Saito, T.
Osaka J. Math.
6 (1969), 303-314

ON A LATTICE ORDERED GROUPOID

Dedicated to Professor Keizo Asano for his 60th birthday

TATSUHIKO SAITO

(Received February 27, 1969)

In most cases a multiplicative partially ordered system satisfies the ditributive law: $a(b\cup c)=ab\cup ac$ (e.g. a lo-semigroup of the ideals in a ring, lo-semigroups of the normal subgroups of a group, etc.). But there are more general examples of multiplicative systems in each of which a weak distributive law: $a(b\cup c)=ab\cup ac\cup(ab)c$ is satisfied. The purpose of the present paper is to develop the theory of normal chain and regular union of a partially ordered groupoid satisfying the weak distributive law.

In §1 we define a lattice ordered groupoid with some conditions and define normal elements and a normal closure in this system and give their properties. In §2 we treat a classification of our system M and show that the classified system also satisfies the same conditions for M. In §3 we define a normal chain in our system and give some results of the chain. In §4 we consider the modularity of our system and give an extension of direct union, called a regular union, and study some results of the union. In §5 we show that the results of the preceding sections are applicable to the family of subgroups of a group and that of the ideals in commutative ring, and list the applied results.

The author is grateful to Professor K. Murata for his many valuable advices.

1. Definitions and elementary properties

Let M be a non-void set with the following five conditions (M1~M5).

M1. M is a commutative groupoid,

M2. M is a complete (upper and lower) lattice,

M3. $ab\leq a\cup b$ for all $a, b\in M$,

M4. $a(b\cup c)=ab\cup ac\cup(ab)c$, if $bc\leq b$ or $bc\leq c$.

An element b of M is said to be *normal* with respect to a, or shortly *a-normal*, if $ba\leq b$. For the greatest element e of M, an *e-normal* element of M is simply said to be normal. We shall denote by N and N_a the set of all normal elements of M and that of all a-normal elements of M respectively.

M5. $(ab)c\leq(bc)a\cup(ca)b$ holds for normal elements a, b and c.

EXAMPLES. (1) Let $\mathfrak{S}$ be a set consisting of subgroups of a group G. Then $\mathfrak{S}$ satisfies the above conditions M1, $\cdots$, M5 under the commutator-product and the set-inclusion[1]. In this case normal subgroups of G are normal elements of $\mathfrak{S}$.

(2) The set $\mathfrak{R}$ consisting of the subrings of a commutative ring R satisfies the above five conditions under the module-product and the set-inclusion. In this case the multiplication is associative, and every ideal is evidently normal.

We shall list some elementary properties of M.

Proposition 1. (1) *$a \leq b$ implies $ac \leq bc$ for all $c \in M$.*
(2) *$(ab)b \leq ab$ for all a, $b \in M$.*
(3) *$N \subseteq N_a$ for every a of M.*
(4) *$a(b \cup c) = ab \cup ac$, if a is normal and b is c-normal.*
(5) *$ab \leq a \cap b$ holds for a, $b \in N$.*
(6) *N is closed under the join, meet and multiplication.*

Proof. For (1), since $ab \leq a \cup b = b$ (by M3), by using M4 we have $bc = (a \cup b)c = ac \cup bc \cup (ac)b \geq ac$. For (2), since $bb \leq b \cup b = b$, by using M4 we have $ab = a(b \cup b) = ab \cup ab \cup (ab)b = ab \cup (ab)b$, and hence $ab \geq (ab)b$. For (3), let b be any element of N, then $b \geq be \geq ba$ (by (1)), hence we have $b \in N_a$. For (4), since a is normal, we have $ab \leq a$ (by (3)). Hence $(ab)c \leq ac$. Therefore we obtain $a(b \cup c) = ab \cup ac \cup (ab)c = ab \cup ac$. (5) is obvious. For (6), let a and b be any two elements of N. Then we have $e(a \cup b) = ea \cup eb \leq a \cup b$. Hence $a \cup b \in N$. Since $e(a \cap b) \leq ea \leq a$ and similarly $e(a \cap b) \leq b$, we have $e(a \cap b) \leq a \cap b$. Hence $a \cap b \in N$. By using 5 we have $e(ab) \leq (ea)b \cup (eb)a \leq ab \cup ab = ab$. Hence $ab \in N$.

DEFINITION 1. The greatest lower bound of the set $\{x \mid x \geq a,\ xe \leq x\}$ is called a *normal closure* of a, and is denoted by $\bar{a}$.

The normal closure has the following properties.

Proposition 2. (1) *$\bar{a}$ is normal,* (2) *$a \leq \bar{a}$,* (3) *$a \leq b$ implies $\bar{a} \leq \bar{b}$,* (4) *$\bar{a} = \bar{\bar{a}}$,* (5) *$\overline{a \cup b} = \bar{a} \cup \bar{b}$,* (6) *$\overline{ab} \leq \bar{a}\bar{b}$.*

Proof. For (1), $\bar{a}e = (\inf\{x \mid x \geq a,\ xe \leq x\})e \leq \inf\{xe \mid x \geq a,\ xe \leq x\} \leq \inf\{x \mid x \geq a,\ xe \leq x\} = \bar{a}$. (2), (3) and (4) are obvious. For (5), since $\bar{a} \cup \bar{b}$ is normal (by Proposition 1 (6)), we have $\overline{\bar{a} \cup \bar{b}} = \bar{a} \cup \bar{b}$ and hence $\overline{a \cup b} \leq \overline{\bar{a} \cup \bar{b}} = \bar{a} \cup \bar{b}$. On the other hand, since $\overline{a \cup b} \geq \bar{a}$ and $\overline{a \cup b} \geq \bar{b}$, we have $\overline{a \cup b} \geq \bar{a} \cup \bar{b}$. Hence we obtain $\overline{a \cup b} = \bar{a} \cup \bar{b}$. For (6), Since $\bar{a}\bar{b}$ is normal (by Proposition 1 (6)), we have $\overline{ab} \leq \overline{\bar{a}\bar{b}} = \bar{a}\bar{b}$.

Lemma 1. *$a \cup ab$ is b-normal for all a, $b \in M$.*

---

1) See § 5 of this paper.

Proof. By Proposition 1 (2) $a(ab)\leq ab$, and hence we have $b(a\cup ab)=ba\cup b(ab)\cup(ab)(ab)=ab\leq a\cup ab$.

