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THE BRAUER GROUPS OF SOME SEPARABLY CLOSED RINGS

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M. Auslander and O. Goldman in [1] laid the foundation for the study of separable extentions of a commutative ring. All unexplained conventions, terminology and notation in this note are as in [1]. One of the most important and difficult problems in this theory, even when the commutative ring is a field, is the explicit computation of the Brauer group. Let K be a commutative ring. Following [2] we call a commutative K-algebra S strongly separable in case it is separable, finitely generated, and projective over K; and we call a commutative ring Ω separably closed in case the only strongly separable Ω algebras are direct sums of copies of Ω. In [2] it is shown that if K is any commutative ring with no idempotents but 0 and 1 then there is a separably closed K-algebra Ω with no idempotents but 0 and 1, called the separable closure of K, which contains an isomorphic copy of every strongly separable K-algebra with no idempotents but 0 and 1, and with the property that any finite subset is contained in a strongly separable K-algebra.

In analogy with the situation in fields, one would expect the Brauer group of a separably closed ring to be trivial. However M. Auslander has discovered a separably closed principal ideal domain whose Brauer group has order two. If K is a commutative ring let $B(K)$ denote the Brauer group of K and if S is a strongly separable K-algebra let $B(S/K)$ denote the kernal of map induced by the correspondence $A \rightarrow S \otimes_K A$ where A is a central separable K-algebra. If K is a semi-local noetherian ring we prove that $B(K) = \cup B(S/K)$ where S ranges over all strongly separable K-algebras. If moreover, K has no idempotents but 0 and 1 and Ω is the separable closure of K then $B(\Omega)$ is trivial.

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Let K be a semi-local ring (commutative ring with a finite number of maximal ideals). If $\{M_i\}_{i=1}^n$ are the maximal ideals of K then $N = \cap_{i=1}^n M_i$ is the radical of K. One can make K into a topological space and complete the space where a neighborhood base for 0 consists of the powers of N. We refer the reader to Chapter II of [4] for the facts about completeness we employ here. We will also frequently make use of the fact that any strongly separable algebra over a semi-local ring is semi-local. Also we will employ the well known Nakayama's Lemma which asserts that if R is any ring with identity and I is a left ideal of R then I is contained in the radical of R if and only if for all finitely generated left R modules M with N a submodule of M, if $IM+N=M$ then $N=M$. Our first result generalizes Theorem 6.3 of [1].

Theorem 1. *Let A be a central separable algebra over the noetherian semi-local ring K, then there is a strongly separable K-algebra S with $S\otimes_K A$ in the zero class of $B(S)$. Moreover, if K has no idempotents but* 0 *and* 1, *we may choose S with no idempotents but* 0 *and* 1.

Proof. Any commutative semi-local ring is a finite direct sum of semi-local rings without idempotents but 0 and 1. Because of this observation we may assume K has no idempotents but 0 and 1.

Let $\bar{A}=A/NA=\bar{A}_1\oplus\cdots\oplus\bar{A}_n$ with the center of $\bar{A}_i$ equal to K/M_iK which we denote $\bar{K}_i$. Since K has no idempotents but 0 and 1, A is a free K-module (Page 377 of [1]). Thus $\mathrm{Rank}_{\bar{K}_i}(\bar{A}_i)=\mathrm{Rank}_K(A)=m^2$ for some integer m. There is a commutative separable algebra F_i over $\bar{K}_i$ of dimension m which is a maximal commutative subalgebra of $\bar{A}_i$. Moreover, there is a $\theta_i\in\bar{A}_i$ so that $F_i=\bar{K}_i(\theta_i)$ and θ_i satisfies a monic polynomial $P_i(x)$ over $\bar{K}_i$ of degree m. The existence of such an F_i and θ_i can be seen in the following way. If $\bar{K}_i$ is finite, then there is a field extention of $\bar{K}_i$ (necessarily separable) which can be imbedded in $\bar{A}_i$ as a maximal commutative subring. There is always a maximal commutative separable subfield of the division algebra component of $\bar{A}_i$ of the form $\bar{K}_i(\beta)$ and when $\bar{K}_i$ is infinite, we can let F_i be the diagonal matrix with entries in $\bar{K}_i(\beta)$, F_i is generated over $\bar{K}_i$ by a matrix θ_i with distinct scalar multiples of β along the main diagonal.

