Goto, M. and Kobayashi, E. T.
Osaka J. Math.
6 (1969), 251-281

ON THE SUBGROUPS OF THE CENTERS OF SIMPLY CONNECTED SIMPLE LIE GROUPS — CLASSIFICATION OF SIMPLE LIE GROUPS IN THE LARGE

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(Received February 3, 1969)
(Revised June 7, 1969)

0. Introduction

A Lie group is said to be simple if its (real) Lie algebra is simple. The purpose of our paper is to classify all connected simple Lie groups. Let G be a simply connected simple Lie group and $\mathfrak{g}$ its Lie algebra. Any subgroup S of the center C of G determines a group G/S locally isomorphic to G, and conversely any connected Lie group locally isomorphic to G is determined in this manner. The problem of enumerating all the nonisomorphic connected Lie groups locally isomorphic to a given G reduces to the study of the action of the group of automorphisms of G on the center C of G. In fact we have:

Lemma. *Let C be the center of a simply connected simple Lie group G and S_1, S_2 subgroups of C. Then G/S_1 and G/S_2 are isomorphic if and only if there is an automorphism σ of G such that $\sigma S_1=S_2$.*

Proof. The "if" part is trivial. For the "only if" part we let σ' be an isomorphism from G/S_1 onto G/S_2. We denote the natural map $G\to G/S_i$ by $\pi_i (i=1, 2)$. Take open sets U_1, U_2 of G containing the identity of G such that $\pi_i | U_i (i=1, 2)$ is a homeomorphism and $\sigma'\pi_1(U_1)=\pi_2(U_2)$. Let σ be the unique homeomorphism from U_1 onto U_2 defined by $\sigma'\pi_1=\pi_2\sigma$. Then σ is a local automorphism of G, and can be extended to an automorphism of G, in virtue of the simple connectedness of G and we shall denote this extended automorphism also by σ. Since G is generated by U_1 the relation $\sigma'\pi_1=\pi_2\sigma$ remains true on G. The only if part now follows from kernel $\pi_i=S_i$ $(i=1, 2)$.
q.e.d.

The center C was studied by Cartan [1] and later by Dynkin and Oniščik [2], Sirota and Solodovnikov [8], Takeuchi [9] and Glaeser [3]. The automor-

* This work was supported in part by the National Science Foundation under grants GP 4503 and GP 6080.

phisms of the simply connected simple Lie group G are in one to one correspondence with the automorphisms of the real simple algebra $\mathfrak{g}$. These automorphisms were studied by Cartan [1] and later by Murakami [6], Takeuchi [9] and Matsumoto [5]. We shall use the results of Dynkin and Oniščik (for compact G), Sirota and Solodovnikov (for noncompact G) and Glaeser, which show that one can pick a set of representatives in a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ which maps onto the center C of simply connected G by the exponential map. These representatives of C in $\mathfrak{h}$ are given in terms of roots suitably imbedded in $\mathfrak{h}$. For an arbitrary automorphism σ of G we have $\sigma\cdot\exp=\exp\cdot d\sigma$, so in view of the fact that G is simply connected, in order to classify the subgroups S of the center C with respect to automorphisms of G, it suffices to study the effect of the automorphisms (in fact only of the outer automorphisms) of $\mathfrak{g}$ on the representatives of C in $\mathfrak{h}$. This study is almost trivial for compact G because Aut $\mathfrak{g}$/Inn $\mathfrak{g}$ is of order 1 or 2 except when $\mathfrak{g}$ is of type D_4, where Aut $\mathfrak{g}$ and Inn $\mathfrak{g}$ are the group of automorphisms and the group of inner automorphisms of $\mathfrak{g}$ respectively. For noncompact G we make use of Murakami's description of Aut $\mathfrak{g}$/Inn $\mathfrak{g}$ as orthogonal transformations on the Cartan subalgebra $\mathfrak{h}$. One should note that [8] and [6] are both based on Gantmacher's classification of real simple Lie algebras, and hence, that the choice of the same Cartan subalgebra $\mathfrak{h}$ in [8] and [6] allows the two studies to be combined here.[0]

1. Real forms of a complex simple Lie algebra

Let $\mathfrak{g}_C$ be a complex simple Lie algebra. The Killing form $(\,,)$ on $\mathfrak{g}_C$ is given by $(x, y)=\mathrm{Tr}\,(\mathrm{ad}\,x)\,(\mathrm{ad}\,y)$ for $x, y\in\mathfrak{g}_C$. Let $\mathfrak{h}_C$ be a Cartan subalgebra of $\mathfrak{g}_C$, Δ the set of all nonzero roots of $\mathfrak{g}_C$ with respect to $\mathfrak{h}_C$ and Π a system of simple roots in Δ. Let $\mathfrak{h}_0$ be the real part of $\mathfrak{h}_C$, i.e., $\mathfrak{h}_0=\{h\in\mathfrak{h}_C\,|\,\alpha(h)$ is real for all $\alpha\in\Delta\}$. Then we have $\mathfrak{h}_C=\mathfrak{h}_0\otimes C$. $(\,,)\,|\,\mathfrak{h}_0$ is positive definite, so Π and Δ can be imbedded in $\mathfrak{h}_0$ by the correspondence $\alpha\mapsto h_\alpha$ given by $(h_\alpha, h)=\alpha(h)$ for all $h\in\mathfrak{h}_0$ (and consequently for all $h\in\mathfrak{h}_C$).

Let $\mathfrak{g}_C=\mathfrak{h}_C+\sum_{\alpha\neq 0}\mathfrak{g}_\alpha$ be the eigenspace decomposition of $\mathfrak{g}_C$ with respec to $\mathfrak{h}_C$. From each $\mathfrak{g}_C$ one can choose a root vector $e_\alpha\neq 0$ so that $(e_\alpha, e_{-\alpha})=-1$ and $N_{\alpha,\beta}=N_{-\alpha,-\beta}$ hold, where $\alpha, \beta\in\Delta$. Here $N_{\alpha,\beta}$ is the structure constant given by $[e_\alpha, e_\beta]=N_{\alpha,\beta}e_{\alpha+\beta}$ if $\alpha, \beta, \alpha+\beta\in\Delta$. We note that $N_{\alpha,\beta}$ are real numbers. We also note that we have $[e_\alpha, e_{-\alpha}]=-h_\alpha$ for $\alpha\in\Delta$, by the choice of e_α.

Let $u_\alpha=e_\alpha+e_{-\alpha}$ and $v_\alpha=i(e_\alpha-e_{-\alpha})$. Then the real linear space spanned by $i\mathfrak{h}_0$, u_α, v_α $(\alpha\in\Delta)$ gives a compact form of $\mathfrak{g}_C$, and as all compact forms of $\mathfrak{g}_C$ are mapped to each other by inner automorphisms of $\mathfrak{g}_C$, one can consider

0) After this work was completed we learned about the paper A.I. Sirota: Classification of real simple Lie groups (in the large). Moskov. Gos. Ped. Inst. Ucen. Zap. No. 243 (1965), 345–365, in which the author carries out the same idea as ours described above. However, the way of obtaining the automorphisms is quite different from ours.

any compact from $\mathfrak{g}_u$ of $\mathfrak{g}_C$ to be given in this manner.

All non-compact real forms $\mathfrak{g}$ of $\mathfrak{g}_C$ are obtained from some compact form $\mathfrak{g}_u$ of $\mathfrak{g}_C$ and some involutory automorphism J of $\mathfrak{g}_u$, namely, if $\mathfrak{k}=\{x\in\mathfrak{g}_u \mid Jx=x\}$ and $\mathfrak{q}=\{x\in\mathfrak{g}_u \mid Jx=-x\}$ then $\mathfrak{g}=\mathfrak{k}+i\mathfrak{q}$ [8, §5] [4, III, §7]. We shall see next that J can be chosen in a specific manner.

Let us start with a compact form $\mathfrak{g}_u$ of $\mathfrak{g}_C$, a Cartan subalgebra $\mathfrak{h}_C$ of $\mathfrak{g}_C$ and root vectors e_α $(\alpha\in\Delta)$ so that $\mathfrak{g}_u$ is spanned by $i\mathfrak{h}_0$, u_α, v_α $(\alpha\in\Delta)$. Fix a system of simple roots $\Pi\subset\mathfrak{h}_0$. We say that two automorphisms of $\mathfrak{g}_u$ are conjugate if one of them is transformed into the other by an inner automorphism of $\mathfrak{g}_u$. An automorphism of any real form of $\mathfrak{g}_C$ can be considered as an automorphism of $\mathfrak{g}_C$. One can show that any involutory automorphism J of $\mathfrak{g}_u$ is conjugate to an automorphism of $\mathfrak{g}_u$ which leaves $\Pi\subset\mathfrak{h}_0$ invariant [6 (2), Proposition 2], so we now assume that J leaves $\Pi\subset\mathfrak{h}_0$ invariant.

In the proof of the fact that J can be chosen to leave $\Pi\subset\mathfrak{h}_0$ invariant, one starts with a maximal abelian subalgebra $\mathfrak{h}'$ of $\mathfrak{k}$ and shows that the maximal abelian subalgebra $\mathfrak{h}''$ of $\mathfrak{g}_u$ containing $\mathfrak{h}'$ is uniquely determined. Because of the compactness of $\mathfrak{g}_u$, $\mathfrak{h}''$ is mapped onto $i\mathfrak{h}_0$ by an inner automorphism S of $\mathfrak{g}_u$. Then SJS^{-1} leaves $i\mathfrak{h}_0$ invariant and induces an orthogonal transformation in $\mathfrak{h}_0$ which permutes elements of Π. So by assuming that J leaves $\Pi\subset\mathfrak{h}_0$ invariant, we are also making the assumption that $i\mathfrak{h}_0\cap\mathfrak{k}$ is maximal abelian in $\mathfrak{k}$. We make use of this fact in §4.

For involutory automorphism J of $\mathfrak{g}_u$ leaving Π invariant we define a normal automorphism J_0 of $\mathfrak{g}_C$ uniquely by the conditions i) $J_0|\mathfrak{h}_C=J|\mathfrak{h}_C$ and ii) $J_0e_\alpha=e_{J(\alpha)}$ for $\alpha\in\Pi$. Note that J_0 depends on the choice of the e_α's. From the construction of J_0 [6 (2) p. 109] one can deduce that $J_0(u_\alpha)=\pm u_{J(\alpha)}$, $J_0(v_\alpha)=\pm v_{J(\alpha)}$ for $\alpha\in\Delta$, and hence $J_0(\mathfrak{g}_u)=\mathfrak{g}_u$. Thus J_0 is an involutory automorphism of $\mathfrak{g}_u$.

Then one can still further show that an involutory automorphism J of $\mathfrak{g}_u$ leaving Π invariant is equal to $J_0 \exp(\operatorname{ad} ih_0)$, where h_0 is some element in $\mathfrak{h}_0$ such that $Jh_0=h_0$ and J_0 is the normal automorphism of $\mathfrak{g}_C$ determined as above [6 (2), Proposition 3].

2. Aut $\mathfrak{g}$/Inn $\mathfrak{g}$ as orthogonal transformations of $\mathfrak{h}_0$

The following is an outline of Murakami's results on Aut $\mathfrak{g}$/Inn $\mathfrak{g}$ [6]. Let $\mathfrak{g}_C$, $\mathfrak{h}_C$, $\Pi\subset\Delta\subset\mathfrak{h}_0$, $\{e_\alpha\}$, $\mathfrak{g}_u=\{i\mathfrak{h}_0, u_\alpha, v_\alpha\}_R$ be as in §1. Then if $\mathfrak{g}$ is a real form of $\mathfrak{g}_C$, we can assume that $\mathfrak{g}$ is determined from $\mathfrak{g}_u$ by $J=J_0\exp(\operatorname{ad} ih_0)$. In particular if $\mathfrak{g}$ is compact we let $J=$identity.

The groups of automorphisms of $\mathfrak{g}$, $\mathfrak{g}_u$ and $\mathfrak{g}_C$ are denoted by Aut $\mathfrak{g}$, Aut $\mathfrak{g}_u$ and Aut $\mathfrak{g}_C$ respectively and Aut $\mathfrak{g}$, Aut $\mathfrak{g}_u$ are considered as subgroups of Aut $\mathfrak{g}_C$. Let $\mathcal{K}$ be Aut $\mathfrak{g}\cap$ Aut $\mathfrak{g}_u$, $\mathcal{K}_0$ the connected component of $\mathcal{K}$ containing the identity and Q the subset of Aut $\mathfrak{g}$ given by $\{\exp \operatorname{ad} x \mid x\in i\mathfrak{q}\}$, where $\mathfrak{g}=\mathfrak{k}+i\mathfrak{q}$ is the decomposition determined by J. Then Aut $\mathfrak{g}=Q\mathcal{K}$ and the group Inn $\mathfrak{g}$ of inner automorphisms of $\mathfrak{g}$ is equal to $Q\mathcal{K}_0$, so Aut $\mathfrak{g}$/Inn $\mathfrak{g}$ $\cong\mathcal{K}/\mathcal{K}_0$. We note that if $\mathfrak{g}$ is compact then $Q=\{e\}$.

Let $\mathcal{K}^*$ denote the subgroup of elements of $\mathcal{K}$ leaving $\mathfrak{h}_C$ invariant. Then $\mathcal{K}=\mathcal{K}_0\mathcal{K}^*$, so if we let $\mathcal{K}_0^*=\mathcal{K}^*\cap\mathcal{K}_0$ we have $\mathcal{K}/\mathcal{K}_0\cong\mathcal{K}^*/\mathcal{K}_0^*$ and Aut $\mathfrak{g}=\mathcal{K}^*$Inn $\mathfrak{g}$.

We note that any automorphism of $\mathfrak{g}_C$ leaving $\mathfrak{h}_C$ invariant leaves Δ invariant, hence induces an orthogonal transformation on $\mathfrak{h}_0$. Hence any σ in $\mathcal{K}^*$ induces an orthogonal transformation on $\mathfrak{h}_0$. If $\sigma|\mathfrak{h}_0$ is the identity then $\sigma\in\mathcal{K}_0^*$. Letting $\mathfrak{T}$ and $\mathfrak{S}$ denote the group of orthogonal transformations on $\mathfrak{h}_0$ induced by automorphisms in $\mathcal{K}^*$ and $\mathcal{K}_0^*$ respectively, we then have $\mathcal{K}^*/\mathcal{K}_0^*\cong\mathfrak{T}/\mathfrak{S}$.

Thus we conclude that Aut $\mathfrak{g}$/Inn $\mathfrak{g}\cong\mathfrak{T}/\mathfrak{S}$.

Let $Je_\alpha=\nu_\alpha e_{J(\alpha)}$ and set

$$\begin{aligned}\Delta_1 &= \{\alpha\in\Delta\,|\,J(\alpha)=\alpha,\ \nu_\alpha=1\}\\ \Delta_2 &= \{\beta\in\Delta\,|\,J(\beta)=\beta,\ \nu_\beta=-1\}\\ \Delta_3 &= \{\xi\in\Delta\,|\,J(\xi)\neq\xi\}\end{aligned}$$

For $\xi\in\Delta_3$ if $(J(\xi),\xi)\neq 0$, then $\xi+J(\xi)\in\Delta_2$.

Theorem. (Murakami)

I. *If τ is an orthogonal transformation of $\mathfrak{h}_0$ then $\tau\in\mathfrak{T}$ if and only if*

(i) $\tau J=J\tau$

(ii) $\tau\Delta_i=\Delta_i$ $(i=1, 2, 3)$

are satisfied.

II. *For $\gamma\in\Delta$, let σ_γ be the reflection of $\mathfrak{h}_0$ defined by*

$$\sigma_\gamma(h) = h-(2\gamma(h)/\gamma(h)_\gamma)h_\gamma \qquad (h\in h_0)\,.$$

Then $\mathfrak{S}$ is generated by

(i) σ_α, $\alpha\in\Delta_1$

(ii) σ_β, *where* $\beta=\xi+J(\xi)$, $\xi\in\Delta_3$ *and* $(J(\xi),\xi)\neq 0$

(iii) $\sigma_{J(\xi)}\sigma_\xi$ *where* $\xi\in\Delta_3$ *snd* $(J(\xi),\xi)=0$.

REMARK. (1) When we apply this theorem in the following sections we consider $\tau\in\mathfrak{T}$ as a linear transformation on $\mathfrak{h}_C$.

(2) Let $J_0e_\alpha=\mu_\alpha e_{J(\alpha)}$. Then we have

$$\nu_\alpha = \mu_\alpha \exp(i\alpha(h_0))\,.$$

This is useful because in the classification of simple real forms h_0 is given explicitly in terms of $\alpha_i(h_0)(\alpha_i\in\Pi_0)$ and often J_0 is equal to the identity.

3. The compact case

Consider connected simply connected compact simple Lie group G whose Lie algebra is $\mathfrak{g}$. Let $\mathfrak{g}_C$ be the complexification of $\mathfrak{g}$. Using the notations in §1 and §2, we can assume J to be the identity and $\mathfrak{g}=\mathfrak{g}_u$ to be spanned by $i\mathfrak{h}_0$, u_α and v_α $(\alpha\in\Delta)$.