Theorem 1. *If $a\cup b=e$, then ab is normal.*

Proof. By Lemma 1 $b(a\cup ab)\leq a\cup ab$, and hence we have $(ab)e=(ab)(a\cup b)=(ab)(a\cup ab\cup b)=(ab)((a\cup ab)\cup b)=(ab)(a\cup ab)\cup(ab)b\cup((ab)(a\cup ab))b$. Since $a(ab)\leq ab$, by using M4 we have $(ab)\,(a\cup ab)=a(ab)\cup(ab)(ab)\cup((ab)a)(ab)\leq ab\cup ab\cup(ab)(ab)=ab$, and hence $((ab)(a\cup ab))b\leq(ab)b\leq ab$. Therefore we obtain $(ab)e\leq ab$.

Theorem 2. *$\bar{a}=a\cup ae$ for any $a\in M$.*

Proof. Since $a\leq\bar{a}$ and $ae\leq\bar{a}e\leq\bar{a}$, we have $\bar{a}\geq a\cup ae$. On the other hand, by Lemma 1 $a\cup ae$ is normal. By the definition of the normal closure we have $\bar{a}\leq a\cup ae$. Therefore we obtain $\bar{a}=a\cup ae$.

Corollary 3. *If $a\cup b=e$, then $\bar{a}=a\cup ab$.*

Proof. By Theorem 2 and M4 we have $\bar{a}=a\cup ae=a\cup a(a\cup b)=a\cup a((a\cup ab)\cup b)=a\cup a(a\cup ab)\cup ab\cup(a(a\cup ab))b$. Since $a(ab)\leq ab$, we have $a(a\cup ab)=aa\cup a(ab)\cup(aa)(ab)\leq a\cup ab$. Since $a\cup ab$ is b-normal (by Lemma 1), we have $(a(a\cup ab))b\leq(a\cup ab)b\leq a\cup ab$. Hence $\bar{a}\leq a\cup ab$. On the other hand, since $\bar{a}=a\cup ae$ we have $\bar{a}\geq a\cup ab$. Therefore we obtain $\bar{a}=a\cup ab$.

Corollary 4. *If $a\cup b=e$, $a\geq n$ and $an\leq n$, then $n\cup ab$ is normal.*

Proof. Since e and ab are normal, we have

$$\begin{aligned}
&e(n\cup ab)=en\cup e(ab) \qquad \text{(by Proposition 1 (3))}\\
&\leq(a\cup b)n\cup ab=(a\cup(ab\cup b))n\cup ab\\
&=an\cup(ab\cup b)n\cup(an)(ab\cup b)\cup ab \qquad \text{(by M4)}\\
&\leq n\cup(ab\cup b)n\cup n(ab\cup b)\cup ab \qquad \text{(because } an\leq n)\\
&=n\cup(ab\cup b)n\cup ab=n\cup(ab)n\cup bn\cup((ab)n)b\cup ab \qquad \text{(by M4)}\\
&=n\cup ab \text{ (because } ((ab)n)b\leq(ab)b\leq ab \text{ and } nb\leq ab)\,.
\end{aligned}$$

Hence $n\cup ab$ is normal.

2. A classification of M

Let a be an arbitrary fixed element of N. We now define an equivalence relation of M by putting $u\sim v(a)$, if $u\cup a=v\cup a$, where $u,\ v\in M$. It is easily verified that this relation is stable for the join and the multiplication. That is, $\sim(a)$ is a congruence relation with respect to the join and the multiplication, which is called an *a-congruence* relation of M. The a-congruence class containing an element u is denoted by $K_a(u)$. The join and the multiplication of the classes are defined by $K_a(u)\cup K_a(v)=K_a(u\cup v)$ and $K_a(u)K_a(v)=K_a(uv)$ respectively. Then the set M/a of the classes forms a partially ordered groupoid with the following properties. (1) $K_a(u)=K_a(a)$ if and only if $u\leq a$. (2) $K_a(u)\leq K_a(v)$ if and only if $u\leq v\cup a$. In particular, $u\leq v$ implies $K_a(u)\leq K_a(v)$. (3) $K_a(e)$ and $K_a(a)$ are the greatest element and least element of M/a, respectively.

Lemma 2. (1) $\sup_\alpha \{K_a(x_\alpha)\} = K_a(\sup_\alpha \{x_\alpha\})$.

(2) $\inf_\alpha \{K_a(x_\alpha)\} = K_a(\inf_\alpha \{x_\alpha\cup a\})$.

Proof. (1) is obvious. For (2), put $b=\inf_\alpha\{x_\alpha\cup a\}$. Then, since $b\leq x_\alpha\cup a$ for all α, we have $K_a(b)\leq K_a(x_\alpha)$ (by (2) of the properties of M/a). Suppose that $K_a(c)$ is any lower bound of the set $\{K_a(x_\alpha)\}$. Then, we have $K_a(c)\leq K_a(x_\alpha)$ for all α, hence $c\leq x_\alpha\cup a$ (again by the property (2) of M/a). From this, we have $c\leq\inf_\alpha\{x_\alpha\cup a\}=b$. Thus $K_a(c)\leq K_a(b)$. That is, $K_a(b)$ is the greatest lower bound of the set $\{K_a(x_\alpha)\}$.

Theorem 5. *M/a satisfies the conditions $M1\sim M5$.*[2)]

Proof. It is evident that M/a satisfies M1, M2, M3 and M4. For M5, we begin by showing that, if $K_a(u)$ is normal in M/a then $u\cup a$ is normal in M. Let $K_a(u)$ be normal, then we have $K_a((u\cup a)e)=K_a(u\cup a)K_a(e)=K_a(u)K_a(e)\leq K_a(u)$. Hence we obtain $(u\cup a)e\leq u\cup a$. Let $K_a(u)$, $K_a(v)$ and $K_a(w)$ be normal, we have

$$\begin{aligned}
&(K_a(u)K_a(v))K_a(w) = (K_a(u\cup a)K_a(v\cup a))K_a(w\cup a)\\
&= K_a(((u\cup a)(v\cup a))(w\cup a))\\
&\leq K_a(((u\cup a)(w\cup a))(v\cup a)\cup((v\cup a)(w\cup a))(u\cup a)) \qquad \text{(by M5)}\\
&=(K_a(u)K_a(w))K_a(v)\cup(K_a(v)K_a(w))K_a(u)\,.
\end{aligned}$$

Lemma 3. $\overline{K_a(b)}=K_a(\bar{b})$ *for all* $b\in M$.

