

(e.g., $C^{(2)}$-boundary). The absorbed process $\hat{X}=(\hat{X}_t, \hat{P}_x)$ on $S\cup\{\delta\}$[19] with δ as trap is given by

$$\hat{X}_t = \begin{cases} X_t, & t<\eta, \\ \delta, & \text{otherwise}, \end{cases}$$
$$\hat{P}_x = P_x,$$

where η is the first hitting time of B. Given a bounded, non-negative, $\mathscr{B}(S)$-measurable function k and a stochastic kernel π on $S\times\mathscr{B}(\hat{\boldsymbol{S}})$ such that $\pi(x, S)\equiv 0$, we let $\boldsymbol{X}$ be the bmp on S possessing the regular fundamental system $[X, k, \pi]$ and absorbing set B. Since this process has the property that whenever a particle hits the boundary of S it is absorbed into $\{\partial\}$ we call $\boldsymbol{X}$ a branching diffusion process with absorbing boundary. Note that X^0 is the $e^{-\int_0^t k(\hat{X}_s)\,ds}$-subprocess of $\hat{X}$, where we extend k as a function on $S\cup\{\delta\}$ by setting $k=0$ on δ.

In order to apply the results of §3 for the exploding case we must show that the conditions of Theorem 3.12 are satisfied. According to Remark 3.15, assuming $k|\Gamma\geq k_1>0$, it suffices to show that

$$\inf_{\substack{x\in\Gamma \\ 0\leq s\leq\delta}} P_x^X(X_s\in\Gamma;\, s<\eta)>0 \tag{4.4}$$

for some $\delta>0$. Since $\Gamma\in\mathscr{B}(S)$, $P_x^X(X_s\in\Gamma;\, s<\eta)=\hat{P}_x(\hat{X}_s\in\Gamma)$. Let p and $\hat{p}$ be the transition density for X and $\hat{X}$ respectively. Then we have the relation

$$\hat{p}(t, x, y) = p(t, x, y)-\int_0^t\int_B p(t-s, z, y)\,\mu_x(ds\,dz)$$

for all $t>0$, x and $y\in S$. Here $\mu_x(ds\,dz)=P_x^X(\eta\in ds, X_\eta\in dz)$. Integrating over Γ, we obtain

$$\begin{aligned}\hat{P}(t, x, \Gamma) &= P(t, x, \Gamma)-\int_0^t\int_B P(t-s, z, \Gamma)\,\mu_x(ds\,dz) \\ &\geq P(t, x, \Gamma)-P_x^X(\eta\leq t)\,.\end{aligned}$$

But we have the lower estimate for p

$$p(t, x, y)\geq M_1 t^{-l/2}\exp\left[-\alpha_1\frac{|x-y|^2}{t}\right]-M_2 t^{-(l/2)+\lambda}\exp\left[-\alpha_2\frac{|x-y|^2}{t}\right]$$

where M_1, M_2, α_1, α_2, and λ are positive constants (cf. Dynkin [1: Theorem 0.5]). Furthermore from a result of Varadhan [8] we obtain the estimate: for every compact subset $K\subset S$, there exists a $\rho>0$ such that for all $x\in K$

19. δ is an isolated point.

$$P_x^X(\eta \leq t) \leq e^{-\rho/t}$$

provided t is sufficiently small. Consequently, if Γ is such that $\bar{\Gamma} \subset S$, (4.4) will be valid if

$$\inf_{\substack{x \in \Gamma \\ 0<t\leq\delta}} \int_{\Gamma} t^{-l/2} \exp\left[-\frac{|x-y|^2}{t}\right] dy > 0 \tag{4.5}$$

for δ sufficiently small. But (4.5) is true iff there exists some positive constant κ such that for every ball B of sufficiently small radius and every $x \in \Gamma$, we have

$$m(\Gamma \cap B_x) \geq \kappa m(B)\,, \tag{4.6}$$

where B_x is the ball B centered at x and m is l-dimensional Lebesgue measure.[20] In particular, (4.6) is true if Γ is itself a ball. We shall only outline the proof of the if statement.

So suppose (4.6) is valid. For $r \in \boldsymbol{R}^1$, $x \in \boldsymbol{R}^l$, and $A \subset \boldsymbol{R}^l$ set

$$\begin{aligned} rA &= \{ry\colon y \in A\} \\ A_x &= \{y+x\colon y \in A\}\,. \end{aligned}$$

Also, let B be the unit ball centered at the origin. Consider the following.

$$\begin{aligned} \int_{\Gamma} t^{-l/2} \exp\left[-\frac{|x-y|^2}{t}\right] dy &= \int_{\frac{1}{\sqrt{t}}\Gamma_{-x}} e^{-|z|^2} dz \\ &\geq \int_{\frac{1}{\sqrt{t}}\Gamma_{-x}\cap B} e^{-|z|^2} dz \geq e^{-1} m\left(\frac{1}{\sqrt{t}}\Gamma_{-x} \cap B\right) \\ &= e^{-1} t^{-l/2} m(\Gamma \cap (\sqrt{t}\,B)_x) \\ &\geq e^{-1} t^{-l/2} \kappa m(\sqrt{t}\,B) = \kappa e^{-1} m(B) \end{aligned}$$

for $x \in \Gamma$ and sufficiently small t. Hence

$$\inf_{\substack{x \in \Gamma \\ 0<t\leq\delta}} \int_{\Gamma} t^{-l/2} \exp\left[-\frac{|x-y|^2}{t}\right] dy \geq \kappa e^{-1} m(B) > 0$$

(provided δ is sufficiently small).