In this case $\Delta=\Delta_1$, $\Delta_2=\phi$, $\Delta_3=\phi$, hence $\mathfrak{T}$ is the set of all orthogonal transformations of $\mathfrak{h}_0$ leaving Δ invariant and $\mathfrak{S}$ is the set of orthogonal transformations generated by σ_α, $\alpha\in\Delta$. Then $\mathfrak{S}\triangleleft\mathfrak{T}$, $\mathfrak{T}=\mathfrak{P}\mathfrak{S}$, $\mathfrak{P}\cap\mathfrak{S}=\{e\}$, where $\mathfrak{P}$ is the subgroup of $\mathfrak{T}$ of all orthogonal transformations of $\mathfrak{h}_0$ leaving Π invariant (cf. Satake [7], p. 292, Corollary). Thus $\mathrm{Aut}\,\mathfrak{g}/\mathrm{Inn}\,\mathfrak{g}$ consists of two elements for $A_n(n\geqq 2)$, $D_n(n\neq 4)$, E_6, is isomorphic to the symmetric group on three letters for D_4, and consists of the identity element only for A_1, B_n, C_n, E_7, E_8, F_4 and G_2.

Consider now the Cartan subgroup H (the maximal toroidal subgroup) of G corresponding to $\mathfrak{h}=i\mathfrak{h}_0$. H contains the center C of G. The exponential map on $\mathfrak{h}$, $\exp\colon\mathfrak{h}\rightarrow H$ is epimorphic. Let $\Gamma_1=\{h\in\mathfrak{h}\,|\,\exp h\in C\}$ and $\Gamma_0=\{h\in\mathfrak{h}\,|\,\exp h=e\}$, where e is the identity of G.

Theorem. (Dynkin and Oniščik [2])

(i) *$h\in\Gamma_1\Leftrightarrow\alpha(h)\equiv 0 \pmod{2\pi i}$ for all $\alpha\in\Delta$.*

(ii) *Γ_0 is the lattice in $\mathfrak{h}$ generated by $\alpha'=(2\pi i/(h_\alpha, h_\alpha))2h_\alpha$, $\alpha\in\Delta$.*

Using this theorem a complete set of representatikes of Γ_1/Γ_0 can be found in $\mathfrak{h}$, which maps onto C by the exponential map [2].

$\sigma\mapsto d\sigma$ is an isomorphism of $\mathrm{Aut}\,G$, the group of automorphisms of G, onto $\mathrm{Aut}\,\mathfrak{g}$ by virtue of the simple connectedness of G. Restricted to $\mathrm{Inn}\,G$, the group of inner automorphisms of G, it is an isomorphism from $\mathrm{Inn}\,G$ onto $\mathrm{Inn}\,\mathfrak{g}$. The inner automorphisms leave the center C of G elementwise fixed. Two subgroups of C are considered equivalent if one is transformed onto the other by an automorphism of G. As $\mathrm{Aut}\,\mathfrak{g}/\mathrm{Inn}\,\mathfrak{g}\cong\mathfrak{T}/\mathfrak{S}\cong\mathfrak{P}$, $C\cong\Gamma_1/\Gamma_0$ and $\sigma\cdot\exp=\exp\cdot d\sigma$ the equivalence of subgroups of C is determined by the action of $\mathfrak{T}/\mathfrak{S}\cong\mathfrak{P}$ on Γ_1/Γ_0. The structure of Γ_1/Γ_0 is well known and we obtain the following table.

Type of $\mathfrak{g}_C$		$C\cong\Gamma_1/\Gamma_0$	Number of inequivalent classes of subgroups of C
A_n	$(n\geqq 1)$	Z_{n+1}	Number of divisors of $n+1$
B_n	$(n\geqq 2)$	Z_2	2
C_n	$(n\geqq 3)$	Z_2	2
D_{2k+1}	$(k\geqq 2)$	Z_4	3
D_{2k}	$(k\geqq 2)$	$Z_2\times Z_2$	3 *if* $k=2$, 4 if $k\geqq 3$
E_6		Z_3	2

E_7	Z_2	2
E_8	Z_1	1
F_4	Z_1	1
G_2	Z_1	1

Here Z_n denotes the cyclic group of order n as usual.

The subgroups of cyclic groups are characteristic, so the only case to be verified in this table is the case of $D_{2k}(k \geqq 2)$. In this case we must find the explicit structure of Γ_1/Γ_0. To find Γ_1, we set $\zeta = \sum s_j \alpha'_j$ and derive conditions on the s_j's imposed by the system of congruences $(\zeta, \alpha_j) \equiv 0 \pmod{2\pi i}$, $j=1, \cdots, n$. Then as $\Gamma_0 = \{\alpha'_1, \cdots, \alpha'_n\}_Z$ a set of representatives of nonzero elements of Γ_1/Γ_0 for D_{2k} is given as

(i) for $k=2$

$$z_1 = (\alpha'_1 + \alpha'_4)/2\,, \quad z_2 = (\alpha'_3 + \alpha'_4)/2\,, \quad z_3 = (\alpha'_1 + \alpha'_3)/2$$

(ii) for $k \geqq 3$

$$z_1 = (\alpha'_1 + \alpha'_3 + \cdots + \alpha'_{2k-3} + \alpha'_{2k-1})/2$$
$$z_2 = (\alpha'_{2k-1} + \alpha'_{2k})/2$$
$$z_3 = (\alpha'_1 + \alpha'_3 + \cdots + \alpha'_{2k-3} + \alpha'_{2k})/2$$

(cf. [2], I, 4).

For $k=2$, $\mathfrak{P}$ is the group of orthogonal transformations of $\mathfrak{h}_0$ determined by the permutations on the roots α_1, α_3, α_4. The group $\mathfrak{P}$ is transitive on $\{z_1, z_2, z_3\}$ so all subgroups of C of order 2 are equivalent. For $k \geqq 3$, $\mathfrak{P} = \{1, (2k-1, 2k)\}$, where $(2k-1, 2k)$ is the orthogonal transformation of $\mathfrak{h}_0$ determined by the interchange of the two roots α_{2k-1} and α_{2k}. The orbits of $\mathfrak{P}$ on $\{z_1, z_2, z_3\}$ are $\{z_1, z_3\}$ and $\{z_2\}$. So there are two inequivalent classes of subgroups of C of order 2.

4. The center for the noncompact case

Let G be a connected simply connected noncompact simple Lie group, whose Lie algebra is $\mathfrak{g}$. Let $\mathfrak{g}_C$ be the complexification of $\mathfrak{g}$. Using the notations in §1 and §2, we can assume $\mathfrak{g}$ to be determined from $\mathfrak{g}_u$ by $J = J_0 \exp(\mathrm{ad}\, ih_0)$. The following is an outline of Sirota and Solodovnikov's result on the center of G [8].

Let $\mathfrak{g}_0$ be the real form of $\mathfrak{g}_C$, determined from $\mathfrak{g}_u$ by J_0 and let $\mathfrak{g}_0 = \mathfrak{k}_0 + i\mathfrak{q}_0$ be its decomposition, where $\mathfrak{k}_0 = \{x \in \mathfrak{g}_u \mid J_0 x = x\}$ and $\mathfrak{q}_0 = \{x \in \mathfrak{g}_u \mid J_0 x = -x\}$. The subalgebra $\mathfrak{k}_0$ is semi-simple and $i\mathfrak{h}_0 \cap \mathfrak{k}_0$ is a maximal abelian subalgebra of $\mathfrak{k}_0$. (This depends on our choice of J which forced $i\mathfrak{h}_0 \cap \mathfrak{k}$ to be maximal abelian in $\mathfrak{k}$). $\mathfrak{k}_0 \otimes C$ has a system of simple roots $\Pi_0 \subset \mathfrak{h}_0 \cap i\mathfrak{k}$ consisting of

$$\tilde{\alpha}_i = (\alpha_i + J(\alpha_i))/2\,, \qquad \alpha_i \in \Pi$$

(cf. Lemma 3, §11, [8]).

Let $\mathfrak{g}=\mathfrak{k}+i\mathfrak{q}$ be the decomposition of $\mathfrak{g}$ determined by J. As $\mathfrak{k}$ is compact, $\mathfrak{k}$ is equal to direct sum $\mathfrak{p}\oplus\mathfrak{v}$, where the ideal $\mathfrak{p}=[\mathfrak{k}, \mathfrak{k}]$ is semi-simple compact and $\mathfrak{v}$ is the center of $\mathfrak{k}$. Any Cartan subalgebra $\mathfrak{h}'$ of $\mathfrak{k}$ is of the form $\mathfrak{h}'=\mathfrak{h}_1+\mathfrak{v}$, where $\mathfrak{h}_1$ is a Cartan subalgebra of $\mathfrak{p}$ and conversely.

Let the subgroups of G corresponding to $\mathfrak{k}$, $\mathfrak{p}$ and $\mathfrak{v}$ be denoted by K, P and V respectively. Here P is simply connected compact semi-simple and we have $K=PV$. Let H_1 be the maximal torus in P corresponding to $\mathfrak{h}_1$. Then the subgroup H' of K corresponding to $\mathfrak{h}'$ is of the form $H'=H_1V$. The center C of G is contained in K (cf. [4], p. 214, Theorem 1.1) and the center decomposes into C_1V, where C_1 is the center of P. As P is compact, $C_1\subset H_1$, so we have $C\subset H'$. The exponential map on $\mathfrak{h}'$, $\exp\colon \mathfrak{h}'\to H'$, is epimorphic. Let now $\mathfrak{h}'=i\mathfrak{h}_0\cap\mathfrak{k}$ (cf. §1), and let $\Gamma_1=\{h\in\mathfrak{h}' \mid \exp h\in C\}$ and $\Gamma_0=\{h\in\mathfrak{h}' \mid \exp h=e\}$.

Theorem. (Sirota and Solodovnikov [8])
(i) $\Gamma_1=\Gamma_1(\mathfrak{g}_u)\cap\mathfrak{h}'$,
where $\Gamma_1(\mathfrak{g}_u)=\{h\in i\mathfrak{h}_0 \mid \alpha(h)\equiv 0 \pmod{2\pi i}$ *for all* $\alpha\in\Delta\}$.
For $h\in\mathfrak{h}'=i\mathfrak{h}_0\cap\mathfrak{k}$, *we have*

$$h\in\Gamma_1 \Leftrightarrow \tilde{\alpha}_i(h)\equiv 0 \pmod{2\pi i} \textit{ for all } \tilde{\alpha}_i\in\Pi_0 .$$

(ii) $\Gamma_0=\Gamma_0(\mathfrak{p})$,
where $\Gamma_0(\mathfrak{p})=\{h\in\mathfrak{h}_1 \mid \exp h=e\}$.

This theorem enables us to pick a complete set of representatives of Γ_1/Γ_0 in $\mathfrak{h}'$ which maps onto the center C of G.

Let us consider how Aut G acts on C. As in §3, because of the simple connectedness of G, the map $\sigma\mapsto d\sigma$ gives isomorphisms Aut $G\cong$ Aut $\mathfrak{g}$ and Inn $G\cong$ Inn $\mathfrak{g}$. Furthermore we have $\sigma\cdot\exp=\exp\cdot d\sigma$ and Aut $\mathfrak{g}=\mathcal{K}^*$ Inn $\mathfrak{g}$ (§2). As Inn G acts trivially on C, in order to study the action of Aut G on C, it suffices to study the action of $\mathcal{K}^*$ on Γ_1/Γ_0. One should note that $\mathcal{K}^*$ leaves Δ, $i\mathfrak{h}_0$ and $\mathfrak{h}'$ invariant (§2), and hence leaves Γ_1 and Γ_0 invariant. Thus it suffices to consider the action of $\mathfrak{T}/\mathfrak{S}$ on Γ_1/Γ_0.

REMARK. (1) For a simple algebra $\mathfrak{g}$, if J_0 is the identity, then $\mathfrak{k}_0=\mathfrak{g}_u$. If $\mathfrak{g}_C$ is one of the classical simple algebras, then the types of $\mathfrak{g}$ for which J_0 is not the identity, are AI_n, AII_n and half of DI_n, DI_n being divided into two parts according to whether J_0 is the identity or not. For these three types, to obtain the system Δ_0 of all non zero roots of $\mathfrak{k}_0\otimes C$ one takes the system $\{\tilde{\alpha} \mid \tilde{\alpha}=(\alpha+J(\alpha))/2,\ \alpha\in\Delta\}$ and excludes those $\tilde{\alpha}$ such that $\alpha=J(\alpha)$ and $e_\alpha+J_0e_\alpha=0$. This exclusion actually occurs only for AI_n (n even), and the $\tilde{\alpha}$ to be excluded are those given by $\alpha=\pm(\lambda_i-\lambda_j)$ where $i+j=n+2$ (cf. §5, 6).

Note also that if J_0=identity, then $i\mathfrak{h}_0\cap\mathfrak{k}=i\mathfrak{h}_0$ so rank $\mathfrak{k}$=rank $\mathfrak{g}_C$.

REMARK. (2) In $\mathfrak{k}=\mathfrak{p}\oplus\mathfrak{v}$, $\dim\mathfrak{v}=1$ or 0. The system $\Delta_{\mathfrak{p}}$ of all roots of $\mathfrak{p}\otimes C$ is given by $\{\tilde{\alpha}\,|\,\tilde{\alpha}=(\alpha+J(\alpha))/2,\ \alpha\in\Delta-\Delta_2\}$ (Δ_2 was defined in §2). Using the theorem of Dynkin and Oniščik (§3), one sees that Γ_0 is generated by

$$\gamma=(2\pi i/(h_{\tilde{\alpha}},h_{\tilde{\alpha}}))2h_{\tilde{\alpha}}\,,\qquad \tilde{\alpha}\in\Delta_{\mathfrak{p}} \tag{*}$$

where $h_{\tilde{\alpha}}$ is given by $(h_{\tilde{\alpha}},h)=\tilde{\alpha}(h)$ for all $h\in\mathfrak{h}_C$.

One should note that $h_{\tilde{\alpha}}\in i\mathfrak{h}_1\subset i\mathfrak{p}$. Let $\mathfrak{p}_i\otimes C$ be a simple factor of $\mathfrak{p}\otimes C$. Actually $\mathfrak{p}\otimes C$ is simple or the direct sum of two simple algebras. (cf. §6) The Killing form $(\ ,\ )$ of $\mathfrak{g}_C$ restricted to $\mathfrak{p}_i\otimes C$ is invariant and non-degenerate, hence, is a constant multiple of the Killing form $\langle\ ,\ \rangle$ on $\mathfrak{p}_i\otimes C$. For a root $\tilde{\alpha}$ of $\mathfrak{p}_i\otimes C$ one can define $k_{\tilde{\alpha}}\in i\mathfrak{h}_1\cap\mathfrak{p}_i\otimes C$ such that $\langle k_{\tilde{\alpha}},h\rangle=\tilde{\alpha}(h)$ for all $h\in i\mathfrak{h}_1\cap\mathfrak{p}_i\otimes C$. Then we have

$$k_{\tilde{\alpha}}/\langle k_{\tilde{\alpha}},k_{\tilde{\alpha}}\rangle=h_{\tilde{\alpha}}/(h_{\tilde{\alpha}},h_{\tilde{\alpha}})$$

which justifies the use of (*) above in the application of the theorem of Dynkin and Oniscik.

The center C of G is cyclic if the Lie algebra $\mathfrak{g}$ of G is a real form of an exceptional complex simple algebra except for one real form of E_7 for which $C\cong Z_2\times Z_2$. But in this case $\mathrm{Aut}\,\mathfrak{g}/\mathrm{Inn}\,\mathfrak{g}$ consists of the identity only (cf. Takeuchi [9]) so we can conclude that the subgroups of the center C of G are characteristic if the Lie algebra $\mathfrak{g}$ of G is a real form of an exceptional complex simple algebra.

In the rest of this paper we will deal with the cases where $\mathfrak{g}$ is a real form of a classical algebra of type A, B, C and D.

5. The structure of $\mathfrak{T}/\mathfrak{S}$ for the classical simple algebras

In [6, (1)] Murakami shows how one can determine the structure of $\mathrm{Aut}\,\mathfrak{g}/\mathrm{Inn}\,\mathfrak{g}\cong\mathfrak{T}/\mathfrak{S}$ when $\mathfrak{g}_C$ is of type A, using his characterization of $\mathfrak{T}$ and $\mathfrak{S}$ given in §2. We shall employ his argument to determine the structure of $\mathfrak{T}/\mathfrak{S}$ when $\mathfrak{g}_C$ is of type B, C and D. The argument for type A is repeated here for the sake of completeness.