Proof. Since $K_a(\bar{b})$ is normal, we have $\overline{K_a(b)}\leq\overline{K_a(\bar{b})}=K_a(\bar{b})$. On the other hand, put $\overline{K_a(b)}=K_a(c)$ then $K_a(b)\leq K_a(c)$, and hence $b\leq c\cup a$. Since $K_a(c)$ is normal in M/a, $c\cup a$ is normal in M. Hence we have $\bar{b}\leq c\cup a$. Therefore we obtain $K_a(\bar{b})\leq K_a(c\cup a)=K_a(c)=\overline{K_a(b)}$.

REMARK. It can be proved that if M is a modular lattice then so is M/a.

Theorem 6. *If* $a\cup b=e$, *then* $\bar{a}\bar{b}=\overline{ab}$.

---

2) The normality and the normal closure of elements of M/a are similarly defined as M.

Proof. By Corollary 3, we have $K_{\overline{ab}}(\bar{a}\bar{b})=K_{\overline{ab}}(\bar{a})K_{\overline{ab}}(\bar{b})=K_{\overline{ab}}(a\cup ab)K_{\overline{ab}}(b\cup ab)=K_{\overline{ab}}(a)K_{\overline{ab}}(b)=K_{\overline{ab}}(ab)$, and hence $\bar{a}\bar{b}\leq\overline{ab}$ (because $K_{\overline{ab}}(\bar{a}\bar{b})=K_{\overline{ab}}(ab)$ is the least element in $M/\overline{ab}$). On the other hand, by Proposition 2 (6) we have $\overline{ab}\leq\bar{a}\bar{b}$.

Theorem 7. *If* $\bigcup_{i=1}^{n} a_i=e$, *then* $\overline{\bigcup_{r\neq s} a_r a_s}=\bigcup_{r\neq s}\bar{a}_r\bar{a}_s$ $(r=1, 2, \cdots, n; s=1, 2, \cdots, n)$.

Proof. First we show that $\bar{a}_1(\bar{a}_2\cup\cdots\cup\bar{a}_n)\leq\bigcup_{r\neq s}\overline{a_r a_s}$. By Theorem 6, we have $\bar{a}_1(\bar{a}_2\cup\cdots\cup\bar{a}_n)=\bar{a}_1(\overline{a_2\cup\cdots\cup a_n})=\overline{a_1(a_2\cup\cdots\cup a_n)}$. Put $a_3\cup\cdots\cup a_n=b_2$, then we have

$$\begin{aligned}
&a_1(a_2\cup b_2)=a_1(a_2\cup(a_2b_2\cup b_2))\\
&=a_1a_2\cup a_1(a_2b_2\cup b_2)\cup(a_1a_2)(a_2b_2\cup b_2) \qquad \text{(by M4)}\\
&\leq\overline{a_1a_2}\cup a_1(\overline{a_2b_2}\cup b_2)\cup(\overline{a_1a_2})(a_2b_2\cup b_2)\\
&=\overline{a_1a_2}\cup a_1(\overline{a_2b_2})\cup a_1b_2\cup(a_1(\overline{a_2b_2}))b_2\leq\overline{a_1a_2}\cup\overline{a_2b_2}\cup\overline{a_1b_2}\,.
\end{aligned}$$

Hence we have $\overline{a_1(a_2\cup b_2)}\leq\overline{\overline{a_1a_2}\cup\overline{a_2b_2}\cup\overline{a_1b_2}}=\overline{a_1a_2}\cup\overline{a_2b_2}\cup\overline{a_1b_2}$. Let us assume that $\overline{a_1(a_2\cup\cdots\cup a_n)}=\bigcup_{\substack{r\neq s\\ r=1,s=1}}^{k}\overline{a_r a_s}\cup(\bigcup_{i=1}^{k}\overline{a_i b_k})$, where $b_k=a_{k+1}\cup\cdots\cup a_n$. Since $\bigcup_{i=1}^{k}\overline{a_i b_k}=\bigcup_{i=1}^{k}\overline{a_i(a_{k+1}\cup b_{k+1})}\leq\bigcup_{i=1}^{k}(\overline{a_i a_{k+1}}\cup\overline{a_i b_{k+1}}\cup\overline{a_{k+1}b_{k+1}})=\bigcup_{i=1}^{k}\overline{a_i a_{k+1}}\cup\bigcup_{i=1}^{k+1}\overline{a_i b_{k+1}}$, we have $\overline{a_1(a_2\cup\cdots\cup a_n)}=\bigcup_{\substack{r\neq s\\ r=1,s=1}}^{k+1}\overline{a_r a_s}\cup(\bigcup_{i=1}^{k+1}\overline{a_i b_{k+1}})$. Putting $k=n-1$ we have $\overline{a_1(a_2\cup\cdots\cup a_n)}\leq\bigcup_{\substack{r\neq s\\ r=1,s=1}}^{n}\overline{a_r a_s}$. Similarly we obtain $\bar{a}_i(\bar{a}_1\cup\cdots\cup\bar{a}_{i-1}\cup\bar{a}_{i+1}\cup\cdots\cup\bar{a}_n)\leq\bigcup_{\substack{r\neq s\\ r=1,s=1}}^{n}\overline{a_r a_s}$. Since $\bigcup_{r\neq s}\bar{a}_r\bar{a}_s=\bar{a}_1(\bar{a}_2\cup\cdots\cup\bar{a}_n)\cup\cdots\cup\bar{a}_i(\bar{a}_2\cup\cdots\cup\bar{a}_{i-1}\cup\bar{a}_{i+1}\cup\cdots\cup\bar{a}_n)\cup\cdots\cup\bar{a}_n(\bar{a}_1\cup\cdots\cup\bar{a}_{n-1})$, we obtain $\bigcup_{r\neq s}\bar{a}_r\bar{a}_s\leq\bigcup_{r\neq s}\overline{a_r a_s}$. On the other hand, by using Proposition 2(6) we have $\bigcup_{r\neq s}\bar{a}_r\bar{a}_s\geq\overline{\bigcup_{r\neq s}a_r a_s}$.

3. Normal chain

In this and the next sections, we shall assume that $ao=o$ for any element a of M and the least element o of M and that $(\sup X)n=\sup(Xn)$ for any subset X of N and any element n of N.

DEFINITION 2. The chain $\{a^{(0)}, a^{(1)}, \cdots, a^{(n-1)}, a^{(n)}, \cdots\}$ with $a^{(0)}=\bar{a}$ and $a^{(n)}=a^{(n-1)}e$ is called a *minimal normal chain* of a determind by e (shortly *a-e-chain*).The chain $\{a^{[0]}, a^{[1]}, \cdots, a^{[n-1]}, a^{[n]},\cdots\}$ with $a^{[0]}=a$ and $a^{[n]}=a^{[n-1]}a$ is called an *a-a-chain*.