Putting all this together, we obtain

(4.7) **Theorem.** *Let X be the branching diffusion process with absorbing boundary as described above. Then*

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20. The symbol B has been used to designate both a sphere in $\boldsymbol{R}^l$ and the absorbing set of a bmp. This should introduce no confusion, however.

(i) $\int_0 \frac{d\xi}{G^*(\xi)} = \infty$ *implies no explosion, where* $G^*(\xi) = \sup_{x\in S} G[x;\, \xi 1]$.

(ii) $\int_0 \frac{d\xi}{G_*(\xi)} < \infty$ *implies explosion starting from* Γ

provided Γ *is such that it satisfies* (4.6), $\bar{\Gamma}\subset S$, *and* $k|\Gamma \geq k_1 > 0$, *where* $G_*(\xi) = \inf_{x\in\Gamma} G[x;\, \xi I_\Gamma]$.

(4.8) REMARK.

1. Since $\hat{p}(t, x, y) > 0$ for all $x, y \in S$ and $t > 0$, then if explosion happens from Γ, it happens from any $x \in S$. (cf. Corollary 3.14).

2. Let $Y = (Y_t, Q_x)$ be any diffusion on some $S \subset E$ and let $\mathfrak{A}$ be its characteristic operator. Suppose that S contains a bounded smooth domain D such that $\mathfrak{A}|D = A|D$, where A is some operator on E satisfying the assumptions of (4.2). Since the absorbing diffusion process $\hat{Y}$ on D is the minimal process, we then have

$$\inf_{\substack{x\in\Gamma \\ 0\leq t\leq\delta}} Q(t, x, \Gamma) > 0$$

for any Γ with $\bar{\Gamma}\subset D$ and satisfying (4.6), all δ sufficiently small. Consequently, we can conclude that for such Γ, explosion happens from Γ for the bmp $\boldsymbol{Y}$ corresponding to the regular fundamental system $[Y, k, \pi]$, if $k|\Gamma \geq k_1 > 0$ and $\int_0 \frac{d\xi}{G_*(\xi)} < \infty$, $G_*(\xi) = \inf_{x\in\Gamma} G[x\colon \xi I_\Gamma]$.

EXAMPLE 5.

Let $S = \boldsymbol{R}$ and X be Brownian motion on S. Let $k = I_F$, where F is the following set. Take $I = [0, 1]$ and $\alpha \in (0, \frac{1}{2})$. Let E_0^1 be the middle open interval of length α removed from I. Inductively we define $E_k^1, \cdots, E_k^{2^k}$ to be the middle open intervals of length $\alpha 2^{-2k}$ removed from $I \setminus \bigcup_{\nu=0}^{k-1} \bigcup_{\mu=1}^{2^\nu} E_\nu^\mu$. Set $F = I \setminus \bigcup_{\nu=0}^{\infty} \bigcup_{\mu=1}^{2^\nu} E_\nu^\mu$. Then F is a perfect nowhere dense set of measure $(1\text{–}2\alpha)$; i.e., it is a "fat" Cantor set. We shall now show that F satisfies (4.6). At the k^{th} stage, the distance between two adjacent sets E_ν^μ, $1 \leq \mu \leq 2^\nu$, $0 \leq \nu \leq k$ is

$$d(k) = \frac{2^k - \alpha(2^{k+1} - 1)}{2^{2k+1}}.$$

Let λ be given such that $0 < \lambda \leq (1 - 2\alpha)$, and let B be the unit ball about the origin. Choose $k = k(\lambda)$ to be the first non-negative integer such that $d(k) \leq \lambda$. Then $d(k-1) > \lambda$. Moreover, if $x \in F$,

$$\frac{m(F\cap\lambda B_x)}{m(\lambda B)}\geq\frac{d(k)-\sum_{\nu=0}^{\infty}2^{\nu}\frac{\alpha}{2^{2(\nu+k+1)}}}{2d(k-1)}$$
$$\geq\frac{1}{4}(1-2\alpha)\,.$$

Consequently F satisfies (4.6).

Now, let π be a stochastic kernel on $S\times\mathcal{B}(\hat{S})$ defined by

$$\pi(x,\,dy)=p_n\delta_{\{\underbrace{[x,\cdots,x]}_{n}\}}(dy)\ \text{if}\ dy\in\mathcal{B}(S^n)\,,\quad n=0,\ 1\,,\cdots,\ +\infty\,,$$

where $0\leq p_n\leq 1$, $0=p_0=p_1=p_\infty$, and $\sum p_n=1$. If $\boldsymbol{X}$ is the bmp on S corresponding to $[X,\,k,\,\pi]$, then according to remark 4.8.2 we can say that explosion happens iff $\int^1\frac{d\xi}{1-F(\xi)}<\infty$, $F(\xi)=\sum_{n\geq 2}p_n\xi^n$. Note that splits only occur on the set F.

PRINCETON UNIVERSIY

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