Let $\theta=\theta_1+\cdots+\theta_n$. $\bar{K}(\theta)$ is a maximal commutative separable subalgebra of $\bar{A}$ and θ satisfies the monic polynomial $p(x)\in\bar{K}[x]$ of degree m given by $p(x)=\Sigma_i p_i(x)$ where $p_i(x)$ are the minimum polynomials over K_i of the θ_i. Let $S=K[x]/(P(x))$ where $P(x)$ is a monic polynomial in $K[x]$ which maps onto $p(x)$ mod N. Since K has no idempotents but 0 and 1, S is a finitely generated free K-algebra and $S/NS=\bar{K}(\theta)$ is separable so S is separable over K. Moreover $S\otimes_K A$ is in the kernal of the map from $B(S)\to B(S/NS)$, so we may from now on assume that A is in the kernal of the map from $B(K)\to B(K/NK)$.

With all notation as before we find an element $\theta\in\bar{A}$ with $\bar{K}(\theta)$ a maximal commutative separable subalgebra of $\bar{A}$ and satisfying a monic polynomial of degree m over $\bar{K}$. Let $\beta\in A$ with β mapping to θ under the map from A to $\bar{A}$. Let $S=K\cdot 1+K\cdot\beta+\cdots+K\cdot\beta^{m-1}$. Since $\{1, \theta, \cdots, \theta^{m-1}\}$ is a free basis for a $\bar{K}$ direct summand of $\bar{A}$, Nakayama's lemma (pg. 377 of [1]) implies that $\{1, \beta, \cdots, \beta^{m-1}\}$ is a free set of generators of S over K which extends to a free set of generators of A over K so S is a K-direct summand of A.

We now show S is a subring of A by showing $\beta^m \in S$. Let $\hat{K}$ be the completion of K and let $\hat{A}=\hat{K}\otimes_K \hat{A}$. Then $A/NA \simeq \hat{A}/N\hat{A}$ so since $\hat{K}$ is a finite direct sum of complete local rings by Corollary 6.2 of [1], A is in the zero class of $B(\hat{K})$. Since A is a free $\hat{K}$-module of rank m^2, $\hat{A} \simeq \mathrm{Hom}_{\hat{K}}(P, P)$ with P a free $\hat{K}$-module of rank m. Since K is noetherian, we may assume A is a subring of $\hat{A}$ so that $\beta \in \hat{A}$. By the Cayley-Hamilton Theorem, β satisfies a monic polynomial of degree m over $\hat{K}$. But S is a K-direct summand of A and since $\beta^m \in \hat{K}\otimes_K S$, $\beta^m \in S$.

S is separable over K since $S \cap NS = S \cap NA$ and thus $S/NS = \bar{K}(\theta)$ which is separable over $\bar{K}$. Let S^* be the commutant of S in A, by Theorem 2 of [3] S^* is a separable finitely generated projective K-algebra and the commutant in A of S^* is S. But $S^* \cap NA$ is a two-sided ideal in S^* so by Corollary 3.2 of [1] there is an ideal $M \subseteq S$ with $MS^* = S^* \cap NA$. But $MS^* \cap S^* = M = (S^* \cap NA) \cap S = NS$ so $M = NS$ and $S^* = NS^* + S$. Therefore by Nakayama's lemma, $S^* = S$, and by Theorem 5.6 of [1], $S\otimes_K A$ is in the zero class of $B(S)$.

For the last statement of the theorem assume K has no idempotents but 0 and 1, then $S \simeq Se_i \oplus \cdots \oplus Se_n$ with Se_i a commutative finitely generated projective separable extention of $Ke_i \simeq K$ and with no idempotents but 0 and 1. Since $S\otimes_K A$ is in the zero class of $B(S)$, there is a finitely generated projective S-module P with $\mathrm{Hom}_S(P, P) \simeq S\otimes_K A$. Then $Se_i \otimes_K A \simeq \mathrm{Hom}_{Se_i}(Pe_i, Pe_i)$ and $Se_i \otimes_K A$ is in the zero class of $B(Se_i)$. This proves the theorem.