Let $\tilde{\mathfrak{T}}$ be the set of all orthogonal transformations of $\mathfrak{h}_0$ leaving Δ invariant and $\tilde{\mathfrak{S}}$ be the set of orthogonal transformations generated by σ_α, $\alpha\in\Delta$. Then $\tilde{\mathfrak{S}}\lhd\tilde{\mathfrak{T}}$, $\tilde{\mathfrak{T}}=\tilde{\mathfrak{P}}\tilde{\mathfrak{S}}$, $\tilde{\mathfrak{P}}\cap\tilde{\mathfrak{S}}=\{e\}$, where $\tilde{\mathfrak{P}}$ is the subgroup of $\tilde{\mathfrak{T}}$ of all orthogonal transformations of $\mathfrak{h}_0$ leaving Π invariant [7]. $\tilde{\mathfrak{S}}$ is the Weyl group of $\mathfrak{g}_C$. The structures of $\tilde{\mathfrak{T}}$ and $\tilde{\mathfrak{S}}$ for the classical simple algebras are well known. The theorems of Murakami (cf. §2) show that $\mathfrak{T}\subset\tilde{\mathfrak{T}}$ and $\mathfrak{S}\subset\tilde{\mathfrak{S}}$, and enable us to determine the coset structure of $\mathfrak{T}/\mathfrak{S}$ from the structures of $\tilde{\mathfrak{T}}$ and $\tilde{\mathfrak{S}}$.

In what follows, the dual space of $\mathfrak{h}_0$ is identified with $\mathfrak{h}_0$ via $(\ ,\ )|\mathfrak{h}_0$ and most of the time we use the same symbol for an element in $\mathfrak{h}_0$ and the corresponding element in the dual space of $\mathfrak{h}_0$.

5.1. If $\mathfrak{g}_C$ is of type A_n, a system of simple roots Π is given by

$$\alpha_1 = \lambda_1 - \lambda_2,\ \alpha_2 = \lambda_2 - \lambda_3, \cdots, \alpha_n = \lambda_n - \lambda_{n+1}$$

and a system of roots Δ is given by

$$\pm(\lambda_i - \lambda_j) = \pm(\alpha_i + \cdots + \alpha_{j-1}) \qquad (i<j).$$

5.1.1. If $\mathfrak{g}$ is of type AI_n, n odd, $n \geqq 3$, then one can let $J_0 \neq E$, $\alpha_{(n+1)/2}(h_0) = \pi$, and $\alpha_i(h_0) = 0$ for $i \neq (n+1)/2$. We then have[1]

$$\begin{aligned} J_0(\lambda_i - \lambda_j) &= \lambda_{n+2-j} - \lambda_{n+2-i} \qquad (i<j) \\ J_0(e_{\lambda_i - \lambda_j}) &= (-1)^{i+j+1} e_{J_0(\lambda_i - \lambda_j)} \end{aligned}$$

from which we derive

$$J_0(\lambda_i - \lambda_j) = \lambda_i - \lambda_j \Leftrightarrow i+j = n+2.$$

Remembering that $n+2$ is odd, we thus have

$$\begin{aligned} \Delta_1 &= \text{empty} \\ \Delta_2 &= \{\pm(\lambda_i - \lambda_j) \mid i+j = n+2\} \\ \Delta_3 &= \{\pm(\lambda_i - \lambda_j) \mid i+j \neq n+2\}. \end{aligned}$$

For $\lambda_i - \lambda_j \in \Delta_3$ we note that i, j, $n+2-i$, $n+2-j$ are all distinct and hence $(\lambda_i - \lambda_j, J_0(\lambda_i - \lambda_j)) = 0$. Thus by Murakami's theorem in § 2 $\mathfrak{S}$ is generated by $\sigma_{J_0(\lambda_i - \lambda_j)}\sigma_{\lambda_i - \lambda_j}$ where $\lambda_i - \lambda_j \in \Delta_3$. These $\sigma_{J_0(\lambda_i - \lambda_j)}\sigma_{\lambda_i - \lambda_j}$ interchange λ_i and λ_j, λ_{n+2-i} and λ_{n+2-j} but leave λ_k fixed, where $k \neq i, j, n+2-i, n+2-j$. We have $\mathfrak{T} = \tilde{\mathfrak{S}} + J_0 \tilde{\mathfrak{S}}$. We know that $\tilde{\mathfrak{S}} \cong S$, where S is the symmetric group on $n+1$ letters, the isomorphism $\psi: \tilde{\mathfrak{S}} \to S$ being given by $s(\lambda_i) = \lambda_{\psi s(i)}$ for $s \in \tilde{\mathfrak{S}}$ and all i. We shall identify $\tilde{\mathfrak{S}}$ with S and write $s(i)$ for $\psi s(i)$. As $-J_0 \in \tilde{\mathfrak{S}}$ we can write $\mathfrak{T} = \tilde{\mathfrak{S}} + (-1)\tilde{\mathfrak{S}}$. Note that $-1 \in \mathfrak{T}$. For $s \in \tilde{\mathfrak{S}}$, we have

$$s \in \mathfrak{T} \Leftrightarrow sJ_0 = J_0 s \Leftrightarrow s(i) + s(n+2-i) = n+2 \qquad \text{for all } i,$$

From this we see that $\mathfrak{T} \cap \tilde{\mathfrak{S}} = \mathfrak{S} + \sigma_{\lambda_a - \lambda_{n+2-a}}\mathfrak{S}$, for any $1 \leqq a \leqq n+1$.[2] Thus we have

$$\mathfrak{T} = \mathfrak{S} + \sigma_{\lambda_a - \lambda_{a+2-n}}\mathfrak{S} + (-1)\mathfrak{S} + \sigma_{\lambda_a - \lambda_{n+2-a}}(-1)\mathfrak{S}.$$

5.1.2. If $\mathfrak{g}$ is of type AI_n, n even, $n \geqq 2$, then we can let $J_0 \neq E$ and $h_0 = 0$. Using what was said for $J_0 \neq E$ in 5.1.1 and remembering that n is even and $h_0 = 0$ now, we have

1) The derivation of the second equation requires computation similar to that in 5.4.2.

2) cf. Appendix

$$\begin{aligned} \Delta_1 &= \text{empty} \\ \Delta_2 &= \{\pm(\lambda_i - \lambda_j) \mid i+j = n+2\} \\ \Delta_3 &= \{\pm(\lambda_i - \lambda_j) \mid i+j \neq n+2\}\,. \end{aligned}$$

For $\lambda_i - \lambda_j \in \Delta_3$, we have $(\lambda_i - \lambda_j,\ J_0(\lambda_i - \lambda_j)) = -(\lambda_i,\ \lambda_{n+2-i}) - (\lambda_j,\ \lambda_{n+2-j})$, hence

$$(\lambda_i - \lambda_j,\ J_0(\lambda_i - \lambda_j)) \begin{cases} = 0 & \text{if } i,\, j\ (n+2)/2 \\ \neq 0 & \text{if } i \text{ or } j = (n+2)/2\,. \end{cases}$$

We have $(\lambda_i - \lambda_{(n+2)/2}) + J_0(\lambda_i - \lambda_{(n+2)/2}) = \lambda_i - \lambda_{n+2-i}$ for all i. Hence $\mathfrak{S}$ is generated by $\sigma_{\lambda_i - \lambda_{n+2-i}}$ $(i \leqq n/2)$ and $\sigma_{\lambda_i - \lambda_j} \sigma_{J_0(\lambda_i - \lambda_j)}$ $(i,\ j \neq (n+2)/2$ and $i+j \neq n+2)$. We have $\tilde{\mathfrak{T}} = \tilde{\mathfrak{S}} + J_0 \tilde{\mathfrak{S}} = \tilde{\mathfrak{S}} + (-1)\tilde{\mathfrak{S}}$. Note that $-1 \in \mathfrak{T}$. For $s \in \tilde{\mathfrak{S}} \cong S$, we have

$$s \in \mathfrak{T} \Leftrightarrow sJ_0 = J_0 s \Leftrightarrow s(i) + s(n+2-i) = n+2 \qquad \text{for all } i\,,$$

thus $\mathfrak{T} \cap \tilde{\mathfrak{S}} = \mathfrak{S}^{2)}$ and $\mathfrak{T} = \mathfrak{S} + (-1)\mathfrak{S}$.

5.1.3. If $\mathfrak{g}$ is of type AII_n, n odd, $n \geqq 3$, then we can let $J_0 \neq E$ and $h_0 = 0$. Using what was said for $J_0 \neq E$ in 5.1.1, and remembering that n is odd and $h_0 = 0$ now, we see that

$$\begin{aligned} \Delta_1 &= \{\pm(\lambda_i - \lambda_j) \mid i+j = n+2\} \\ \Delta_2 &= \text{empty} \\ \Delta_3 &= \{\pm(\lambda_i - \lambda_j) \mid i+j \neq n+2\} \end{aligned}$$

and $(\lambda_i - \lambda_j,\ J_0(\lambda_i - \lambda_j)) = 0$ for $\lambda_i - \lambda_j \in \Delta_3$. $\mathfrak{S}$ is generated by $\sigma_{\lambda_i - \lambda_j}$ $(i+j = n+2)$ and $\sigma_{\lambda_i - \lambda_j} \sigma_{J_0(\lambda_i - \lambda_j)}$ $(i+j \neq n+2)$. We have $\tilde{\mathfrak{T}} = \tilde{\mathfrak{S}} + J_0 \tilde{\mathfrak{S}} = \tilde{\mathfrak{S}} + (-1)\tilde{\mathfrak{S}}$ and $-1 \in \mathfrak{T}$ as before. For $s \in \tilde{\mathfrak{S}} \cong S$, we have again

$$s \in \mathfrak{T} \Leftrightarrow sJ_0 = J_0 s \Leftrightarrow s(i) + s(n+2-i) = n+2 \qquad \text{for all } i\,,$$

so as before we again have $\mathfrak{T} \cap \tilde{\mathfrak{S}} = \mathfrak{S}^{2)}$ and $\mathfrak{T} = \mathfrak{S} + (-1)\mathfrak{S}$.

5.1.4. If $\mathfrak{g}$ is of type $AIII_n$, $n \geqq 1$, then we can let $J_0 = E$, $\alpha_m(h_0) = \pi$, $\alpha_i(h_0) = 0$ for $i \neq m$. For each m, $1 \leqq m \leqq [(n+1)/2]$, we have a real form of $\mathfrak{g}_C$ of type A_n. Distinct values of m determine nonisomorphic real forms. Using $\nu_\alpha = \mu_\alpha \exp(i\alpha_0(h_0))$ (cf. §2), we see that

$$\begin{aligned} \Delta_1 &= \{\pm(\lambda_i - \lambda_j) \mid i < j \leqq m \text{ or } m < i < j\} \\ \Delta_2 &= \{\pm(\lambda_i - \lambda_j) \mid i \leqq m < j\} \\ \Delta_3 &= \text{empty.} \end{aligned}$$

We have $\mathfrak{S} = \mathfrak{S}_1 \times \mathfrak{S}_2$. Here, if $m \neq 1$ and $n \neq 1$, then $\mathfrak{S}_1$ is generated by $\sigma_{\lambda_i - \lambda_j}$, $i < j \leqq m$, and is isomorphic to the symmetric group on m letters, $1, \cdots, m$, while, if $n \neq 1$, then $\mathfrak{S}_2$ is generated by $\sigma_{\lambda_i - \lambda_j}$, $m < i < j$, and is isomorphic to the symmetric group on $n-m+1$ letters, $m+1, \cdots, n+1$. The isomorphisms $\psi_r (r=1, 2)$ are given by $s(\lambda_i)=\lambda_{\psi_r s(i)}$ for $s \in \mathfrak{S}_r$. For $m=1$, $\mathfrak{S}_1=\{1\}$. For $n=1$, $\mathfrak{S}_1=\mathfrak{S}_2=\{1\}$. For $n \neq 1$, we have $\mathfrak{T}=\tilde{\mathfrak{S}}+J_0\tilde{\mathfrak{S}}=\tilde{\mathfrak{S}}+(-1)\tilde{\mathfrak{S}}$ and $-1 \in \mathfrak{T}$. For $s \in \tilde{\mathfrak{S}} \cong S$, we have

$$s \in \mathfrak{T} \Leftrightarrow \begin{cases} s \in \mathfrak{S}_1 \times \mathfrak{S}_2 & \text{if } n+1 \neq 2m \\ s \in (\mathfrak{S}_1 \times \mathfrak{S}_2)+\sigma_{\pi_0}(\mathfrak{S}_1 \times \mathfrak{S}_2) & \text{if } n+1=2m \end{cases}$$

where $\sigma_{\pi_0}=\sigma_{\lambda_1-\lambda_{m+1}}\sigma_{\lambda_2-\lambda_{m+2}}\cdots\sigma_{\lambda_m-\lambda_{n+1}}$. Hence

$$\mathfrak{T} = \begin{cases} \mathfrak{S}+(-1)\mathfrak{S} & \text{if } n+1 \neq 2m \\ \mathfrak{S}+(-1)\mathfrak{S}+\sigma_{\pi_0}\mathfrak{S}+\sigma_{\pi_0}(-1)\mathfrak{S} & \text{if } n+1=2m . \end{cases}$$

For $n=1$, $\mathfrak{T}=\tilde{\mathfrak{S}} \cong S=$symmetric group on two letters, and $\mathfrak{S}=\{1\}$. Thus

$$\mathfrak{T} = \{1, \sigma_{\lambda_1-\lambda_2}\} .$$

5.2. If $\mathfrak{g}_C$ is of type B_n a system of simple roots Π is given by

$$\alpha_1 = \lambda_1-\lambda_2,\ \alpha_2 = \lambda_2-\lambda_3,\ \cdots,\ \alpha_{n-1} = \lambda_{n-1}-\lambda_n,\ \alpha_n = \lambda_n$$

and a system of roots Δ is given by

$$\begin{aligned} &\pm(\lambda_i-\lambda_j) = \pm(\alpha_i+\cdots+\alpha_{j-1}) && (i<j) \\ &\pm\lambda_i = \pm((\lambda_i-\lambda_n)+\lambda_n) = \pm(\alpha_i+\cdots+\alpha_{n-1}+\alpha_n) \\ &\pm(\lambda_i+\lambda_j) = \pm(\lambda_i-\lambda_n)+(\lambda_j-\lambda_n)+2\lambda_n) && (i<j) \\ &\qquad = \pm((\alpha_i+\cdots+\alpha_n)+(\alpha_j+\cdots+\alpha_n)) . \end{aligned}$$

5.2.1. If $\mathfrak{g}$ is of type BI_n, $n \geqq 2$, then one can let $J_0=E$, $\alpha_m(h_0)=\pi$, $\alpha_i(h_0)=0$ for $i \neq m$. For each m, $1 \leqq m \leqq n$, we have a real form of $\mathfrak{g}_C$ of type B_n. Distinct values of m determine nonisomorphic real forms. We see that

$$\begin{aligned} &\Delta_1 = \{\pm(\lambda_i-\lambda_j),\ \pm(\lambda_i+\lambda_j) \text{ for } i<j \leqq m \text{ or } m<i<j \text{ and } \pm\lambda_i \text{ for } i>m\} \\ &\Delta_2 = \Delta-\Delta_1 \\ &\Delta_3 = \text{empty} \end{aligned}$$

Hence $\mathfrak{S}=\mathfrak{D}_1{}^+\mathfrak{S}_1 \times \mathfrak{D}_2\mathfrak{S}_2$, where $\mathfrak{S}_1$ and $\mathfrak{S}_2$ are as in 5.1.4, except that the indices for $\mathfrak{S}_2$ run from $m+1$ to n now, and where $\mathfrak{D}_1{}^+=\{d \mid d(\lambda_i)=\varepsilon_i\lambda_i$, $\varepsilon_i=\pm 1$ for $i \leqq m$, $\varepsilon_i=1$ for $m<i$, $\Pi\varepsilon_i=1\}$ and $\mathfrak{D}_2=\{d \mid d(\lambda_i)=\varepsilon_i\lambda_i$, $\varepsilon_i=1$ for $i \leqq m$, $\varepsilon_i=\pm 1$ for $m<i\}$. For $m=n-1$, $\mathfrak{S}_2=\{1\}$, for $m=n$, $\mathfrak{D}_2=\mathfrak{S}_2=\{1\}$. For $m=1$, $\mathfrak{D}_1{}^+=\mathfrak{S}_1=\{1\}$. We have $\mathfrak{T}=\tilde{\mathfrak{S}}=\tilde{\mathfrak{D}}\tilde{\mathfrak{S}}_0$, where $\tilde{\mathfrak{D}}$ is the subgroup of the elements d such that $d(\lambda_i)=\varepsilon_i\lambda_i$, $\varepsilon_i=\pm 1$, $\tilde{\mathfrak{S}}_0$ is the subgroup generated by $\sigma_{\lambda_i-\lambda_j}$ and is isomorphic to the symmetric group on n letters. We have $\tilde{\mathfrak{D}}\Delta_1 \subset \Delta_1$ so $\tilde{\mathfrak{D}} \subset \mathfrak{T}$ and $\tilde{\mathfrak{S}}_0 \cap \mathfrak{T}=\mathfrak{S}_1 \times \mathfrak{S}_2$. Hence

$$\mathfrak{T} = \mathfrak{S}+\rho_1\mathfrak{S}$$

where $\rho_k = d \in \tilde{\mathfrak{D}}$ such that $d(\lambda_k) = -\lambda_k$ and $d(\lambda_i) = \lambda_i$ for $i \neq k$.