The following properties are immediate.

(1) $a^{(n)}$ is normal and $a^{(n)}\geq a^{(n+1)}$ for every whole number n.

(2) $a^{[n]}$ is a-normal and $a^{[n]} \geq a^{[n+1]}$ for every whole number n.

Theorem 8. $(\bigcup_{i=1}^{n} a_i)^{(p)} = \bigcup_{i=1}^{n} a_i^{(p)}$ *for any* $a \in M$.

Proof. By Proposition 2 (5) $(\bigcup_{i=1}^{n} a_i)^{(0)} = \overline{\bigcup_{i=1}^{n} a_i} = \bigcup_{i=3}^{n} \bar{a}_i = \bigcup_{i=1}^{n} a_i^{(0)}$. Hence the theorem holds for $p=0$. Let us assume that the theorem holds for $p=k-1$. Then we have $(\bigcup_{i=1}^{n} a_i)^{(k)} = (\bigcup_{i=1}^{n} a_i)^{(k-1)} e = (\bigcup_{i=1}^{n} a_i^{(k-1)})e = \bigcup_{i=1}^{n} (a_i^{(k-1)} e) = \bigcup_{i=1}^{n} a_i^{(k)}$. This completes the proof.

Theorem 9. $e^{[p-1]} a \leq a^{(p)}$ *for any* $a \in M$.

Proof. If $p=1$, this is trivial. Let us now assume that this holds for $p=k-1$. Then we have

$$
\begin{aligned}
& e^{[k-1]} a \leq (e^{[k-2]} e) \bar{a} \\
& \leq (e\bar{a}) e^{[k-2]} \cup (\bar{a} e^{[k-2]}) e \qquad \text{(by M5)} \\
& \leq a^{(1)} e^{[k-2]} \cup \bar{a}^{(k-1)} e \qquad \text{(by the assumption)} \\
& \leq (a^{(1)})^{(k-1)} \cup a^{(k)} = a^{(k)} \cup a^{(k)} = a^{(k)} .
\end{aligned}
$$

This completes the proof.

Theorem 10. *If* $\bigcup_{i=1}^{n} a_i = e$, *then* $(\bigcup_{r \neq s} a_r a_s)^{(p)} = (\bigcup_{r \neq s} \bar{a}_r \bar{a}_s)^{(p)}$.

Proof. This is easily verified by the induction on p.

DEFINITION 3. The least upper bound of the set $\{x \mid xe=o, x \in N\}$ is called an *annihilator* of e. A chain $o=c_0 \leq c_1 \leq c_2 \leq \cdots \leq c_n \leq \cdots$ is called an *upper normal chain*, if c_n is normal and $K_{c_n}(c_{n+1})$ is an annihilator of $K_{c_n}(e)$ in M/c_n for every whole number n.

Lemma 4. *Let a be an annihilator of e. Then the equality $ae=o$ holds.*

Proof. Since a is normal (by the definition of the annihilator), $ae = (\sup \{x \mid xe=o, x \in N\})e = \sup \{xe\} = o$ (by the assumption of this section).

Theorem 11. *Let $o=c_0 \leq c_1 \leq c_2 \leq \cdots \leq c_n \leq \cdots$ be an upper normal chain. Then $(c_n)^{(n)}=o$, and if $a^{(n)}=o$ for some $a \in M$ then $a \leq c_n$.*

Proof. We show that $c_n^{(k)} \leq c_{n-k}$. Since $K_{c_{n-1}}(c_n)$ is an annihilator of $K_{c_{n-1}}(e)$, we have $K_{c_{n-1}}(c_n^{(1)}) = K_{c_{n-1}}(c_n e) = K_{c_{n-1}}(c_n) K_{c_{n-1}}(e) = K_{c_{n-1}}(c_{n-1})$. Hence we obtain $c_n^{(1)} \leq c_{n-1}$. Let us assume that $c_n^{(k-1)} \leq c_{n-k+1}$. Then we have $c_n^{(k)} = c_n^{(k-1)} e \leq c_{n-k+1}^{(1)} \leq c_{n-k}$. Therefore we obtain $c_n^{(n)} \leq c_0 = o$ if $k=n$.

For the second part of the theorem, we show that $c_k \geq a^{(n-k)}$. By the assumption $a^{(n)} = a^{(n-1)} e = o$, we have $K_{c_0}(a^{(n-1)}) K_{c_0}(e) = K_{c_0}(o)$. Since $K_{c_0}(c_1)$

is an annihilator of $K_{c_0}(e)$, by the definition of the annihilator we have $K_{c_0}(c_1) \geq K_{c_0}(a^{(n-1)})$. This shows that $c_1 \geq c_0 \cup a^{(n-1)}$, and hence $c_1 \geq a^{(n-1)}$. Let us assume that $c_{k-1} \geq a^{(n-k+1)}$. Then, since $a^{(n-k+1)} = a^{(n-k)}e$ we have $K_{c_{k-1}}(c_{k-1}) \geq K_{c_{k-1}}(a^{(n-k)})K_{c_{k-1}}(e)$. Since $K_{c_{k-1}}(c_k)$ is an annihilator of $K_{c_{k-1}}(e)$, we have $K_{c_{k-1}}(c_k) \geq K_{c_{k-1}}(a^{(n-k)})$. Hence $c_k = c_k \cup c_{k-1} \geq a^{(n-k)}$. Putting $k=n$, we obtain $c_n \geq a^{(0)} = \bar{a} \geq a$, as desired.

DEFINITION 4. An element a is said to be *nilpotent* if $a^{[n]} = o$ for some positive integer n. An element a is said to be *semi-nilpotent* if there exists a finite chain $a = a_0 \geq a_1 \geq a_2 \geq \cdots \geq a_n = o$ with $a_{i-1}a_{i-1} \geq a_i$ $(i = 1, 2, \cdots, n)$.

Proposition 3. (1) *If a is nilpotent, then a is semi-nilpotent.*

(2) *If e is nilpotent, then a is nilpotent for all $a \in M$ and $K_b(e)$ is nilpotent in M/b for all $b \in N$.*

Proof. (1) If a is nilpotent, then $a = a^{[0]} \geq a^{[1]} \geq \cdots \geq a^{[n]} = o$ and $a^{[i-1]}a^{[i-1]} \leq aa^{[i-1]} = a^{[i]}$. Therefore a is semi-nilpotent.