Corollary 2. *If K is a semi-local noetherian ring then $B(K) = UB(S/K)$ where S ranges over all strongly separable K-algebras.*

If K has no idempotents but 0 and 1 then by Corollary 2, Theorem A. 15 of [1] and the fact that any strongly separable K-algebra without idempotents but 0 and 1 is contained in a Galois extention of K [2]; the computation of $B(K)$ is reduced to the computation of $H^2(G, U(S))$ for each Galois extention S of K. Here $U(S)$ denotes the multiplicative units in S, G is the Galois group of S over K, and $H^2(G, U(S))$ is the second cohomology group of G acting on $U(S)$.

We now give E. Ingraham's proof of a module theoretic fact about the separable closure of a semi-local ring which has no idempotents but 0 and 1.

Proposition 3. *Let K be a semi-local ring with no idempotents but 0 and 1 and let Ω be the separable closure of K, then every finitely generated projective Ω-module is Ω-free.*

Proof. Let E be a finitely generated projective Ω-module. Then there exists an Ω-free module F of rank n, and a finitely generated projective Ω-module L with $E\oplus L = F$. Let $x_1 \cdots x_n$ be a free bases for F over Ω, let $y_1, \cdots, y_m$ be a minimal set of generators for E over Ω, and let $y_{m+1}, \cdots, y_t$ be a minimal set of generators for L over Ω.

To prove the Proposition we will show that if $\sum_{i=1}^{m} \rho_i y_i=0$ with $\rho_i \in \Omega$ then $\rho_i=0$ for all i.

Now there are $\alpha_{ij} \in \Omega$ with $y_i=\sum_{j=1}^{n} \alpha_{ij} x_j$ and $\beta_{ij} \in \Omega$ with $x_i=\sum_{j=1}^{t} \beta_{ij} y_j$ since the $y_1 \cdots y_t$ generate all of F. Let $T=\{\alpha_{ij}, \beta_{ij}, \rho_i\}$. This is a finite set so there is a strongly separable K-subalgebra S of Ω with $T \subseteq S$. Let $F_S= Sx_1+\cdots+Sx_n$, $E_S=Sy_1+\cdots+Sy_m$ and $L_S=Sy_{m+1}+\cdots+Sy_t$. F_S is a free S-module since the x_i are linearly independent over Ω and hence over S. F_S and L_S are submodules of F_S since $T \subseteq S$. Since $F_S \subseteq E$ and $L_S \subseteq L$, $E_S \cap L_S=0$. Again since $T \subseteq S$, $E_S+L_S=F_S$ so $E_S \oplus L_S=F_S$. Thus E_S is a finitely generated projective S-module. But S is a strongly separable K-subalgebra of Ω so S is a semi-local ring without idempotents but 0 and 1. Also $y_1 \cdots y_m$ is a minimal set of generators for E_S over S since they were minimal for E over Ω.

It is not a hard exercise using Nakayama's lemma to show that in this situation $y_1 \cdots y_m$ must be free basis for E_S over S. (see for example pg. 377 of [1]). Thus $y_1 \cdots y_m$ are linearly independent over S so all the $\rho_i=0$ which is what we wanted to show.

We can now prove out final result.

Theorem 4. *If K is a semi-local noetherian ring without idempotents but* 0 *and* 1 *and Ω is the separable closure of K, then $B(\Omega)$ is trivial.*

Proof. Let A be a central separable Ω algebra. A is free as an Ω-module by Proposition 3, so let $x_1, \cdots, x_n$ be a free Ω-basis for A. Let $\{c_{ij}^k\}$ be the multiplication constants for the algebra A with respect to the basis $x_1, \cdots, x_n$. That is, $x_i x_j=\sum_k c_{ij}^k x_k$ with $c_{ij}^k \in \Omega$. Let S be a strongly separable K-subalgebra of Ω containing $\{c_{ij}^k\}$. Define the central separable S-algebra A_S by letting A_S be the free S-module $Sx_1+\cdots Sx_n$ with multiplication constants $\{c_{ij}^k\}$. Since K is a semi-local noetherian ring S is, so by theorem 1 there is a strongly separable S-algebra T with no idempotents but 0 and 1 so that $T \otimes_S A_S$ is in the zero class of $B(T)$. But T is strongly separable over K so we may identify T with a K-subalgebra of Ω. Since $A \simeq \Omega \otimes_T (T \otimes_S A_S)$, we conclude A is in the zero class of $B(\Omega)$.

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