5.3. If $\mathfrak{g}_C$ is of type C_n a system of simple roots Π is given by

$$\alpha_1 = \lambda_1 - \lambda_2, \cdots, \alpha_{n-1} = \lambda_{n-1} - \lambda_n, \alpha_n = 2\lambda_n$$

and a system of roots Δ is given by

$$\begin{aligned} \pm(\lambda_i - \lambda_j) &= \pm(\alpha_i + \cdots + \alpha_{j-1}) & (i<j) \\ \pm(\lambda_i + \lambda_j) &= \pm((\lambda_i - \lambda_n) + (\lambda_j - \lambda_n) + 2\lambda_n) & \\ &= \pm((\alpha_i + \cdots + \alpha_{n-1}) + (\alpha_j + \cdots + \alpha_{n-1}) + \alpha_n) & (i=j \text{ allowed here}) \end{aligned}$$

5.3.1. If $\mathfrak{g}$ is of type CI_n, $n \geqq 3$, then we can let $J_0 = E$, $\alpha_n(h_0) = \pi$, $\alpha_i(h_0) = 0$ for $i \neq n$. Then we have

$$\begin{aligned} \Delta_1 &= \{\pm(\lambda_i - \lambda_j)\} \\ \Delta_2 &= \{\pm(\lambda_i + \lambda_j)\} \\ \Delta_3 &= \text{empty} \end{aligned}$$

We see that $\mathfrak{S}$ is isomorphic to the symmetric group on n letters. We have $\mathfrak{T} = \tilde{\mathfrak{S}} = \tilde{\mathfrak{D}}\tilde{\mathfrak{S}}_0$ and $\mathfrak{T} \cap \tilde{\mathfrak{D}} = \{1, -1\}$. Hence $\mathfrak{T} = \mathfrak{S} + (-1)\mathfrak{S}$.

5.3.2. If $\mathfrak{g}_n$ is of type CII_n, $n \geqq 3$, then we can let $J_0 = E$, $\alpha_m(h_0) = \pi$, $\alpha_i(h_0) = 0$ for $i \neq m$. For each m, $1 \leqq m \leqq [n/2]$, we have a real form of $\mathfrak{g}_C$ of type C_n. Distinct values of m determine nonisomorphic real forms. We see that

$$\begin{aligned} \Delta_1 &= \{\pm(\lambda_i - \lambda_j), \pm(\lambda_i + \lambda_j) \mid i \leqq j \leqq m \text{ or } m \leqq i \leqq j\} \\ \Delta_2 &= \Delta - \Delta_1 \\ \Delta_3 &= \text{empty} \end{aligned}$$

Hence we get $\mathfrak{S} = \mathfrak{D}_1\mathfrak{S}_1 \times \mathfrak{D}_2\mathfrak{S}_2 = \tilde{\mathfrak{D}}(\mathfrak{S}_1 \times \mathfrak{S}_2)$, where the subgroups are as in 5.2.1. except that the elements of $\mathfrak{D}_1$ do not have the restriction $\Pi\varepsilon_i = 1$, which those of $\mathfrak{D}_1^+$ have. For $m=1$ we let $\mathfrak{D}_1 = \mathfrak{S}_1 = \{1\}$. Here $\mathfrak{T} = \tilde{\mathfrak{S}} = \tilde{\mathfrak{D}}\tilde{\mathfrak{S}}_0$ and $\tilde{\mathfrak{D}} \subset \mathfrak{T}$ so we have

$$\mathfrak{T} \cap \tilde{\mathfrak{S}}_0 = \begin{cases} \mathfrak{S}_2 \times \mathfrak{S}_1 & \text{if } n \neq 2m \\ (\mathfrak{S}_1 \times \mathfrak{S}_2) + \sigma_{\pi_0}(\mathfrak{S}_1 \times \mathfrak{S}_2) & \text{if } n = 2m, \end{cases}$$

where $\sigma_{\pi_0} = \sigma_{\lambda_1 - \lambda_{m+1}} \sigma_{\lambda_2 - \lambda_{m+2}} \cdots \sigma_{\lambda_m - \lambda_n}$. Hence

$$\mathfrak{T} = \begin{cases} \mathfrak{S} & \text{if } n \neq 2m \\ \mathfrak{S} + \sigma_{\pi_0}\mathfrak{S} & \text{if } n = 2m. \end{cases}$$

5.4. If $\mathfrak{g}_C$ is of type D_n a system of simple roots Π is given by

$$\alpha_1 = \lambda_1 - \lambda_2, \alpha_2 = \lambda_2 - \lambda_3, \cdots, \alpha_{n-1} = \lambda_{n-1} - \lambda_n, \alpha_n = \lambda_{n-1} + \lambda_n$$

and a system of roots Δ is given by

$$\begin{aligned} \pm(\lambda_i-\lambda_j) &= \pm(\alpha_i+\cdots+\alpha_{j-1}) && (i<j) \\ \pm(\lambda_i+\lambda_j) &= \pm((\lambda_1-\lambda_{n-1})+(\lambda_j-\lambda_n)+(\lambda_{n-1}+\lambda_n)) \\ &= \pm((\alpha_i+\cdots+\alpha_{n-2})+(\alpha_j+\cdots+\alpha_{n-1})+\alpha_n) && (i<j) \end{aligned}$$

5.4.1. If $\mathfrak{g}$ is of type DI_n, $n\geqq 4$, and $\dot{J}_0=E$ then we can let $\alpha_m(h_0)=\pi$, $\alpha_i(h_0)=0$ for $i\neq m$. For each m, $1\leqq m\leqq[n/2]$, we have a real form of $\mathfrak{g}_C$ of type D_n. Distinct values of m determine nonisomorphic real forms. We see that

$$\begin{aligned} \Delta_1 &= \{\pm(\lambda_i-\lambda_j),\ \pm(\lambda_i+\lambda_j) \mid \geqq i<jm \text{ or } m<i<j\} \\ \Delta_2 &= \Delta-\Delta_1 \\ \Delta_3 &= \text{empty} \end{aligned}$$

Hence as in 5.2.1 we get $\mathfrak{S}=\mathfrak{D}_1{}^+\mathfrak{S}_1\times\mathfrak{D}_2{}^+\mathfrak{S}_2$, where $\mathfrak{D}_2{}^+$ is the subgroup of $\mathfrak{D}_2$ of elements satisfying $\Pi\varepsilon_i=1$. If $m=1$, we let $\mathfrak{D}_1{}^+=\mathfrak{S}_1=\{1\}$.

(i) For $n\geqq 5$ we have $\tilde{\mathfrak{T}}=\tilde{\mathfrak{S}}+\rho_n\tilde{\mathfrak{S}}$, where the notation ρ_n was introduced in 5.2.1. Furthermore $\tilde{\mathfrak{S}}=\tilde{\mathfrak{D}}^+\tilde{\mathfrak{S}}_0$, where $\tilde{\mathfrak{D}}^+$ is the subgroup of $\tilde{\mathfrak{D}}$ of elements satisfying $\Pi\varepsilon_i=1$. Thus $\tilde{\mathfrak{T}}=\tilde{\mathfrak{D}}\tilde{\mathfrak{S}}_0$. As $\tilde{\mathfrak{D}}\subset\mathfrak{T}$, to determine $\mathfrak{T}$ we only have to consider $\mathfrak{T}\cap\tilde{\mathfrak{S}}$ and see that

$$\mathfrak{T}\cap\tilde{\mathfrak{S}} = \begin{cases} \mathfrak{S}_1\times\mathfrak{S}_2 & \text{if } n\neq 2m \\ (\mathfrak{S}_1\times\mathfrak{S}_2)+\sigma_{\pi_0}(\mathfrak{S}_1\times\mathfrak{S}_2) & \text{if } n=2m \end{cases}$$

where σ_{π_0} was given in 5.3.2. Hence

$$\mathfrak{T} = \begin{cases} \mathfrak{S}+\rho_1\mathfrak{S}+\rho_n\mathfrak{S}+\rho_1\rho_n\mathfrak{S} & \text{if } n\neq 2m \\ \mathfrak{S}+\rho_1\mathfrak{S}+\rho_n\mathfrak{S}+\rho_1\rho_n\mathfrak{S}+\sigma_{\pi_0}\mathfrak{S}+\sigma_{\pi_0}\rho_1\mathfrak{S}+\sigma_{\pi_0}\rho_n\mathfrak{S}+\sigma_{\pi_0}\rho_1\rho_n\mathfrak{S} & \text{if } n=2m\,. \end{cases}$$

(ii) For $n=4$ we have $\tilde{\mathfrak{T}}=S_{(3)}\tilde{\mathfrak{S}}$, where $S_{(3)}$ is the group consisting of elements keeping α_2 fixed and permuting α_1, α_3, α_4. We have $\tilde{\mathfrak{S}}=\tilde{\mathfrak{D}}^+\tilde{\mathfrak{S}}_0$ as above. We consider the cases $m=1$ and $m=2$ separately.

(a) If $m=1$, then

$$\Delta_1 = \{\pm\alpha_2,\ \pm(\alpha_2+\alpha_3),\ \pm\alpha_3,\ \pm(\alpha_2+\alpha_3+\alpha_4),\ \pm(\alpha_2+\alpha_4),\ \pm\alpha_4\}\,.$$

Let $d\in\tilde{\mathfrak{D}}^+$, $s\in\tilde{\mathfrak{S}}_0$ and suppose

$$ds\Delta_1 = \{\pm(\lambda_1-\lambda_i),\ \pm(\lambda_1-\lambda_j),\ \pm(\lambda_i-\lambda_j),\ \pm(\lambda_1+\lambda_i),\ \pm(\lambda_1+\lambda_j),\ \pm(\lambda_i+\lambda_j)\}\,.$$

Note that $\lambda_1+\lambda_2=\alpha_1+\alpha_2+\alpha_2+\alpha_3+\alpha_4$ and $\lambda_1+\lambda_3=\alpha_1+\alpha_2+\alpha_3+\alpha_4$. As $ds\Delta_1$ contains $\lambda_1+\lambda_2$ and/or $\lambda_1+\lambda_3$, and as all $\sigma\in S_{(3)}$ leave both of these fixed, we have $\sigma ds\Delta_1\neq\Delta_1$ for all $\sigma\in S_{(3)}$. Hence if $\sigma ds\Delta_1=\Delta_1$ for $\sigma\in S_{(3)}$, $d\in\tilde{\mathfrak{D}}^+$ and $s\in\tilde{\mathfrak{S}}_0$, then $s\in\mathfrak{S}_2$ and $\sigma=1$ or $\sigma(\alpha_3, \alpha_4)$, where by $\sigma(\alpha_i, \alpha_j)$ we shall denote the element of $S_{(3)}$ which permutes α_i and α_j and leaves α_k $(k\neq i, j)$ fixed.

Note that $\sigma(\alpha_3, \alpha_4)=\rho_4$. If we now denote the element $d\in\tilde{\mathfrak{D}}$ such that $d(\lambda_i)=-\lambda_i$, $d(\lambda_j)=-\lambda_j$ and $d(\lambda_k)=\lambda_k$ for $k\neq i, j$, by $\rho_{i,j}$, then we can write

$$\mathfrak{T}=\mathfrak{S}+\rho_{1,2}\mathfrak{S}+\rho_4\mathfrak{S}+\rho_4\rho_{1,2}\mathfrak{S}\,.$$

(b) If $m=2$, then

$$\Delta_1=\{\pm\alpha_1,\ \pm\alpha_3,\ \pm((\alpha_1+\alpha_2)+(\alpha_2+\alpha_3)+\alpha_4),\ \pm\alpha_4\}\,,$$

so $S_{(3)}\Delta_1=\Delta_1$, hence $S_{(3)}\subset\mathfrak{T}$. It is clear that $\tilde{\mathfrak{D}}^+\subset\mathfrak{T}$. We observe that

$$\mathfrak{T}\cap\tilde{\mathfrak{S}}=(\mathfrak{S}_1\times\mathfrak{S}_2)+\sigma_{\pi_0}(\mathfrak{S}_1\times\mathfrak{S}_2)$$

where $\sigma_{\pi_0}=\sigma_{\lambda_1-\lambda_3}\sigma_{\lambda_2-\lambda_4}$. Hence we conclude that

$$\mathfrak{T}=S_{(3)}(\mathfrak{S}+\rho_{1,4}\mathfrak{S}+\sigma_{\pi_0}\mathfrak{S}+\rho_{1,4}\sigma_{\pi_0}\mathfrak{S})\,.$$

5.4.2. If $\mathfrak{g}$ is of type DI_n, $n\geqq 4$, and $J_0\neq E$ then we can let $\alpha_m(h_0)=\pi$, $\alpha_i(h_0)=0$ for $i\neq m$ if $m\neq 0$, and let $h_0=0$ if $m=0$. For each m, $0\leqq m\leqq[(n-1)/2]$, we have a real form of $\mathfrak{g}_C$ of type D_n. Distinct values of m determine non-isomorphic real forms. In order to determine Δ_i $(i=1, 2, 3)$ we shall first compute the value of μ_α (cf. §2). By [6, (1) p. 128] μ_α must satisfy

$$\begin{aligned}
&(m1)\quad \mu_\alpha\mu_{-\alpha}=1\\
&(m2)\quad \mu_{\alpha+\beta}=(N_{J_0(\alpha),\,J_0(\beta)}/N_{\alpha,\beta})\mu_\alpha\mu_\beta\\
&(m3)\quad \mu_{\alpha_i}=1.
\end{aligned}$$

We find for $i<j<k$

$$\begin{aligned}
&(e\,1)\quad [e_{\lambda_i-\lambda_j}, e_{\lambda_j-\lambda_k}]=e_{\lambda_i-\lambda_k}\\
&(e\,2)\quad [e_{\lambda_i-\lambda_j}, e_{\lambda_j+\lambda_k}]=e_{\lambda_i+\lambda_j}\\
&(e\,3)\quad [e_{\lambda_i-\lambda_k}, e_{\lambda_j+\lambda_k}]=e_{\lambda_i+\lambda_j}\\
&(e\,4)\quad [e_{\lambda_i+\lambda_k}, e_{\lambda_j-\lambda_k}]=e_{\lambda_i+\lambda_j}\,.
\end{aligned}$$

For $i<j-1$ we have by $(m2)$

$$\mu_{\lambda_i-\lambda_j}=(N_{J_0(\lambda_i-\lambda_{j-1}),J_0(\lambda_{j-1}-\lambda_j)}/N_{\lambda_i-\lambda_{j-1},\lambda_{j-1}-\lambda_j})\mu_{\lambda_i-\lambda_{j-1}}\mu_{\lambda_{j-1}-\lambda_j}\,.$$

So using $(e1)$, $(e2)$ and $(m3)$ we have

$$\mu_{\lambda_i-\lambda_j}=1\qquad\text{for } i<j \tag{1}$$

For $i<n-1$ we have from $(m2)$

$$\mu_{\lambda_i+\lambda_n}=(N_{J_0(\lambda_i-\lambda_{n-1}),\,J_0(\lambda_{n-1}+\lambda_n)}/N_{\lambda_i-\lambda_{n-1},\,\lambda_{n-1}+\lambda_n})\mu_{\lambda_i-\lambda_{n-1}}\mu_{\lambda_{n-1}+\lambda_n}$$

so using $(e1)$, $(e2)$, $(m3)$ and (1) we get

$$\mu_{\lambda_i+\lambda_j}=1\qquad\text{for } i<n \tag{2}$$

For $i<j<n$ we have from ($m2$)

$$\mu_{\lambda_i+\lambda_j} = (N_{J_0(\lambda_i-\lambda_n),\, J_0(\lambda_j+\lambda_n)}/N_{\lambda_i-\lambda_n,\, \lambda_j+\lambda_n})\mu_{\lambda_i-\lambda_n}\mu_{\lambda_j+\lambda_n}\,.$$

Using ($e3$), ($e4$), (1) and (2) we conclude that $\mu_{\lambda_i+\lambda_j}=1$. Finally we use ($m1$) and have $\mu_\alpha=1$ for all $\alpha\in\Delta$. Now we find

$$\begin{aligned}\Delta_1 &= \{\pm(\lambda_i-\lambda_j),\ \pm(\lambda_i+\lambda_j)\,|\,i<j\leqq m \text{ or } m<i<j<n\}\\ \Delta_2 &= \{\pm(\lambda_i-\lambda_j),\ \pm(\lambda_i+\lambda_j)\,|\,i\leqq m<j<n\}\\ \Delta_3 &= \{\pm(\lambda_i-\lambda_n),\ \pm(\lambda_i+\lambda_n)\,|\,i<n\}\end{aligned}$$

Note that $(\lambda_i-\lambda_n,\ \lambda_i+\lambda_n)=0$ and that

$$\sigma_{\lambda_i+\lambda_n}\sigma_{\lambda_i-\lambda_n}(\lambda_k) = \begin{cases} \lambda_k & \text{if } k\neq i,\ n\\ -\lambda_i & \text{if } k=i\\ -\lambda_n & \text{if } k=n\end{cases}$$

Now we see that $\mathfrak{S}=\tilde{\mathfrak{D}}^+(\mathfrak{S}_1\times\mathfrak{S}_3)$, where as before $\tilde{\mathfrak{D}}^+$ is the group of elements d such that $d(\lambda_i)=\varepsilon_i\lambda_i$, $\varepsilon_i=\pm1$ for $1\leqq i\leqq n$ with $\Pi\varepsilon_i=1$, while $\mathfrak{S}_1$ is the group generated by $\sigma_{\lambda_i-\lambda_j}$ for $1\leqq i<j\leqq m$ and $\mathfrak{S}_3$ is the group generated by $\sigma_{\lambda_i-\lambda_j}$ for $m<i<j<n$. If $m=0$ or 1 then $\mathfrak{S}_1=\{1\}$. If $n=4$ and $m=2$ then $\mathfrak{S}_3=\{1\}$.