(2) Since $a^{[i]} \leq e^{[i]}$ and $(K_b(e))^{[i]} = K_b(e^{[i]})$, this is obvious.

Theorem 12. *If $e(\neq o)$ is nilpotent, then the annihilator of e is not o.*

Proof. Suppose that $e^{[n]} = o$ for some positive integer n. Then $e^{[i]} = o$ and $e^{[i-1]} \neq o$ for some i $(1 \leq i \leq n)$. Since $e^{[i-1]}e = e^{[i]} = o$, $e^{[i-1]}$ precedes the annihilator of e.

4. Regular unions

In this section we shall assume the following condition.

M6. If $b \leq a \cup c$ and $ac \leq a$ or $ac \leq c$, then $b \leq (a \cap (b \cup c)) \cup (c \cap (a \cup b))$ for $a, b, c \in M$.

Lemma 5. *If $a \cup c = b \cup c$, $a \cap c = b \cap c$, $a \leq b$ and $ac \leq a$, then $a = b$.*

Proof. Since $b \leq a \cup c$ and $ac \leq a$, by M6 we have $b \leq (a \cap (b \cup c)) \cup (c \cap (a \cup b)) = (a \cap (a \cup c)) \cup (c \cap b) = a \cup (a \cap c) = a$. Hence we obtain $a = b$.

Lemma 6. *If a and c are b-normal and $a \leq c$, then $a \cup (b \cap c) = (a \cup b) \cap c$.*

Proof. Put $a' = a \cup (b \cap c)$, $b' = (a \cup b) \cap c$ and $c' = b$. Then, since $a(b \cap c) \leq ab \leq a$, by M4 we have $a'c' = (a \cup (b \cap c))b = ab \cup b(b \cap c) \cup (ab)(b \cap c)$. Since a and $b \cap c$ are b-normal, $ab \leq a$ and $b(b \cap c) \leq b \cap c$, and we have $a(b \cap c) \leq ab \cap ac \leq a \cap c$. Therefore $a'c' \leq a \cup (b \cap c) \cup (a \cap c) = a \cup (b \cap c) = a'$. And we have $a' \cup c' = b' \cup c'$, $a' \cap c' = b' \cap c'$ and $a' \leq b'$. Hence by using Lemma 5, we obtain $a' = b'$.

DEFINITION 5. A finite number of elements $a_1, a_2, \cdots, a_n$ of M is said to be *normally independent*, if $a_i \cap \overline{(a_1 \cup \cdots \cup a_{i-1} \cup a_{i+1} \cup \cdots \cup a_n)} = o$ for $i=1, 2, \cdots, n$.

DEFINITION 6. An element b is called a *regular union* of $a_1, a_2, \cdots, a_n$, and is denoted by $b=a_1 \cup^{(R)} a_2 \cup^{(R)} \cdots \cup^{(R)} a_n$, if $b=a_1 \cup a_2 \cup \cdots \cup a_n$ and if $a_1, a_2, \cdots, a_n$ are normally independent.

An element b is called a *k-th nilpotent union* of $a_1, a_2, \cdots, a_n$, and is denoted by $b=a_1 \cup^{(k)} a_2 \cup^{(k)} \cdots \cup^{(k)} a_n$, if $b=a_1 \cup^{(R)} a_2 \cup^{(R)} \cdots \cup^{(R)} a_n$ and if $(\bigcup_{r \neq s} \bar{a}_r \bar{a}_s)^{(k)} = o$ but $(\bigcup_{r \neq s} \bar{a}_r \bar{a}_s)^{(k-1)} \neq o$ $(r, s=1, 2, \cdots, n)$. In particular, 0–th nilpotent union is called a *direct union*.

An element b is called a *free union* of $a_1, a_2, \cdots, a_n$, and is denoted by $b=a_1 \cup^{(F)} a_2 \cup^{(F)} \cdots \cup^{(F)} a_n$, if $b=a_1 \cup^{(R)} a_2 \cup^{(R)} \cdots \cup^{(R)} a_n$ and if $(\bigcup_{r \neq s} \bar{a}_r \bar{a}_s)^{(m)} \neq o$ and $(\bigcup_{r \neq s} \bar{a}_r \bar{a}_s)^{(m)} \lneqq (\bigcup_{r \neq s} \bar{a}_r \bar{a}_s)^{(m-1)}$ for every whole number m.

Lemma 7. $\bigcup_{r \neq s} \bar{a}_r \bar{a}_s \leq \overline{a_1 \cup \cdots \cup a_{i-1} \cup a_{i+1} \cup \cdots \cup a_n}$ $(r, s=1, 2, \cdots, n)$ *for each* i $(1 \leq i \leq n)$.

Proof. Since $\bar{a}_r \bar{a}_i \leq \bar{a}_r$ and $\bar{a}_r \bar{a}_s \leq \bar{a}_r \cup \bar{a}_s$, we have $\bigcup_{r \neq s} \bar{a}_r \bar{a}_s = \bar{a}_1 \bar{a}_i \cup \cdots \cup \bar{a}_{i-1} \bar{a}_i \cup \bar{a}_{i+1} \bar{a}_i \cup \cdots \cup \bar{a}_n \bar{a}_i \cup (\bigcup_{\substack{r \neq s \\ r \neq i, s \neq i}} \bar{a}_r \bar{a}_s) \leq \bar{a}_1 \cup \cdots \cup \bar{a}_{i-1} \cup \bar{a}_{i+1} \cup \cdots \cup \bar{a}_n = \overline{a_1 \cup \cdots \cup a_{i-1} \cup a_{i+1} \cup \cdots \cup a_n}$.

Lemma 8. *If the elements* $a_1, a_2, \cdots, a_n$ *are normally independent and* $a_i \geq c_i$ $(i=1, 2, \cdots, n)$, *then* $c_1, c_2, \cdots, c_n$ *are normally independent.*

Proof. $c_i \cap \overline{(c_1 \cup \cdots \cup c_{i-1} \cup c_{i+1} \cup \cdots \cup c_n)} \leq a_i \cap \overline{(a_1 \cup \cdots \cup a_{i-1} \cup a_{i+1} \cup \cdots \cup a_n)} = o$.

Lemma 9. *If the elements* $a_1, a_2, \cdots, a_n$ *are normally independent and* $c \leq \bigcup_{r \neq s} \bar{a}_r \bar{a}_s$, *then* $K_c(a_1), K_c(a_2), \cdots, K_c(a_n)$ *are normally independent, where* $c \in N$.