(i) For $n\geqq5$ as in 5.4.1 we have $\tilde{\mathfrak{T}}=\tilde{\mathfrak{D}}\tilde{\mathfrak{S}}_0$. As $\tilde{\mathfrak{D}}\Delta_1=\Delta_1$ we have $\tilde{\mathfrak{D}}\subset\mathfrak{T}$. Furthermore

$$\mathfrak{T}\cap\tilde{\mathfrak{S}} = \begin{cases}\mathfrak{S}_1\times\mathfrak{S}_3 & \text{if } n-1\neq 2m\\ (\mathfrak{S}_1\times\mathfrak{S}_3)+\sigma_{\pi_1}(\mathfrak{S}_1\times\mathfrak{S}_3) & \text{if } n-1=2m\end{cases}$$

where $\sigma_{\pi_1}=\sigma_{\lambda_1-\lambda_{m+1}}\sigma_{\lambda_2-\lambda_{m+2}}\cdots\sigma_{\lambda_m-\lambda_{n-1}}$. Hence

$$\mathfrak{T} = \begin{cases}\mathfrak{S}+\rho_n\mathfrak{S} & \text{if } n-1\neq 2m\\ \mathfrak{S}+\rho_n\mathfrak{S}+\sigma_{\pi_1}\mathfrak{S}+\sigma_{\pi_1}\rho_n\mathfrak{S} & \text{if } n-1=2m\,.\end{cases}$$

(ii) For $n=4$ as in 5.4.1 we have $\tilde{\mathfrak{T}}=S_{(3)}\tilde{\mathfrak{S}}=S_{(3)}\ \tilde{\mathfrak{D}}^+\tilde{\mathfrak{S}}_0$. We have two separate cases: $m=0$ and 1.

(a) If $m=0$ then

$$\Delta_1 = \{\pm(\lambda_i-\lambda_j),\ \pm(\lambda_i+\lambda_j)\,|\,i<j\leqq3\}$$

We note that the following three elements of Δ_1,

$$\begin{aligned}\lambda_1+\lambda_2 &= \alpha_1+2\alpha_2+\alpha_3+\alpha_4\\ \lambda_1+\lambda_3 &= \alpha_1+\alpha_2+\alpha_3+\alpha_4\\ \lambda_2-\lambda_3 &= \alpha_2\end{aligned}$$

are all fixed by any $\sigma\in S_{(3)}$. Thus if $\sigma ds\Delta_1=\Delta_1$ for $\sigma\in S_{(3)}$, $d\in\tilde{\mathfrak{D}}^+$ and $s\in\tilde{\mathfrak{S}}_0$

then $ds\Delta_1$ contains $\pm(\lambda_1+\lambda_2)$, $\pm(\lambda_1+\lambda_3)$, $\pm(\lambda_2-\lambda_3)$, hence $s\in\mathfrak{S}_3$ and $\sigma\Delta_1=\Delta_1$. The remaining three positive elements of Δ_1 not listed above are

$$\lambda_1-\lambda_2=\alpha_1,\ \lambda_1-\lambda_3=\alpha_1+\alpha_2,\ \lambda_2+\lambda_3=\alpha_2+\alpha_3+\alpha_4$$

so the condition $\sigma\Delta_1=\Delta_1$ implies $\sigma=1$ or $\sigma=\sigma(\alpha_3, \alpha_4)$. Hence we have

$$\mathfrak{T}=\mathfrak{S}+\sigma(\alpha_3, \alpha_4)\mathfrak{S}\,.$$

(b) If $m=1$ then

$$\Delta_1=\{\pm(\lambda_2-\lambda_3),\ \pm(\lambda_2+\lambda_3)\}=\{\pm\alpha_2,\ \pm(\alpha_2+\alpha_3+\alpha_4)\}\,.$$

For $\sigma\in S_{(3)}$ we note that $\sigma(\lambda_2-\lambda_3)=\lambda_2-\lambda_3$, so if $\sigma ds\Delta_1=\Delta_1$ for $\sigma\in S_{(3)}$, $d\in\tilde{\mathfrak{D}}^+$ and $s\in\tilde{\mathfrak{S}}$ then $ds\Delta_1$ contains $\pm(\lambda_2-\lambda_3)$, and thus $s\in\mathfrak{S}_3$ and $\sigma=1$ or $\sigma(\alpha_3, \alpha_4)$. Hence

$$\mathfrak{T}=\mathfrak{S}+\sigma(\alpha_3, \alpha_4)\mathfrak{S}\,.$$

5.4.3. If $\mathfrak{g}$ is of type $DIII_n$, $n\geqq 5$, then we can let $J_0=E$, $\alpha_n(h_0)=\pi$, $\alpha_i(h_0)=0$ for $i\neq n$. Then we see that

$$\begin{aligned}\Delta_1&=\{\pm(\lambda_i-\lambda_j)\}\\ \Delta_2&=\{\pm(\lambda_i+\lambda_j)\}\\ \Delta_3&=\text{empty}\end{aligned}$$

We have $\mathfrak{S}=\tilde{\mathfrak{S}}_0\simeq S$. As in 5.4.1 we have $\tilde{\mathfrak{T}}=\tilde{\mathfrak{D}}\tilde{\mathfrak{S}}_0$. As $\mathfrak{T}\cap\tilde{\mathfrak{D}}=\{1, -1\}$ we have

$$\mathfrak{T}=\mathfrak{S}+(-1)\mathfrak{S}\,.$$

6. The structure of $\mathfrak{k}_0$ and $\mathfrak{k}$. The action of $\mathfrak{T}/\mathfrak{S}$ on Γ_1/Γ_0.

In this section we determine the action of $\mathfrak{T}/\mathfrak{S}$ on Γ_1/Γ_0 when $\mathfrak{g}_C$ is a classical simple algebra, using the structure of Γ_1/Γ_0 given by Sirota and Solodovnikov in [8] and the explicit coset decomposition of $\mathfrak{T}/\mathfrak{S}$ determined in §5. In order that this section be self-contained, we shall elaborate on some details that were omitted in [8]. In particular we shall indicate how to derive the structures of $\mathfrak{p}\otimes C$ and $\mathfrak{k}_0\otimes C$. In some cases we choose representatives of Γ_1/Γ_0 different from those in [8].[3)]

In §4 we have seen that Γ_0 is generated by $\gamma=(2\pi i/(h_\alpha, h_\alpha))2h_\alpha$, $\tilde{\alpha}\in\Delta_\mathfrak{q}$. Note that if $J_0=E$, then we have $\gamma=\alpha'=(2\pi i/(h_\alpha, h_\alpha))2h_\alpha$. This is the case if $\mathfrak{g}$ is one of the following types: $AIII_n$, BI_n, CI_n, CII_n, DI_n with $J_0=E$, $DIII_n$.

6.1.1. If $\mathfrak{g}$ is of type AI_n (denoted I_n in [8]), n odd, $n\geqq 3$, then $\mathfrak{k}_0\otimes C$ is

3) We have corrected the errors in [8] that were pointed out by H. Freudenthal in Zentralblatt 102, 21–22.

of type $C_{(n+1)/2}$, and $\mathfrak{k}=\mathfrak{p}$ is of type $D_{(n+1)/2}$. In fact we know by [8], §11, Lemma 3, that $\mathfrak{k}_0\otimes C$ is semi-simple and that $\Pi_0=\{\tilde{\alpha}_1, \cdots, \tilde{\alpha}_{(n-1)/2}, \tilde{\alpha}_{(n+1)/2}\}$ is a system of simple roots for it. The Killing form (,) of $\mathfrak{g}_C$ restricted to $\mathfrak{k}_0\otimes C$ is invariant and nondegenerate. If $\mathfrak{k}_0\otimes C$ were not simple, then Π_0 would decompose into disjoint proper subsets, orthogonal to each other with respect to the restriction of (,) to $\mathfrak{k}_0\otimes C$. But computation shows that this is not the case, so we conclude that $\mathfrak{k}_0\otimes C$ is simple and that $(,)|\mathfrak{k}_0\otimes C$ is a constant multiple of the Killing form of $\mathfrak{k}_0\otimes C$. Then

$$(\tilde{\alpha}_1, \tilde{\alpha}_1) = \cdots = (\tilde{\alpha}_{(n-1)/2}, \tilde{\alpha}_{(n-1)2}) = (\tilde{\alpha}_{(n+1)/2}, \tilde{\alpha}_{(n+1)/2})/2$$

shows that $\mathfrak{k}_0\otimes C$ is of type $C_{(n+1)/2}$. To determine the structure of $\mathfrak{k}$, we note that $\Delta-\Delta_2=\Delta_3$ because $\Delta_1=\phi$, and hence that the root system of $\mathfrak{p}\otimes C$ is given by $\{\tilde{\alpha}\,|\,\alpha\in\Delta_3\}$ (cf. §4, Remark (2)). Then we find that

$$\Pi_{\mathfrak{p}} = \{\tilde{\alpha}_{(n-1)/2}, \tilde{\alpha}_{(n-3)/2}, \cdots, \tilde{\alpha}_1, \beta\}$$

is a system of simple roots for $\mathfrak{p}\otimes C$, where

$$-\beta = \tilde{\alpha}_1+2\tilde{\alpha}_2+\cdots+2\tilde{\alpha}_{(n-1)/2}+\tilde{\alpha}_{(n+1)/2}\,.^{4)}$$

As rank $\mathfrak{p}\otimes C\leqq$rank $\mathfrak{k}_0\otimes C$ we conclude that $\mathfrak{v}=\{0\}$ and $\mathfrak{k}=\mathfrak{p}$. Furthermore an argument similar to that for $\mathfrak{k}_0$, using the restriction of the Killing form of $\mathfrak{g}_C$ to $\mathfrak{k}\otimes C$, will show the simplicity of $\mathfrak{k}\otimes C$ and then we can determine its type.

We let $\gamma_j=(2\pi i/(h_{\tilde{\alpha}_j}, h/_{\tilde{\alpha}_j}))2h_{\tilde{\alpha}_j}$ $(j=1, \cdots, (n+1)/2)$ and note that

$$-(2\pi i/(h_\beta, h_\beta))2h_\beta = \gamma_1+2\gamma_2+\cdots+2\gamma_{(n-1)/2}+2\gamma_{(n+1)/2}$$
$$\text{(which we shall write } -\gamma_\beta)\,.$$

Then we have

$$\begin{aligned}\Gamma_0 &= \{\gamma_{(n-1)/2}, \gamma_{(n-3)/2}, \cdots, \gamma_1, \gamma_\beta\}_Z\\ &= \{\gamma_{(n-1)/2}, \gamma_{(n-3)/2}, \cdots, \gamma_1, 2\gamma_{(n+1)/2}\}_Z\end{aligned}$$

To obtain Γ_1 we have first $\Gamma_1=\{\zeta\,|\,(\zeta, \tilde{\alpha}_j)\equiv 0 \pmod{2\pi i}, j=1, \cdots, (n+1)/2\}$. Writing $\zeta=\sum s_j\gamma_j$ we can find conditions imposed on $s_j(j=1, \cdots, (n+1)/2)$ in order that $\zeta\in\Gamma_1$. From this we see that

$$\Gamma_1 = \{\gamma_1, \cdots, \gamma_{(n+1)/2}, z\}_Z$$

where

$$z = (\gamma_1+\gamma_3+\cdots+\gamma_{(n-3)/2}+\gamma_{(n+1)/2})/2 \qquad \text{if } (n+1)/2 \text{ odd}$$

4) For $\alpha=\alpha_i+\cdots+\alpha_{j-1}$ $(i<j)$
i) If $i\leqq j-1<(n+1)/2$ then $\tilde{\alpha}=\tilde{\alpha}_i+\cdots+\tilde{\alpha}_{j-1}$
ii) If $i\leqq(n+1)/2<j-1$ (and $i<n+2-j$) then $\tilde{\alpha}=-\beta-\tilde{\alpha}_1-2\tilde{\alpha}_2-\cdots-2\tilde{\alpha}_{i-1}-\tilde{\alpha}_i-\cdots-\tilde{\alpha}_{n+1-j}$

$$z = (\gamma_1+\gamma_3+\cdots+\gamma_{(n-1)/2})/2 \qquad \text{if } (n+1)/2 \text{ even.}$$

Thus the center C is given by

$$C \cong \Gamma_1\Gamma_0 \begin{cases} =\langle z+\Gamma_0\rangle \cong Z_4 & \text{if } (n+1)/2 \text{ odd} \\ =\langle z+\Gamma_0\rangle\times\langle z_1+\Gamma_0\rangle \cong Z_2\times Z_2 & \text{if } (n+1)/2 \text{ even} \end{cases}$$

where $z_1=\gamma_{(n+1)/2}$.

The outer automorphisms to consider are -1 and $\sigma=\sigma_{\lambda_{(n+1)}-\lambda_{(n+3/2)}}$. The action of -1 on C is clear. The action of σ on C is determined by the following relations. For $(n+1)/2$ odd, we have

$$\sigma z+z = \gamma_1+\gamma_3+\cdots+\gamma_{(n-3)/2}\in\Gamma_0\,,$$

and for $(n+1)/2$ even, we have

$$\sigma z-z = \gamma_{(n+1)/2} = z_1 \quad \text{and} \quad \sigma z_1 = -z_1\,.$$

We consider two subgroups of C equivalent if one transforms to the other by an automorphism of G. Using the action of $\mathfrak{T}/\mathfrak{S}$ on Γ_1/Γ_0 we determine the number of inequivalent classes of subgroups of the center C and list it in the following table. Here and in the following tables the asterisks * mark the cases where there are classes containing more than one subgroup of C.

order of subgroup	1	2	4	Total
$(n+1)/2$ odd	1	1	1	3
$(n+1)/2$ even	1	2*	1	4

6.1.2. If $\mathfrak{g}$ is of type AI_n, n even, $n\geqq 2$, then as $h_0=0$ we have $J=J_0$ and hence $\mathfrak{k}=\mathfrak{k}_0$. Consequently $\mathfrak{k}$ is semi-simple and $\mathfrak{v}=\{0\}$ and $\mathfrak{k}=\mathfrak{p}$. The system of roots for $\mathfrak{t}_0\otimes C=\mathfrak{k}\otimes C=\mathfrak{p}\otimes C$ is given by $\{\tilde{\alpha}\,|\,\alpha\in\Delta_3\}$ (because $\Delta_1=\phi$ in this case) and we see that $\Pi_0=\Pi_{\mathfrak{p}}=\{\tilde{\alpha}_1, \tilde{\alpha}_2, \cdots, \tilde{\alpha}_{n/2}\}$ is a system of simple roots.[5] Using the Killing form of $\mathfrak{g}_C$ restricted to $\mathfrak{k}\otimes C$ and arguing as in 6.1.1, we conclude that $\mathfrak{k}\otimes C$ is simple. Then

$$(\tilde{\alpha}_1, \tilde{\alpha}_1)=\cdots=(\tilde{\alpha}_{(n-2)/2}, \tilde{\alpha}_{(n-2)/2}) = 2(\tilde{\alpha}_{n/2}, \tilde{\alpha}_{n/2})$$

shows that $\mathfrak{k}\otimes C$ is of type $B_{n/2}$.

Letting $\gamma_j=(2\pi i/(h_{\tilde{\alpha}_j}, h_{\tilde{\alpha}_j}))2h_{\tilde{\alpha}_j}$ $(j=1, \cdots, n/2)$, we have

$$\Gamma_0 = \{\gamma_1, \cdots, \gamma_{(n-2)/2}, \gamma_{n/2}\}_Z$$

and as in 6.1.1 from $\Gamma_1=\{\zeta\,|\,(\zeta, \tilde{\alpha}_j)\equiv 0 \pmod{2\pi i}, j=1, \cdots, n/2\}$ we get

5) For $\alpha=\alpha_i+\cdots+\alpha_{j-1}$ $(i<j)$

i) If $i\leqq j-1\leqq n/2$ then $\tilde{\alpha}=\tilde{\alpha}_i+\cdots+\tilde{\alpha}_{j-1}$

ii) If $i\leqq n/2<j-1$ (and $i<n+2-j$) then $\tilde{\alpha}=\tilde{\alpha}_i+\cdots+\tilde{\alpha}_{n+1-j}+2\tilde{\alpha}_{n+2-j}+\cdots+2\tilde{\alpha}_{n/2}$

$$\Gamma_1=\{\gamma_1,\cdots,\gamma_{(n-2)/2},(\gamma_{n/2})/2\}_Z.$$

Thus the center C of G is given by

$$C\cong\Gamma_1/\Gamma_0=\langle z_2+\Gamma_0\rangle\cong Z_2.$$

where $z_2=(\gamma_{n/2})/2$. The only outer automorphism to consider is -1 and the action on C is trivial.