Proof. We have

$$\begin{aligned}
& K_c(a_i) \cap \overline{(K_c(a_1) \cup \cdots \cup K_c(a_{i-1}) \cup K_c(a_{i+1}) \cup \cdots \cup K_c(a_n))} \\
= \ & K_c(a_i) \cap (K_c(\bar{a}_1) \cup \cdots \cup K_c(\bar{a}_{i-1}) \cup K_c(\bar{a}_{i+1}) \cup \cdots \cup K_c(\bar{a}_n)) \\
& \qquad \text{(by Proposition 2 (5) and Lemma 3)} \\
= \ & K_c((a_i \cup c) \cap (\bar{a}_1 \cup \cdots \cup \bar{a}_{i-1} \cup \bar{a}_{i+1} \cup \cdots \cup \bar{a}_n \cup c)) \qquad \text{(by Lemma 2)} \\
= \ & K_c((a_i \cup c) \cap \overline{(a_1 \cup \cdots \cup a_{i-1} \cup a_{i+1} \cup \cdots \cup a_n)}) \qquad \text{(by Lemma 7)} \\
= \ & K_c(c \cup (a_i \cap \overline{(a_1 \cup \cdots \cup a_{i-1} \cup a_{i+1} \cup \cdots \cup a_n)})) \qquad \text{(by Lemma 6)} \\
= \ & K_c(c \cup o) = K_c(c).
\end{aligned}$$

Theorem 13. *If* $\bigcup_{i=1}^{n}{}^{(R)} a_i = b$, $c_i \leq a_i$ $(i=1, 2, \cdots, n)$ *and* $\bigcup_{i=1}^{n} c_i = d$, *then* $\bigcup_{i=1}^{n}{}^{(R)} c_i = d$.

Proof. This is obvious by Lemma 8.

Theorem 14. *If* $\overset{n}{\underset{i=1}{\bigcup}}{}^{(R)}a_i=b$ *and* $\bar{c}\leq\bigcup_{r\neq s}\bar{a}_r\bar{a}_s$, *then* $\overset{n}{\underset{i=1}{\bigcup}}{}^{(R)}K_{\bar{c}}(a_i)=K_{\bar{c}}(b)$.

Proof. This is obvious by Lemma 9.

Theorem 15. *If* $a\cup^{(R)}b=e$ *and* a *is normal, then* $a\cup^{(0)}b=e$ *and* b *is normal.*

Proof. Since $\bar{a}\bar{b}\leq\bar{a}\cap\bar{b}=a\cap\bar{b}=o$, we have $a\cup^{(0)}b=e$. And we have $be\leq b(a\cup b)=ab\cup bb\cup(ab)b\leq b$ (because $ab=o$). Hence b is normal.

Theorem 16. *If* $\overset{n}{\underset{i=1}{\bigcup}}{}^{(F)}a_i=b$, *then* $\overset{n}{\underset{i=1}{\bigcup}}{}^{(k)}(K_{(\bigcup_{r\neq s}\bar{a}_r\bar{a}_s)^{(k)}}(a_i))=K_{(\bigcup_{r\neq s}a_ra_s)^{(k)}}(b)$.

Proof. Put $c=(\bigcup_{r\neq s}\bar{a}_r\bar{a}_s)^{(k)}$. Then, by Theorem 14 $\overset{n}{\underset{i=1}{\bigcup}}{}^{(R)}(K_c(a_i))=K_c(b)$. And we have $(\bigcup_{r\neq s}\overline{K_c(a_r)}\,\overline{K_c(a_s)})^{(k)}=((\cdots(\bigcup_{r\neq s}\overline{K_c(a_r)}\,\overline{K_c(a_s)})K_c(e))\cdots)K_c(e))=K_c(((\cdots(\bigcup_{r\neq s}\bar{a}_r\bar{a}_s)e)\cdots)e))=K_c((\bigcup_{r\neq s}\bar{a}_r\bar{a})_s{}^{(k)})=K_c(c)$, but we have $(\bigcup_{r\neq s}\overline{K_c(a_r)}\,\overline{K_c(a_s)})^{(k-1)}=K_c((\bigcup_{r\neq s}\bar{a}_r\bar{a}_s)^{(k-1)})\neq K(c)$. This completes the proof.

Corollary 17. *If* $\overset{n}{\underset{i=1}{\bigcup}}{}^{(R)}a_i=b$, *then* $\overset{n}{\underset{i=1}{\bigcup}}{}^{(0)}(K_{\bigcup_{r\neq s}\bar{a}_r\bar{a}_s}(a_i))=K_{\bigcup_{r\neq s}\bar{a}_r\bar{a}_s}(b)$.

Proof. This is obvious by Theorem 16.

Theorem 18. *If* $a\cup^{(k)}b=e$ *and* $a=\overset{n}{\underset{i=1}{\bigcup}}a_i$, *then* $(\bigcup_{r\neq s}\bar{a}_r\bar{a}_s)^{(p)}=(\bigcup_{r\neq s}\bar{a}_r^a\bar{a}_s^a)^{(p)}$ *for* $p\geq k-1$, *where* $\bar{a}_i^a=a_i\cup a_ia$ $(i=1, 2, \cdots, n)$.

Proof. By Corollary 4 $\bar{a}_i^a\cup ab$ is normal and $a_i\leq\bar{a}_i^a\cup ab$, and hence we have $\bar{a}_i\leq\bar{a}_i^a\cup ab$. By using Proposition 1 (4) and M4, we have $\bigcup_{r\neq s}\bar{a}_r\bar{a}_s\leq\bigcup_{r\neq s}((\bar{a}_r^a\cup ab)(\bar{a}_s^a\cup ab))=\bigcup_{r\neq s}((\bar{a}_r^a\cup ab)\bar{a}_s^a\cup(\bar{a}_r^a\cup ab)(ab))=\bigcup_{r\neq s}(\bar{a}_r^a\bar{a}_s^a\cup\bar{a}_s^a(ab)\cup(\bar{a}_r^a\bar{a}_s^a)(ab)\cup(\bar{a}_r^a\cup ab)(ab))\leq\bigcup_{r\neq s}(\bar{a}_r^a\bar{a}_s^a\cup(ab)e)=(\bigcup_{r\neq s}\bar{a}_r^a\bar{a}_s^a)\cup(ab)e$. By using Theorem 8, we have $(\bigcup_{r\neq s}\bar{a}_r^a\bar{a}_s^a)^{(p)}\leq(\bigcup_{r\neq s}\bar{a}_r\bar{a}_s)^{(p)}\leq(\bigcup_{r\neq s}\bar{a}_r^a\bar{a}_s^a\cup(ab)e)^{(p)}=(\bigcup_{r\neq s}\bar{a}_r^a\bar{a}_s^a)^{(p)}\cup((ab)e)^{(p)}=(\bigcup_{r\neq s}\bar{a}_r^k\bar{a}_s^a)^{(p)}\cup(ab)^{(p+1)}=(\bigcup_{r\neq s}\bar{a}_r^a\bar{a}_s^a)^{(p)}$ (because $(ab)^{(k)}=o$). This completes the proof.

Theorem 19. *If* $\overset{n}{\underset{i=1}{\bigcup}}{}^{(k)}a_i=e$, *then* $e^{[k-1]}(\bigcup_{r\neq s}\bar{a}_r\bar{a}_s)=o$.

Proof. By using Theorem 9, we have $e^{[k-1]}(\bigcup_{r\neq s}\bar{a}_r\bar{a}_s)\leq(\bigcup_{r\neq s}\bar{a}_r\bar{a}_s)^{(k)}=o$.

Theorem 20. *If* $a\cup^{(k)}b=e$, *then* $\bar{a}^{[k]}\bar{b}=o$

Proof. We shall show that $\bar{a}^{[p]}\bar{b}\leq(ab)^{(p)}$. Since $\bar{a}\bar{b}=\overline{ab}$ (by Theorem 6), we have $\bar{a}^{[0]}\bar{b}=\bar{a}\bar{b}=\overline{ab}=(ab)^{(0)}$. Let us assume that $a^{[p-1]}b\leq(ab)^{(p-1)}$. Then we have

$$\begin{aligned}
\bar{a}^{[p]}\bar{b} &= (\bar{a}\bar{a}^{[p-1]})\bar{b} & \\
&\leq (\bar{a}\bar{b})a^{[p-1]} \cup (\bar{a}^{[p-1]}\bar{b})\bar{a} & \text{(by using M5)} \\
&\leq (\bar{a}\bar{b})e^{[p-1]} \cup (\bar{a}\bar{b})^{(p-1)}e & \text{(by the assumption)} \\
&\leq (\bar{a}\bar{b})^{(p)} \cup (\bar{a}\bar{b})^{(p)} & \text{(by Theorem 9)} \\
&= (ab)^{(p)} . &
\end{aligned}$$

Putting $p=k$, we obtain $\bar{a}^{[k]}\bar{b} \leq (ab)^{(k)}=o$.

5. Applications

(1) Application to groups

Let G be any group and let $A_1, A_2, \cdots, A_n$ be a finite number of subgroups of G. The following notations will be used:
$[A_1, A_2]$; the commutator subgroup of A_1 and A_2,
$\{A_1, A_2, \cdots, A_n\}$; the subgroup which is generated by $A_1, A_2, \cdots, A_n$,
$\bar{A}_1$; the normal subgroup which is generated by A_1,
$\{[A_r, A_s]\}$; the subgroup which is generated by all commutator subgroups $[A_r, A_s]$ $r \neq s$, $r, s=1, 2, \cdots, n$,
$A_1^{(p)}$; the commutator subgroup $[[\cdots\overbrace{[[\bar{A}_1, G], G], \cdots], G]}^{p}$,
$A_1^{[p]}$; the commutator subgroup $[[\cdots\overbrace{[[A_1, A_1], A_1], \cdots], A_1]}^{p}$,
$A_1 \wedge A_2$; the intersection of A_1 and A_2.

Lemma 10. *Let A, B and C be any subgroups of a group G. Then A, B and C have the following properties*:

(1) $[A, B]=[B, A]$,
(2) $[A, B] \subseteq \{A, B\}$,
(3) *If* $[B, C] \subseteq B$, *then* $[A, \{B, C\}]=\{[A, B], [A, C], [[A, B], C]\}$,
(4) *If A, B and C are normal subgroups of G, then* $[[A, B], C] \subseteq [[B, C]A][[C, A]B]$,
(5) *If* $B \subseteq \{A, C\}$ *and* $[A, C] \subseteq A$, *then* $B \subseteq \{A \wedge \{B, C\}, C \wedge \{A, B\}\}$.

Proof. The proofs of (1) and (4) are well-known. For (3), since $[B, C] \subseteq B$, for any elements $b \in B$ and $c \in C$ there exists an element $b' \in B$ such that $bc=cb'$. Therefore the generator of the commutator subgroup $[\{B, C\}, A]$ can be represented in the form $[bc, a]$, where $a \in A$, $b \in B$, $c \in C$. And we have $[bc, a] =[b, a][[b, a], c][c, a]$. Hence $[bc, a]$ belongs to $\{[B, A], [C, A], [[B, A], C]\}$. Thus $[\{B, C\}, A] \subseteq \{[B, A], [C, A], [[B, A], C]\}$. On the other hand, we have $[[b, a], c]=[a, b]\,[bc, a]\,[a, c]$, hence $[[b, a], c]$ belongs to $[\{B, C\}, A]$. The generator of the commutator subgroup $[[B, A], C]$ can be represented in the form $[u_1 u_2 \cdots u_m, c]$, where u_i are of the form $[b_i, a_i]$, $a_i \in A$, $b_i \in B$ $(i=1, 2, \cdots, m)$.

Since $[u_1u_2\cdots u_m, c]=(u_1u_2\cdots u_m)^{-1}c^{-1}u_1u_2\cdots u_mc=(u_1u_2\cdots u_m)^{-1}c^{-1}u_1cc^{-1}u_2c\cdots c^{-1}u_mc$, where $u_1u_2\cdots u_m$ and $c^{-1}u_ic$ belong to $[\{B, C\}, A]$, $[u_1u_2\cdots u_m, c]$ belongs to $[\{B, C\}, A]$, and hence $[[B, A], C]\subseteq[\{B, C\}, A]$. Therefore we obtain $[\{B, C\}, A]=\{[B, A], [C, A], [[B, A], C]\}$. For (5), let b be any element of B. Then there exist two elements $a\in A$ and $c\in C$ such that $b=ac$. Since $a=bc^{-1}$ and $c=a^{-1}b$, we have $a\in A\wedge\{B, C\}$, $c\in C\wedge\{A, B\}$. Thus b belongs to $\{A\wedge\{B, C\}, C\wedge\{A, B\}\}$. Hence we have $B\subseteq\{A\wedge\{B, C\}, C\wedge\{A, B\}\}$. (2) is obvious.