6.1.3. If $\mathfrak{g}$ is of type AII_n (denoted J_n in [8]), n odd, $n\geqq 3$, then $h_0=0$, hence $J=J_0$ and $\mathfrak{k}=\mathfrak{k}_0$, so $\mathfrak{d}=\{0\}$ and $\mathfrak{k}=\mathfrak{p}$. The system of roots for $\mathfrak{k}_0\otimes C=\mathfrak{k}\otimes C=\mathfrak{p}\otimes C$ is given by $\{\tilde{\alpha}\mid\alpha\in\Delta\}$ (in this case $\Delta_2=\phi$). Using the same argument as above we conclude that $\Pi_0=\Pi_{\mathfrak{p}}=\{\tilde{\alpha}_1,\cdots,\tilde{\alpha}_{(n+1)/2}\}$[6] is a simple system of roots, and that $\mathfrak{k}\otimes C$ is simple. Then

$$(\tilde{\alpha}_1,\tilde{\alpha}_1)=\cdots=(\tilde{\alpha}_{(n-1)/2},\tilde{\alpha}_{(n-1)/2})=(\tilde{\alpha}_{(n+1)/2},\tilde{\alpha}_{(n+1)/2})/2$$

shows that $\mathfrak{k}\otimes C$ is of type $C_{(n+1)/2}$.

Letting $\gamma_j=(2\pi i/(h_{\tilde{\alpha}_j},h_{\tilde{\alpha}_j}))2h_{\tilde{\alpha}_j}$ $(j=1,\cdots,(n+1)/2)$, we have

$$\Gamma_0=\{\gamma_1,\cdots,\gamma_{(n+1)/2}\}_Z.$$

As in 6.1.1 we derive from $\Gamma_1=\{\zeta\mid(\zeta,\alpha_j)\equiv 0 \pmod{2\pi i},\ j=1,\cdots,(n+1)/2\}$ that

$$\Gamma_1=\{\gamma_1,\cdots,\gamma_{(n+1)/2},z\}_Z.$$

where

$$\begin{aligned} z&=(\gamma_1+\gamma_3+\cdots+\gamma_{(n-3)/2}+\gamma_{(n+1)/2})/2 &&\text{if } (n+1)/2 \text{ odd}\\ z&=(\gamma_1+\gamma_3+\cdots+\gamma_{(n-1)/2})/2 &&\text{if } (n+1)/2 \text{ even}\end{aligned}$$

Thus the center C of G is given by

$$C\cong\Gamma_1/\Gamma_0=\langle z+\Gamma_0\rangle\cong Z.$$

The only outer automorphism to consider is -1 and its action on C is trivial.

6.1.4. If $\mathfrak{g}$ is of type $AIII_n$ (denoted A_n^m in [8]), $n\geqq 1$, then $J_0=E$, hence $\mathfrak{k}_0=\mathfrak{g}_u$. We have $\Pi_0=\{\alpha_1,\cdots,\alpha_n\}$. We have $\mathfrak{k}=\mathfrak{p}_1\oplus\mathfrak{p}_2\oplus\mathfrak{d}$, where $\mathfrak{d}=iRh_0$, and $\mathfrak{p}_1\otimes C$ and $\mathfrak{p}_2\otimes C$ are simple of types A_{m-1} and A_{n-m} respectively, except that $\mathfrak{p}_1=\{0\}$ if $m=1$, and $\mathfrak{p}_1=\mathfrak{p}_2=\{0\}$ if $n=1$. To verify this, we first note that Δ_3 being empty the root system of $\mathfrak{p}\otimes C$ is given by Δ_1, which is empty if $n=1$ and which is the disjoint union of two subsystems $\{\pm(\lambda_i-\lambda_j)\mid i,j\leqq m\}$ and

6) For $\alpha=\alpha_1+\cdots+\alpha_{j-1}$ $(i<j)$

i) If $i\leqq -1\leqq(n+1)/2$ then $\tilde{\alpha}=\tilde{\alpha}_i+\cdots+\tilde{\alpha}_{j-1}$

ii) If $i\leqq(n+1)/2\leqq j-1$ (and $i\leqq n+2-j$) then

$$\tilde{\alpha}=\tilde{\alpha}_i+\cdots+\tilde{\alpha}_{n+1-j}+2\tilde{\alpha}_{n+2-j}+\cdots+2\tilde{\alpha}_{(n-1)/2}+\tilde{\alpha}_{(n+1)/2}$$

$\{\pm(\lambda_i-\lambda_j)\,|\,m<i<j\}$ if $n>1$. Thus $\{\alpha_1, \cdots, \alpha_{m-1}\}$ and $\{\alpha_{m+1}, \cdots, \alpha_n\}$ are systems of simple roots for simple algebras $\mathfrak{p}_1\otimes C$ and $\mathfrak{p}_2\otimes C$ such that $\mathfrak{p}={}_1\mathfrak{p}\oplus\mathfrak{p}_2$. One should also note that the Killing form on $\mathfrak{p}\otimes C$ is the restriction of that for $\mathfrak{g}_C$. From $\alpha_i(h_0)=0$ for $i\neq m$ and the structure of $\mathfrak{p}$ we see that $[h_0, \mathfrak{p}]=0$.

We now let $\gamma_j=(2\pi i/(h_{\alpha_j}, h_{\alpha_j}))2h_{\alpha_j}$ $(j=1, \cdots, n)$. For $n=1$, we have $\Gamma_0=\{0\}$ and $\Gamma_1\{\gamma_1/2\}_Z$ and the center C is given by

$$C\cong\Gamma_1/\Gamma_0=\langle\gamma_1/2\rangle\cong Z\,.$$

The action of $\mathfrak{T}$ on C is given by $\sigma_{\lambda_1-\lambda_2}(\gamma_1/2)=-\gamma_1//2$. For $n>1$, we have

$$\Gamma_0=\{\gamma_1, \cdots, \gamma_{m-1}, \gamma_{m+1}, \cdots, \gamma_n\}_Z\,.$$

From $\Gamma_1=\{\zeta\,|\,(\zeta, \alpha_j)\equiv 0 \pmod{2\pi i}, j=1, \cdots, n\}$ we obtain

$$\Gamma_1=\{\gamma_1, \cdots, \gamma_n, u_1\}_Z$$

where $u_1=(1/n+1)\sum_1^n k\gamma_k$. Here we could replace u_1 by $u_2=(1/n+1)\times\sum_1^n(n-k+1)\gamma_k$ just as well. Then the center C is given by

$$C\cong\Gamma_1/\Gamma_0=\langle z_1+\Gamma_0\rangle\times\langle z_2+\Gamma_0\rangle\cong Z_d\times Z\,,$$

where $d=(m, n+1)$ and z_1, z_2 are given by

$$\begin{aligned}
z_1 &= (m/d)u_2-(n-m+1/d)u_1\\
z_2 &= M_1u_1+M_2u_2 \qquad (M_1, M_2\in Z \text{ satisfying } M_1m+M_2(n-m+1)=d)\,.
\end{aligned}$$

Here we have chosen z_1 and z_2 so that if we write $z_i=\sum s_{j\,j}$ then $s_m=0$ for z_1 and $s_m=d/(n+1)$ for z_2.

If $n+1\neq 2m$ the only outer automorphism to consider is -1. The action of -1 is clear. If $n+1=2m$, then $n-m+1=m=d$ and we have

$$\begin{aligned}
z_1 = u_2-u_1 &= (1/n+1)\Big(\sum_1^n(n+1)\gamma_k-2\sum_1^n k\gamma_k\Big)\\
&\equiv (-2/(n+1))\sum_{k\neq m}k\gamma_k \qquad (\mathrm{mod}\ \Gamma_0)\,.
\end{aligned}$$

We can let $M_1=1$, $M_2=0$. Then we have $z_2=u_1$. We only have to consider the action of -1 and σ_{π_0}. The action of -1 is clear. As for σ_{π_0} we have

$$\begin{aligned}
\pi_{\pi_0}(z_1)&\equiv(-2/(n+1))\Big(\sum_{k=m+1}^n(k-m)\gamma_k+\sum_{k=1}^{m-1}(k+m)\gamma_k\Big)\\
&\equiv(-2/(n+1))\sum_{k\neq m}k\gamma_k=z_1 \qquad (\mathrm{mod}\ \Gamma_0)\,.
\end{aligned}$$

To find $\sigma_{\pi_0}(z_2)$, consider

$$u_1+u_2\equiv\gamma_m \qquad (\mathrm{mod}\ \Gamma_0)$$

Because $\sigma_{\pi_0}(\alpha_m)=-(\alpha_1+\cdots+\alpha_n)$ we have

$$\sigma_{\pi_0}(u_1+u_2) \equiv -(u_1+u_2) \qquad (\mathrm{mod}\ \Gamma_0)\,.$$

We also have

$$\sigma_{\pi_0}(z_1) = \sigma_{\pi_0}(u_2-u_1) \equiv z_1 = u_2-u_1 \qquad (\mathrm{mod}\ \Gamma_0)\,,$$

hence

$$\sigma_{\pi_0}(z_2) = \sigma_{\pi_0}(u_1) \equiv (1/2)(-(u_1+u_2)-(u_2-u_1)) = -u_2 = -z_1-z_2 \qquad (\mathrm{mod}\ \Gamma_0)\,.$$

For $n=1$, each non-negative integer gives a subgroup of C and distinct integers give subgroups which are inequivalent under automorphisms of G. For $n>1$, the subgroups of $C \cong \Gamma_1/\Gamma_0$ are of the form

$$\langle az_1+\Gamma_0\rangle \times \langle b_1z_1+b_2z_2+\Gamma_0\rangle \cong Z_{d/a}\times Z \text{ or } 1\times Z$$

where a, b_1 and b_2 are non-negative integers such that if $a\neq 0$, then $a\,|\,d$, $0\leqq b_1<a$, if $a=0$, then $0\leqq b_1<d$, and if $b_2=0$, then $b_1=0$. If $n+1\neq 2m$, then the only outer automorphism to consider is -1, so for each choice of (a, b_1, b_2) we have a subgroup of C, distinct triples defining subgroups which are inequivalent under automorphisms of G. If $n+1=2m$, then we have to consider σ_{π_0} along with -1 and the subgroups of C given by (a, b_1, b_2) and (a, b_1', b_2') are sent onto each other by σ_{π_0} if and only if

1) $a=a'\neq 0\,,\ b_2=b_2'$ and $b_1-b_2\equiv -b_1' \quad (\mathrm{mod}\ a)$

or 2) $a=a'=0\,,\ b_2=b_2'$ and $b_1-b_2\equiv -b_1' \quad (\mathrm{mod}\ d)\,.$

6.2.1. If $\mathfrak{g}$ is of type BI_n (denoted B_n^{2m} in [8]), $n\geqq 2$, then $J_0=E$ and we have $\mathfrak{k}_0=\mathfrak{g}_u$ and $\Pi_0=\{\alpha_1, \cdots, \alpha_n\}$. For $m=1$, we have $\mathfrak{k}=\mathfrak{p}\oplus\mathfrak{v}$, where $\mathfrak{p}\otimes C$ is simple of type B_{n-1}, while $\mathfrak{v}=iRh_0$. In fact as the system of roots for $\mathfrak{p}\otimes C$ is $\Delta_1=\{\pm(\lambda_i\pm\lambda_j),\ 1<i<j;\ \pm\lambda_k,\ 1<k\}$ we see that $\{\alpha_2, \cdots, \alpha_n\}$ is a system of simple roots for $\mathfrak{p}\otimes C$, and thus by the argument in 6.1.1 we can derive the simplicity and the type of $\mathfrak{p}\otimes C$. Then from $\alpha_i(h_0)=0$, $i\neq 1$, we conclude that $[h_0, \mathfrak{p}]=0$. For $1<m<n$, $\mathfrak{k}=\mathfrak{p}_1\oplus\mathfrak{p}_2$, where $\mathfrak{p}_1\otimes C$ and $\mathfrak{p}_2\otimes C$ are simple and of types D_m and B_{n-m} respectively. This can be seen by observing that Δ_1 decomposes into two disjoint subsystems $\{\pm(\lambda_i\pm\lambda_j)\,|\,i<j\leqq m\}$ and $\{\pm(\lambda_i\pm\lambda_j)\,|\,m<i<j\}\cup\{\pm\lambda_i\,|\,m<i\}$, orthogonal to each other with respect to the Killing form on $\mathfrak{g}_C$, then picking systems of simple roots $\{\alpha_{m-1}, \cdots, \alpha_2, \alpha_1, \beta\}$, where $-\beta=\lambda_1+\lambda_2=\alpha_1+2\alpha_2+\cdots+2\alpha_n$,[7] and $\{\alpha_{m+1}, \cdots, \alpha_{n-1}, \alpha_n\}$ for the subsystems and finally applying the argument in 6.1.1 for each subsystem. From rank $\mathfrak{p}_1$+rank $\mathfrak{p}_2=n=$rank $\mathfrak{k}$ we conclude $\mathfrak{v}=\{0\}$. For $m=n$, we get $\mathfrak{k}=\mathfrak{p}$, where $\mathfrak{p}\otimes C$ is simple and of type D_n, by the same agrument as in 6.1.1.

Let $\gamma_j=(2\pi i/(h_{\alpha_j}, h_{\alpha_j}))2h_{\alpha_j}$ $(j=1, \cdots, n)$ and $\gamma_\beta=(2\pi i/(h_\beta, h_\beta))2h_\beta$. Then

7) If $i<j\leqq m$ then $\lambda_i+\lambda_j=-\beta-(\alpha_1+2\alpha_2+\cdots 2\alpha_{i-1}+\alpha_i+\cdots+\alpha_{j-1})$.

$-\gamma_\beta=\gamma_1+2\gamma_2+\cdots+2\gamma_{n-1}+\gamma_n$. From $\Gamma_1=\{\zeta\,|\,(\zeta, \alpha_j)\equiv 0 \pmod{2\pi i}, j=1,\cdots,n\}$ we get

$$\Gamma_1=\{\gamma_1, \gamma_2, \cdots, \gamma_{n-1}, \gamma_n/2\}_Z .$$

If $m=1$, then $\Gamma_0=\{\gamma_2, \cdots, \gamma_n\}_Z$ so the center C is given by

$$C\cong\Gamma_1/\Gamma_0=\langle z_1+\Gamma_0\rangle\times\langle z_2+\Gamma_0\rangle\cong Z\times Z_2 ,$$

where $z_1=\gamma_1$ and $z_2=\gamma_n/2$. If $1<m<n$ then $\Gamma_0=\{\gamma_1, \cdots, \gamma_{m-1}, \gamma_{m+1}, \cdots, \gamma_n, \gamma_\beta\}_Z=\{\gamma_1, \cdots, \gamma_{m-1}, 2\gamma_m, \gamma_{m+1}, \cdots, \gamma_n\}_Z$, hence the center C is given by

$$C\cong\Gamma_1/\Gamma_0=\langle z_1+\Gamma_0\rangle\times\langle z_2+\Gamma_0\rangle\cong Z_2\times Z_2$$

where $z_1=\gamma_m$ and $z_2=\gamma_n/2$. If $m=n$ then $\Gamma_0=\{\gamma_1, \cdots, \gamma_{n-1}, \gamma_\beta\}_Z=\{\gamma_1, \cdots, \gamma_{n-1}, \gamma_n\}_Z$ and thus the center C is given by

$$C\cong\Gamma_1/\Gamma_0=\langle z_2+\Gamma_0\rangle\cong Z_2$$

where $z_2=\gamma_n/2$. The outer automorphism to be considered is ρ_1. We have

$$\begin{aligned}\rho_1 z_2 &= z_2\\ \rho_1 z_1 &= z_1 \qquad \text{if } m>1 .\end{aligned}$$

If $m=1$, then $\alpha_m=\alpha_1$ and

$$\rho_1\alpha_1=\rho_1(\lambda_1-\lambda_2)=-\lambda_1-\lambda_2=-(\alpha_1+2(\alpha_2+\cdots+\alpha_n)) ,$$

hence

$$\rho_1 z_1=\rho_1\gamma_1=-\gamma_1+2(\gamma_2+\cdots+\gamma_{n-1})-\gamma_n=-z_1 \qquad (\mathrm{mod}\,\Gamma_0) .$$

For $m=1$, the subgroups of C are of the form

$$\langle b_1z_1+b_2z_2+\Gamma_0\rangle\times\langle az_2+\Gamma_0\rangle\cong Z\times Z_2 \text{ or } Z\times 1 .$$

Here b_1 is a non-negative integer, a and b_2 take values 0 and 1. If $a=0$, then either $b_1=b_2=0$ or $b_1>0$. If $a=1$, then $b_2=0$. Each of these subgroups is stable by ρ_1, so they are all inequivalent under the automorphisms of G. For $m>1$, the subgroups of C are all pointwise fixed by automorphisms of G.

6.3.1. If $\mathfrak{g}$ is of type CI_n (denoted IC_n in [8]), $n\geqq 3$, then $J_0=E$ and $\mathfrak{k}_0=\mathfrak{g}_u$ and $\Pi_0=\{\alpha_1, \cdots, \alpha_n\}$. We have $\mathfrak{k}=\mathfrak{p}\oplus\mathfrak{v}$, where $\mathfrak{p}\otimes C$ is simple and of type A_{n-1} and $\mathfrak{v}=iRh_0$. To show this we just have to observe that the system of roots $\Delta-\Delta_2=\Delta_1=\{\pm(\lambda_i-\lambda_j)\}$($\Delta_3$ is empty) of $\mathfrak{p}\otimes C$ has a system of simple roots $\{\alpha_1, \cdots, \alpha_{n-1}\}$ and apply the argument in 6.1.1. We again see that $[h_0, \mathfrak{p}]=0$ from $\alpha_i(h_0)=0$, for $i\neq n$.

Let $\gamma_j=(2\pi i/(h_{\alpha_j}, h_{\alpha_j}))2h_{\alpha_j}$ $(j=1, \cdots, n)$. We have $\Gamma_0=\{\gamma_1, \cdots, \gamma_{n-1}\}_Z$ and from $\Gamma_1=\{\zeta\,|\,(\zeta, \alpha_j)\equiv 0 \pmod{2\pi i}, j=1, \cdots, n\}$ we get

$$\Gamma_1 = \{\gamma_1, \cdots, \gamma_n, z\}_Z$$

where

$$z = (\gamma_1+\gamma_3+\cdots+\gamma_n)/2 \qquad \text{if } n \text{ odd}$$
$$z = (\gamma_1+\gamma_3+\cdots+\gamma_{n-1})/2 \qquad \text{if } n \text{ even.}$$

Hence the center C is given by

$$C \cong \Gamma_1/\Gamma_0 = \begin{cases} \langle z+\Gamma_0\rangle \cong Z & \text{if } n \text{ odd} \\ \langle z+\Gamma_0\rangle \times \langle z_1+\Gamma_0\rangle \cong Z_2\times Z & \text{if } n \text{ even} \end{cases}$$

where $z_1=\gamma_n$.

The outer automorphism to consider is -1, so the action is clear. Hence, if n is odd, then each non-negative integer gives a subgroup of C, inequivalent under automorphisms of G, and if n is even, then the enumeration of subgroups is the same as in the case of BI_n, $m=1$ (6.2.1) and the subgroups are all inequivalent under automorphisms of G.

6.3.2. If $\mathfrak{g}$ is of type CII_n (denoted C_n^{2m} in [8]), $n\geqq 3$, then $J_0=E$ and $\mathfrak{k}_0=\mathfrak{g}_u$ and $\Pi_0=\{\alpha_1, \cdots, \alpha_n\}$. We have $\mathfrak{k}=\mathfrak{p}_1\oplus\mathfrak{p}_2$, where $\mathfrak{p}_1\otimes C$ and $\mathfrak{p}_2\otimes C$ are simple and of types C_m and C_{n-m} respectively. In fact, the root system Δ_1 of $\mathfrak{p}\otimes C$ decomposes into two subsystems $\{\pm(\lambda_i-\lambda_j)\,|\,i\leqq j<m\}$ and $\{\pm(\lambda_i-\lambda_j)\,|\,m<i\leqq j\}$. The two subsystems are orthogonal to each other with respect to the Killing form of $\mathfrak{g}_C$. The first one has $\{\alpha_{m-1}, \cdots, \alpha_1, \beta\}$ where $-\beta=2\alpha_1+\cdots+2\alpha_{n-1}+\alpha_n$, as a system of simple roots, while the second one has $\{\alpha_{m+1}, \cdots, \alpha_{n-1}, \alpha_n\}$, as a system of simple roots. We derive the simplicity using the argument in 6.1.1 and the types follow from

$$(\alpha_1, \alpha_1) = \cdots = (\alpha_{n-1}, \alpha_{n-1}) = (\alpha_n, \alpha_n)/2 = (\beta, \beta)/2\,.$$

Letting $\gamma_j=(2\pi i/(h_{\alpha_j}, h_{\alpha_j}))2h_{\alpha_j}$ and $\gamma_\beta=(2\pi i/(h_\beta, h_\beta))2h_\beta$ we have $-\gamma_\beta=\gamma_1+\cdots+\gamma_{n-1}+\gamma_n$. We have then

$$\Gamma_0 = \{\gamma_{m-1}, \cdots, \gamma_1, \gamma_\beta, \gamma_{m+1}, \cdots, \gamma_{n-1}, \gamma_n\}_Z = \{\gamma_1, \cdots, \gamma_n\}_Z$$

and as Γ_1 is exactly the same as in 6.3.1, i.e., $\Gamma_1=\{\gamma_1, \cdots, \gamma_n, z\}_Z$, $z=(\gamma_1+\gamma_3+\cdots)/2$, we see that the center C is given by

$$C \cong \Gamma_1/\Gamma_0 = \langle z+\Gamma_0\rangle \cong Z_3\,.$$

The only outer automorphism to consider is σ_{π_0} and it occurs only when $n=2m$. The action of σ_{π_0} on z is trivial in this case. At any rate the center is pointwise fixed by all automorphisms of G.

6.4.1. If $\mathfrak{g}$ is of type DI_n, $n\geqq 4$, and $J_0=E$ (denoted D_n^{2m} in [8]), then $\mathfrak{k}_0=\mathfrak{g}_u$ and $\Pi_0=\{\alpha_1,\cdots, \alpha_n\}$. We let $1\leqq m\leqq [n/2]$. If $m>1$, then $\mathfrak{k}=\mathfrak{p}=\mathfrak{p}_1\oplus\mathfrak{p}_2$, where $\mathfrak{p}_1\otimes C$ and $\mathfrak{p}_2\otimes C$ are simple and of types D_m and D_{n-m} respectively,[8] and if $m=1$, then $\mathfrak{k}=\mathfrak{p}\oplus\mathfrak{v}$, where $\mathfrak{p}\otimes C$ is simple and of type D_{n-1}. To see the structure of $\mathfrak{k}$, we observe that the root system Δ_1 of $\mathfrak{p}\otimes C$ decomposes into two subsystems $\{\pm(\lambda_i\pm\lambda_j)\,|\,i<j\leqq m\}$ and $\{\pm(\lambda_i\pm\lambda_j)\,|\,m<i<j\}$, orthogonal to each other with respect to the Killing form of $\mathfrak{g}_C$, and that the first subsystem is empty if $m=1$. For $m>1$ letting $\beta=-(\lambda_1+\lambda_2)$ we see that $\{\alpha_{m-1},\cdots,\alpha_1,\beta\}$ is a system of simple roots for the first subsystem,[9] while $\{\alpha_{m+1},\cdots,\alpha_{n-1},\alpha_n\}$ is a system of simple roots for the second. The rest of the argument goes as before. For $m=1$, the empty first subsystem is replaced by $\mathfrak{v}=iRh_0$. We have $[h_0,\mathfrak{p}]=0$ from $\alpha_i(h_0)=0$ for $i\neq 1$.

Letting $\gamma_j=(2\pi i/(h_{\alpha_j},h_{\alpha_j}))2h_{\alpha_j}$ $(j=1,\cdots,n)$ and $\gamma_\beta=(2\pi i/(h_\beta,h_\beta))2h_\beta$ we have $\gamma_\beta=\gamma_1+2(\gamma_2+\cdots+\gamma_{n-2})+\gamma_{n-1}+\gamma_n$. From $\Gamma_1=\{\zeta\,|\,(\zeta,\alpha_j)\equiv 0 \pmod{2\pi i}$, $j=1,\cdots,n\}$ we obtain $\Gamma_1=\{\gamma_1,\cdots,\gamma_{n-2},z,z_1\}_Z$, where

$$z=(\gamma_{n-1}+\gamma_n)/2$$
$$z_1=\begin{cases}(\gamma_1+\gamma_3+\cdots+\gamma_{n-2})/2+(\gamma_{n-1}-\gamma_n)/4 & \text{if } n \text{ odd}\\ (\gamma_1+\gamma_3+\cdots+\gamma_{n-3})/2+\gamma_{n-1}/2 & \text{if } n \text{ even}\end{cases}$$

For $m=1$ we have $\Pi_{\mathfrak{p}}=\{\alpha_2,\cdots,\alpha_n\}$, hence $\Gamma_0=\{\gamma_2,\cdots,\gamma_n\}_Z$ and thus the center C is given by

$$C\cong\Gamma_1/\Gamma_0=\langle z+\Gamma_0\rangle\times\langle z_1+\Gamma_0\rangle\cong Z_2\times Z\,.$$

For $m>1$ we have $\Pi_{\mathfrak{p}}=\{\alpha_{m-1},\cdots,\alpha_1,\beta\}\cup\{\alpha_{m+1},\cdots,\alpha_{n-1},\alpha_n\}$, hence $\Gamma_0=\{\gamma_{m-1},\cdots,\gamma_1,\gamma_\beta,\gamma_{m+1},\cdots,\gamma_{n-1},\gamma_n\}_Z=\{\gamma_1,\cdots,\gamma_{m-1},2\gamma_m,\gamma_{m+1},\cdots,\gamma_n\}_Z$. Thus we can write $\Gamma_1=\{z,z_1,z_4,\Gamma_0\}_Z$, where $z_4=\gamma_m$. If n is odd the center C is given by

$$C\cong\Gamma_1/\Gamma_0=\langle z_1+\Gamma_0\rangle\times\langle z_4+\Gamma_0\rangle\cong Z_4\times Z_2\,.$$

If n is even and m is odd the center C is given by

$$C\cong\Gamma_1/\Gamma_0=\langle z_1+\Gamma_0\rangle\times\langle z+\Gamma_0\rangle\cong Z_4\times Z_2\,.$$

If n is even and m is even the center C is given by

$$C\cong\Gamma_1/\Gamma_0=\langle z_1+\Gamma_0\rangle\times\langle z+\Gamma_0\rangle\times\langle z_4+\Gamma_0\rangle\cong Z_2\times Z_2\times Z_2\,.$$

(i) For $n\geqq 5$, if $n\neq 2m$, then we have to consider the action of ρ_1 and ρ_n, while if $n=2m$, then we have to consider the action of ρ_1, ρ_n and σ_{π_0}.

(a) If $n\geqq 5$ and $m=1$, then

$$\rho_1(z)=z\,.$$

8) Except $\mathfrak{p}_1$ is not simple for $m=2$, and $\mathfrak{p}_2$ is not simple for $n=4$, $m=2$.

9) If $i<j\leqq m$ then $\lambda_i+\lambda_j=\{(\alpha_1+\cdots+\alpha_{i-1})+(\alpha_2+\cdots+\alpha_j)+\beta\}$.

If furthermore n is odd, then

$$\rho_1(z_1)+z_1\equiv(\rho_1\gamma_1+\gamma_1+\gamma_{n-1}+\gamma_n)/2=-\gamma_2-\cdots-\gamma_{n-2}\equiv 0 \qquad (\mathrm{mod}\ \Gamma_0)$$

and if n is even, then

$$\rho_1(z_1)+z_1+z\equiv(\rho_1\gamma_1+\gamma_1+\gamma_{n-1}+\gamma_n)/2\equiv 0 \qquad (\mathrm{mod}\ \Gamma_0)\,.$$

For ρ_n, regardless of the parity of n, we have

$$\begin{aligned}&\rho_n(z)=z\\&\rho_n(z_1)-z_1=-(\gamma_{n-1}-\gamma_n)/2\equiv z \qquad (\mathrm{mod}\ \Gamma_0)\,.\end{aligned}$$

The subgroups of $C\cong\Gamma_1/\Gamma_0$ are of the form

$$\langle az+\Gamma_0\rangle\times\langle b_1z+b_2z_1+\Gamma_0\rangle\cong Z_2\times Z \text{ or } 1\times Z\,.$$

Here b_2 is a non-negative integer and a and b_1 take values 0 and 1. If $a=0$, then either $b_1=b_2=0$ or $b_2>0$. If $a=1$, then $b_1=0$. The subgroups given by the triple (a, b_1, b_2) are stable under the automorphisms except for those given by $(0, 0, b_2)$ and $(0, 1, b_2)$, where b_2 is odd, which map onto each other by ρ_n.

(b) If $n\geqq 5$, n odd and $m>1$, then

$$\begin{aligned}\rho_1(z_1)+z_1&\equiv(\rho_1\gamma_1+\gamma_1+\gamma_{n-1}+\gamma_n)/2+\begin{cases}\gamma_m & \text{if } m \text{ odd}\\ 0 & \text{if } m \text{ even}\end{cases}\\&\equiv-(\gamma_2+\cdots+\gamma_{m-1})-(\gamma_{m+1}+\cdots+\gamma_{n-2})+\begin{cases}0 & \text{if } m \text{ odd}\\ -\gamma_m & \text{if } m \text{ even}\end{cases}\\&\equiv\begin{cases}0 & \text{if } m \text{ odd}\\ z_4 & \text{if } m \text{ even}\end{cases} \qquad (\mathrm{mod}\ \Gamma_0)\end{aligned}$$

$$\rho_1(z_4)=z_4$$

$$\rho_n(z_1)+z_1\equiv\begin{cases}\gamma_m=z_4 & \text{if } m \text{ odd}\\ 0 & \text{if } m \text{ even}\end{cases} \qquad (\mathrm{mod}\ \Gamma_0)$$

$$\rho_n(z_4)=z_4\,.$$

The number of inequivalent classes of subgroups of the center C under the automorphisms of G are given in the following table.

order of subgroup	1	2	4	8	Total
number of classes	1	3	2*	1	7

(c) If $n\geqq 5$, n even, $m>1$ and m odd, then

$$\begin{aligned}\rho_1(z_1)-z_1+z_4+z&\equiv(\rho_1\gamma_1-\gamma_1)/2+\gamma_m+(\gamma_{n-1}+\gamma_n)/2\\&=-(\gamma_2+\cdots+\gamma_{m-1})-(\gamma_{m+1}+\cdots+\gamma_{n-2})\\&\equiv 0 \qquad (\mathrm{mod}\ \Gamma_0)\end{aligned}$$

$$\rho_1(z) = z$$
$$\rho_n(z_1) - z_1 + z \equiv \gamma_n \equiv 0 \qquad (\mathrm{mod}\ \Gamma_0)$$
$$\rho_n(z) = z$$

Moreover, if $n = 2m$, then $z_1 = (\gamma_1 + \gamma_3 + \cdots + \gamma_m + \cdots + \gamma_{n-1})/2$. Taking note especially that $\sigma_{\pi_0}(\alpha_m) = -(\alpha_1 + \cdots + \alpha_{n-1})$, we find that

$$\sigma_{\pi_0}(z_1) \equiv -z_1 \qquad (\mathrm{mod}\ \Gamma_0)$$

and finally

$$\sigma_{\pi_0}(z_1) - z \equiv \gamma_{m-1} + \cdots + \gamma_{n-2} \equiv \gamma_m \equiv 2z_1 \qquad (\mathrm{mod}\ \Gamma_0)\,.$$

The number of inequivalent classes of subgroups of C under the automorphisms of G are given in the following table.

order of subgroup	1	2	4	8	Total
$n \neq 2m$	1	3	2*	1	7
$n = 2m$	1	2*	2*	1	6

(d) If $n \geqq 5$, n even, $m > 1$ and m even, then

$$\rho_1(z_1) + z_1 + z_4 + z \equiv 0 \quad (\mathrm{mod}\ \Gamma_0) \quad (\text{as in (c)})$$
$$\rho_1(z) = z\,, \quad \rho_1(z_4) = z_4$$
$$\rho_n(z_1) \equiv z_1 + z \ (\mathrm{mod}\ \Gamma_0)\,, \quad \rho_n(z) = z\,, \quad \rho_n(z_4) = z_4\,.$$

Moreover, if $n = 2m$, then noting that $\sigma_{\pi_0}(\alpha_m) = -(\alpha_1 + \cdots + \alpha_{n-1})$ and that $\sigma_{\pi_0}(\alpha_n) = \alpha_{m-1} + 2(\alpha_m + \cdots + \alpha_{n-2}) + \alpha_{n-1} + \alpha_n$, we obtain

$$\sigma_{\pi_0}(z_1) = z_1\,, \quad \sigma_{\pi_0}(z) \equiv z + z_4\,, \quad \sigma_{\pi_0}(z_4) \equiv -z_4 \ (\mathrm{mod}\ \Gamma_0)\,.$$

The number of inequivalent classes of subgroups of C under the automorphisms of G are given in the following table.

order of subgroup	1	2	4	8	Total
$n \neq 2m$	1	4*	4*	1	10
$n = 2m$	1	3*	3*	1	8

(ii) Let us consider the case for $n=4$ now

(a) If $n=4$ and $m=1$, then the automorphisms to be considered are $\rho_{1,2}$ and ρ_4. The center is given by

$$C \cong \Gamma_1/\Gamma_0 = \langle z + \Gamma_0 \rangle \times \langle z_1 + \Gamma_0 \rangle \cong Z_2 \times Z\,,$$

where $z = (\gamma_3 + \gamma_4)/2$ and $z_1 = (\gamma_1 + \gamma_3)/2$. We have

$$\rho_{1,2} z = z \quad \rho_{1,2} z_1 \equiv -z_1 \ (\mathrm{mod}\ \Gamma_0)$$
$$\rho_4 z = z \quad\ \ \rho_4 z_1 \equiv z_1 + z \ (\mathrm{mod}\ \Gamma_0)$$

As in (i) (a) the subgroups of C are of the form

$$\langle az+\Gamma_0\rangle\times\langle b_1z+b_2z_1\Gamma_0\rangle\cong Z_2\times Z \text{ or } 1\times Z\,.$$

They are stable under the automorphisms of G, except those given by $(0, 0, b_2)$ and $(0, 1, b_2)$, where b_2 is odd, which map onto each other by ρ_4.