By Lemma 10, the results of the preceding sections are applicable to groups. That is, the results in §§1 and 2 illustrate the properties of the subgroups (general subgroups, normal subgroups, commutator subgroups, etc.) and factor-groups of a group. The results in §§3 and 4 can be applied to the theory of solvable groups and nilpotent groups and theory of direct products, free products, regular products and k-th nilpotent products[3] of the subgroups.

We shall list briefly the applied results.

(1) If $\{A_1, A_2\}=G$, then $[A_1, A_2]=\overline{[A_1, A_2]}=[\bar{A}_1, \bar{A}_2]$.
(2) $\bar{A}=A[A, G]$ for any subgroup A of G.
(3) If $\{A_1, A_2\}=G$, then $\bar{A}_1=A_1[A_1, A_2]$.
(4) If $\{A_1, A_2\}=G$ and N is a normal subgroup of A_1 then $N[A_1, A_2]$ is a normal subgroup of G.
(5) If $\{A_1, A_2,\cdots,A_n\}=G$, then $\{\overline{[A_r, A_s]}\}=\{[\bar{A}_r, \bar{A}_s]\}$.
(6) $\{A_1, A_2, \cdots, A_n\}^{(p)}=\{A_1^{(p)}, A_2^{(p)},\ldots,A_n^{(p)}\}$.
(7) $[G^{[p-1]}, A]\subseteq A^{(p)}$ for any subgroup A of G.
(8) If $G=\{A_1, A_2,\cdots,A_n\}$, then $\{[A_r, A_s]\}^{(p)}=\{[\bar{A}_r, \bar{A}_s]\}^{(p)}$
(9) Let $Z_0=1\subseteq Z_1\subseteq Z_2\subseteq\cdots\subseteq Z_n\subseteq\cdots$ be an increasing central chain of G, where 1 is a unit group. Then $(Z_n)^{(n)}=1$, and if $A^{(n)}=1$ for some subgroup A of G then $A\subseteq Z_n$.
(10) The center of any nilpotent group is not the unit group.
(11) If G is a regular product of its subgroups A and B, and A is a normal subgroup of G, then G is a direct product of A and B, and B is a normal subgroup of G.
(12) If G is a k-th nilpotent product of its subgroups A and B and $A=\{A_1, A_2, \cdots, A_n\}$, then $\{[\bar{A}_r, \bar{A}_s]\}^{(p)}=\{[\bar{A}_r^A, \bar{A}_s^A]\}^{(p)}$, where $\bar{A}_i^A$ are the normal subgroups of A which are generated by A_i ($i=1, 2, \cdots, n$).
(13) If G is a k-th nilpotent product of its subgroups $A_1, A_2, \cdots, A_n$, then $[G^{[k-1]}, \{[A_r, A_s]\}]$ is a unit group.
(14) If G is a k-th nilpotent product of its subgroups A and B, then $[\bar{A}^{[k]}, \bar{B}]$ is a unit group.

The proofs are obvious by the following correspondences;

(3) Cf. 1.

(1) $\Leftrightarrow$ Theorems 1 and 6, (2) $\Leftrightarrow$ Theorem 2, (3) $\Leftrightarrow$ Corollary 3,
(4) $\Leftrightarrow$ Corollary 4, (5) $\Leftrightarrow$ Theorem 7, (6) $\Leftrightarrow$ Theorem 8,
(7) $\Leftrightarrow$ Theorem 9, (8) $\Leftrightarrow$ Theorem 10, (9) $\Leftrightarrow$ Theorem 11,
(10) $\Leftrightarrow$ Theorem 12, (11) $\Leftrightarrow$ Theorem 15, (12) $\Leftrightarrow$ Theorem 18,
(13) $\Leftrightarrow$ Theorem 19, (14) $\Leftrightarrow$ Theorem 20.

(2) Applicaton to commutative rings

Let R be any commutative ring with or without unity quantity and let $A_1, A_2, \cdots, A_n$ be a finite number of subrings of R. The following notations will be used:

$\{A_1, A_2, \cdots, A_n\}$; the subring which is generated by $A_1, A_2, \cdots, A_n$,
A_1A_2; the module-product of A_1 and A_2,
$\bar{A}_1$; the ideal in R which is generated by A_1,
$A^m = \overbrace{AA\cdots A}^{m}$.

It is easily verified that the set $\mathfrak{R}$ consisting of the subrings of a ring R satisfies the conditions M1$\sim$M5 in §1. Hence the results of the preceding sections can be applied to the set $\mathfrak{R}$.

We shall list briefly the appied results.

(1) If $\{A_1, A_2\} = R$, then $A_1A_2 = \overline{A_1A_2} = \bar{A}_1\bar{A}_2$.
(2) $\bar{A} = \{A, AR\}$ for any subring A of R.
(3) If $\{A_1, A_2\} = R$, then $\bar{A}_1 = \{A_1, A_1A_2\}$.
(4) If $\{A_1, A_2\} = R$ and B is an ideal of A_1, then $\{B, A_1A_2\}$ is an ideals in R.
(5) Let $\{A_rA_s\}$ be a subring which is generated by all module-products A_rA_s, $r \neq s$, $r, s = 1, 2, \cdots, n$. If $\{A_1, A_2, \cdots, A_n\} = R$, then $\{\overline{A_rA_s}\} = \{\bar{A}_r\bar{A}_s\}$.
(6) $\{A_1, A_2, \cdots, A_n\}^m = \{A_1^m, A_2^m, \cdots, A_n^m\}$.
(7) If $\{A_1, A_2, \cdots, A_n\} = R$, then $\{A_rA_s\}^m = \{\bar{A}_r\bar{A}_s\}^m$, $r \neq s$, $r, s = 1, 2, \cdots, n$.

The proofs are obvious by the following correspondences;

(1) $\Leftrightarrow$ Theorems 1 and 6, (2) $\Leftrightarrow$ Theorem 2, (3) $\Leftrightarrow$ Corollary 3,
(4) $\Leftrightarrow$ Corollary 4, (5) $\Leftrightarrow$ Theorem 7, (6) $\Leftrightarrow$ Theorem 8,
(7) $\Leftrightarrow$ Theorem 10.

UNIVERSITY OF FISHERIES, SHIMONOSEKI

References

[1] O.N. Golovin: *Nilpotent prooduct of groups*, Mat. Sb. **27** (1950), 427–454.