(b) If $n=4$ and $m=2$, then the automorphisms to be considered are $\rho_{1,4}$, σ_{π_0} and those of $S_{(3)}$. The center is given by

$$C\cong\Gamma_1/\Gamma_0=\langle z_1+\Gamma_0\rangle\times\langle z+\Gamma_0\rangle\times\langle z_4+\Gamma_0\rangle\cong Z_2\times Z_2\times Z_2\,,$$

where $z_1=(\gamma_1+\gamma_3)/2$, $z=(\gamma_3+\gamma_4)/2$ and $z_4=\gamma_2$. The action of the automorphisms of G is given, mod Γ_0, by the following:

$$\begin{array}{lll}
\rho_{1,4}z_1=-z_1-z_4 & \rho_{1,4}z=z & \rho_{1,4}z_4=z_4\\
\sigma_{\pi_0}z_1=z_1 & \sigma_{\pi_0}z\equiv z_4+z & \sigma_{\pi_0}z_4\equiv z_4\\
\sigma(\alpha_1,\alpha_3)z_1=z_1 & \sigma(\alpha_1,\alpha_3)z\equiv z_1+z & \sigma(\alpha_1,\alpha_3)z_4=z_4\\
\sigma(\alpha_1,\alpha_4)z_1=z & \sigma(\alpha_1,\alpha_4)z=z_1 & \sigma(\alpha_1,\alpha_4)z_4=z_4
\end{array}$$

The number of inequivalent classes of subgroups of the center C under the automorphisms of G are given in the following table.

order of subgroups	1	2	4	8	Total
number of classes	1	2*	2*	1	6

6.4.2. If $\mathfrak{g}$ is of type DI_n, $n\geqq 4$ and $J_0\neq E$ (denoted $D_n^{2m}+1$ in [8]), then $\mathfrak{k}_0\otimes C$ is simple and of type B_{n-1} and $\mathfrak{k}=\mathfrak{k}_0$ for $m=0$, while $\mathfrak{k}=\mathfrak{p}=\mathfrak{p}_1\oplus\mathfrak{p}_2$ for $m\geqq 1$, where $\mathfrak{p}_1\otimes C$ and $\mathfrak{p}_2\otimes C$ are simple of types B_m and B_{n-m-1} respectively. Here note that $0\leqq m\leqq[(n-1)/2]$. We have found that $\mu_\alpha=1$ for all $\alpha\in\Delta$ in 5.4.2. Hence the root system of $\mathfrak{k}_0\otimes C$ is $\{\tilde{\alpha}\,|\,\alpha\in\Delta\}$. The simple system of roots $\Pi_0=\{\tilde{\alpha}_1,\cdots,\tilde{\alpha}_{n-2},\tilde{\alpha}_{n-1}\}$, where $\tilde{\alpha}_i=\alpha_i$ for $i=1,\cdots,n-2$ and $\tilde{\alpha}_{n-1}=(\alpha_{n-1}+\alpha_n)/2$, does not decompose into two mutually orthogonal subsystems with respect to the Killing form of $\mathfrak{g}_C$ so we know that $\mathfrak{k}_0\otimes C$ is simple, and we verify the type by observing that

$$(\alpha_1,\alpha_1)=\cdots=(\alpha_{n-2},\alpha_{n-2})=2(\tilde{\alpha}_{n-1},\tilde{\alpha}_{n-1})\,.$$

To determine the structure of $\mathfrak{k}$ we note that the system of roots for $\mathfrak{k}\otimes C$ is $\{\tilde{\alpha}\,|\,\alpha\in\Delta_1\cup\Delta_3\}=\{\pm(\lambda_i\pm\lambda_j)\,|\,i<j\leqq m \text{ or } m<i<j<n\}\cup\{\pm\lambda_i\,|\,i<n\}$. For $m\geqq 1$, we can decompose this into two subsystems, orthogonal to each other with respect to the Killing form of $\mathfrak{g}_C$. $\{\alpha_{m-1},\alpha_{m-2},\cdots,\alpha_1,\beta\}$ is a system of simple roots for one of the subsystems, while $\{\alpha_{m+1},\cdots,\alpha_{n-2},\tilde{\alpha}_{n-1}\}$ is a system of simple roots for the other. Here $\beta=-\lambda_1=-(\alpha_1+\alpha_2+\cdots+\alpha_{n-2}+\tilde{\alpha}_{n-1})$. $\Pi_\mathfrak{p}$ is the union of the two systems of simple roots. The two subsystems give the two subalgebras $\mathfrak{p}_1$ and $\mathfrak{p}_2$ and the simplicity and type of each $\mathfrak{p}_i\otimes C$ are

obtained by applying the argument of 6.1.1 on each subsystem.

Letting $\gamma_j=(2\pi i/(h_{\tilde{\alpha}_j}, h_{\tilde{\alpha}_j}))2h_{\tilde{\alpha}_j}$ $(j=1, \cdots, n-1)$ and $\gamma_\beta=(2\pi i/(h_\beta, h_\beta))2h_\beta$ we have $\gamma_\beta=-2(\gamma_1+\gamma_2+\cdots+\gamma_{n-2})-\gamma_{n-1}$. From $\Gamma_1=\{\zeta \mid \zeta, \tilde{\alpha}_j)\equiv 0 \pmod{2\pi i}$, $j=1, \cdots, n-1\}$ we get $\Gamma_1=\{\gamma_1, \cdots, \gamma_{n-2}, \gamma_{n-1}/2\}_Z$. If $m=0$, we have $\Gamma_0=\{\gamma_1, \cdots, \gamma_{n-1}\}_Z$. If $m\geqq 1$, we have

$$\begin{aligned}\Gamma_0 &= \{\gamma_{m-1}, \cdots, \gamma_1, \gamma_\beta\}_Z \cup \{\gamma_{m+1}, \cdots, \gamma_{n-2}, \gamma_{n-1}\}_Z \\ &= \{\gamma_1, \cdots, \gamma_{m-1}, 2\gamma_m, \gamma_{m+1}, \cdots, \gamma_{n-1}\}_Z\end{aligned}$$

Hence the center C is given by

$$C \cong \Gamma_1/\Gamma_0 = \begin{cases} \langle z+\Gamma_0\rangle \cong Z_2 & \text{if } m=0 \\ \langle z+\Gamma_0\rangle \times \langle z_4+\Gamma_0\rangle \cong Z_2\times Z_2 & \text{if } m\geqq 1 \end{cases}$$

where $z=\gamma_{n-1}/2$ and $z_4=\gamma_m$.

(i) For $n\geqq 5$, the outer automorphisms that we have to consider are ρ_n if $n-1\neq 2m$, and ρ_n and σ_{π_1} if $n-1=2m$. We have

$$\rho_n z = z\,, \quad \rho_n z_4 = z_4\,,$$

and if $n-1=2m$, then

$$\sigma_{\pi_1} z - z \equiv z_4\,, \quad \sigma_{\pi_1} z_4 \equiv z_4 \pmod{\Gamma_0}\,.$$

The number of inequivalent classes of subgroups of the center C under the automorphisms of G are given in the following table.

order of subgroup	1	2	4	Total
$m=0$	1	1	0	2
$m\geqq 1$, $n-1\neq 2m$	1	3	1	5
$m\geqq 1$, $n-1=2m$	1	2*	1	4

(ii) For $n=4$, the only outer automorphism we have to consider is $\sigma(\alpha_3, \alpha_4)$. We have, for $m=1$, $z=\gamma_3/2$ and $z_4=\gamma_1$ and both are fixed by $\sigma(\alpha_3, \alpha_4)$. Hence all subgroups of the center C are stable under the automorphisms of G. Thus, if $m=1$, then there are three subgroups of order 2, inequivalent under the automorphisms of G.

6.4.3. If $\mathfrak{g}$ is of type $DIII_n$ (denoted JD_n in [8]), $n\geqq 5$, then $J_0=E$, $\mathfrak{k}_0=\mathfrak{g}_u$ and $\Pi_0=\{\alpha_1, \cdots, \alpha_n\}$. We have $\mathfrak{k}=\mathfrak{p}\oplus\mathfrak{d}$, where $\mathfrak{p}\otimes C$ is simple and of type A_{n-1}. The root system for $\mathfrak{p}\otimes C$ is $\Delta_1=\{\pm(\lambda_i-\lambda_j)\}$($\Delta_3$ is empty) and $\Pi_\mathfrak{p}=\{\alpha_1, \cdots, \alpha_{n-1}\}$ is a system of simple roots for Δ_1. We have $\mathfrak{d}=iRh_0$ and $[h_0, \mathfrak{p}]=0$.

Letting $\gamma_j=(2\pi i/(h_{\alpha_j}, h_{\alpha_j}))2h_{\alpha_j}$ $(j=1, \cdots, n)$, we have $\Gamma_0=\{\gamma_1, \cdots, \gamma_{n-1}\}_Z$ and $\Gamma_1=\{\gamma_1, \cdots, \gamma_{n-2}, z, z_1\}_Z$ as in 6.4.1. The center C of G is given by

$$C \cong \Gamma_1/\Gamma_0 = \begin{cases} \langle z_1+\Gamma_0\rangle \cong Z & \text{if } n \text{ odd} \\ \langle z_1+\Gamma_0\rangle \times \langle z+\Gamma_0\rangle \cong Z_2\times Z & \text{if } n \text{ even} \end{cases}$$

where z and z_1 are as defined in 6.4.1. The only outer automorphism we have to consider is -1. If n odd, then each non-negative integer gives a subgroup of C. If n even, then each triple (a, b_1, b_2) gives a subgroup of C. Here b_2 is a non-negative integer and a and b_1 take values 0 and 1; if $a=0$, then either $b_1=b_2=0$ or $b_2>0$; if $a=1$, then $b_1=0$. All subgroups of the center C are stable under the automorphisms of G.

7. Table of number of inequivalent classes of subgroups

We shall now collect the results of §6 on the subgroups of the center C. In the table below $N(r)$ means that the subgroups of order r of the center C of noncompact G are partitioned into N inequivalent classes under the automorphisms of G. As before, the asterisk $*$ indicates the non-trivial action of Aut G. In particular, by $N(r)^*$ we mean that amongst the N inequivalent classes of subgroups of order r some contain more than one subgroup of C, and by countable* we mean that amongst the countably many inequivalent classes there are some that contain more than one subgroup of C.

$\mathfrak{g}$		C		Number of inequivalent classes of subgroups of C
AI_n	n odd, $n\geqq 3$			
	$(n+1)/2$ odd	Z_4		1(1) 1(2) 1(4)
	$(n+1)/2$ even	$Z_2\times Z_2$		1(1) 2(2)* 1(4)
	n even, $n\geqq 2$	Z_2		1(1) 1(2)
AII_n	n odd, $n\geqq 3$	Z_2		1(1) 1(2)
$AIII_n$	$n=1$	Z_2		countable
	$n>1$	$Z_d\times Z$		
			$n+1\neq 2m$	countable
			$n+1=2m$	countable*
BI_n	$n\geqq 2$			
	$m=1$	$Z_2\times Z$		countable
	$1<m<n$	$Z_2\times Z_2$		1(1) 3(2) 1(4)
	$m=n$	Z_2		1(1) 1(2)
CI_n	n odd, $n\geqq 3$	Z		countable
	n even, $n\geqq 3$	$Z_2\times Z$		countable
CII_n	$n\geqq 3$	Z_2		1(1) 1(2)
DI_n, $J=E_0$				
	(i) $n\geqq 5$			
	$m=1$	$Z_2\times Z$		countable*
	$m>1$, n odd	$Z_4\times Z_2$		1(1) 3(2) 2(4)* 1(8)

$m>1$, m odd, n even	$Z_2\times Z_4$		
		$n\neq 2m$	1(1) 3(2) 2(4)* 1(8)
		$n=2m$	1(1) 2(2)* 2(4)* 1(8)
$m>1$, m even, n even	$Z_2\times Z_2\times Z_2$		
		$n\neq 2m$	1(1) 4(2)* 4(4)* 1(8)
		$n=2m$	1(1) 3(2)* 3(4)* 1(8)
(ii) $n=4$			
$m=1$	$Z_2\times Z$		countable*
$m=2$	$Z_2\times Z_2\times Z_2$		1(1) 2(2)* 2(4)* 1(8)
DI_n, $J_0\neq E$			
(i) $n\geqq 5$			
$m=0$	Z_2		1(1) 1(2)
$m\geqq 1$	$Z_2\times Z_2$		
		$n-1\neq 2m$	1(1) 3(2) 1(4)
		$n-1=2m$	1(1) 2(2)* 1(4)
(ii) $n=4$			
$m=1$	Z_2		1(1) 1(2)
$m=0$	$Z_2\times Z_2$		1(1) 3(2) 1(4)
$DIII_n$ n odd	Z		countable
n even	$Z_2\times Z$		countable

Appendix

In 5.1.1, 5.1.2 and 5.1.3 we made use of the following lemma which we shall now prove.

Lemma. *Let S be the symmetric group on $n+1$ letters and H the subgroup of S defined by $H=\{s\in S \mid s(i)+s(n+2-i)=n+2$ for all $i=1, \cdots, n+1\}$. Then H is generated by the following permutations:*
1) $(i, j)\ (n+2-i,\ n+2-j)$, *where* $1\leqq i<j\leqq n+1$, $i+j\neq n+2$ *and if* n *even,* $i, j\neq (n+2)/2$.
2) $(i,\ n+2-i)$, *where* $1\leqq i\leqq n+1$.
It suffices to have all of 1) *and one* $(i,\ n+2-i)$ *in* 2) *to generate* H.

Proof. Consider a fixed i, $1\leqq i\leqq n+1$, and a fixed $s\in H$. When s is written as a product of disjoint cycles, let a be the cycle containing i and b be the cycle containing $i'=n+2-i$. Then either a and b are disjoint or $a=b$.

If a and b are disjoint, then a and b have the same length, say k, and we have

$$a=(i,\ s(i),\ s^2(i),\ \cdots,\ s^{k-1}(i))=(i,\ s(i))(s(i),\ s^2(i))\cdots(s^{k-2}(i),\ s^{k-1}(i))$$
$$b=(i',\ s(i'),\ \cdots,\ s^{k-1}(i'))=(i',\ s(i'))(s(i'),\ s^2(i'))\cdots(s^{k-2}(i'),\ s^{k-1}(i'))$$

Hence the product ab can be written as the product of permutations in 1), namely those of the form $(s^{j-1}(i),\ s^{j}(i))(s^{j-1}(i'),\ s^{j}(i'))$, $j=1, \cdots, k-1$.

If $a=b$, then choose the smallest t such that $s^t(i)=i'$. Then we have $s^t(i')=i$ and the action on i by s and its powers is

$$i \to s(i) \to s^2(i) \to \cdots \to s^{t-1}(i) \to i' \to s(i') \to \cdots \to s^{t-1}(i') \to i$$

where all terms are distinct in this sequence, except the first and the last are the same. We see that a can be written as

$$\begin{aligned} a &= (i,\ s(i),\ \cdots,\ s^{t-1}(i),\ i',\ s(i'),\ \cdots,\ s^{t-1}(i')) \\ &= (i,\ s(i))(i',\ s(i')) \cdots (s^{t-2}(i),\ s^{t-1}(i))(s^{t-2}(i'),\ s^{t-1}(i'))(i,\ i') \end{aligned}$$

so again the cycle a is a product of permutations in 1) and 2).

The last claim is proved by noting that if $j+j'=n+2$, then $(i,\ i')=(i,\ j)$ $(i',\ j')\ (j,\ j')\ (i,\ j)\ (i',\ j')$. q.e.d.

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