Fujiwara, Tsuyoshi
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On Mappings between Algebraic Systems

By Tsuyoshi FUJIWARA

Since the mappings between algebraic systems play one of the most important rôles in the theory of algebraic systems, the construction of the general theory of mappings seems to be interesting and useful in the study of algebraic systems. In the present paper, we shall first introduce a family $\boldsymbol{P}$ of basic mapping-formulas as a generalization of the defining formulas of homomorphisms: $\varphi(x*y)=\varphi(x)*\varphi(y)$, and that of the defining formulas of (φ, ψ)-derivations[1]: $D(xy)=D(x)\varphi(y)+\psi(x)D(y)$, $D(x+y)=D(x)+D(y)$. And we define $\boldsymbol{P}$-mappings as the mappings which satisfy the family $\boldsymbol{P}$ of basic mapping-formulas. And we shall try to construct the general theory of $\boldsymbol{P}$-mappings.

Our theory can be divided into two parts. The first part (§§ 1–4) deals with a general theory of $\boldsymbol{P}$-mappings which contains the results with respect to homomorphisms, derivations and others. In this part, the algebraic Taylor's expansion theorem[2] (Theorem 1.1) will play the fundamental rôles, since, by this theorem, any $\boldsymbol{P}$-mappings from an algebraic system $\mathfrak{A}$ into another algebraic system $\mathfrak{B}$ can be reduced to a homomorphism from $\mathfrak{A}$ into the $\boldsymbol{P}$-product system over $\mathfrak{B}$. In the second part (§ 5), we shall give a characterization of the defining formulas of homomorphisms by considering the absolutely universal family of basic mapping-formulas.

§ 1. Fundamental properties of $\boldsymbol{P}$-mappings.

First we shall explain terminology and notations with respect to free algebraic systems[3], for the convenience of our discussion.

The finitary compositions will be denoted by $v, w, \cdots$, and we denote by $N(v)$ the number N such that the composition v is N-ary. Let $V=\{v, \cdots\}$ be a set of finitary compositions. By a ϕ_V-algebraic system $\mathfrak{A}$, we shall always mean an algebraic system $\mathfrak{A}$ which is defined by V so that $v(a_1, \cdots, a_{N(v)})$ is assigned a single element in $\mathfrak{A}$, for any com-

1) Cf. [4; P. 170].
2) Cf. [5; § 12].
3) Cf. [1; P. viii], [2], [3], [6], [7; Chapter II, § 1] and [8; § 4].

position v in V and any elements $a_1, \cdots, a_{N(v)}$ in $\mathfrak{A}$. An absolutely free ϕ_V-algebraic system[4] is simply called a free ϕ_V-algebraic system. And the free ϕ_V-algebraic system with a free generator system $\{a_1, \cdots, a_r\}$[5] is denoted by $F(\{a_1, \cdots, a_r\}, \phi_V)$. An element of the free ϕ_V-algebraic system $F(\{a_1, \cdots, a_r\}, \phi_V)$ is called a V-word or a V-polynomial, and denoted by $f(a_1, \cdots, a_r)$, $g(a_1, \cdots, a_r)$, $\cdots$. A V-word in $F(\{a_{11}, \cdots, a_{1r}, \cdots, a_{s1}, \cdots, a_{sr}\}, \phi_V)$ is not only denoted by $f(a_{11}, \cdots, a_{1r}, \cdots, a_{s1}, \cdots, a_{sr})$, but also denoted by

$$f\begin{pmatrix} a_{11}, \cdots, a_{1r} \\ \cdots\cdots\cdots\cdots \\ a_{s1}, \cdots, a_{sr} \end{pmatrix} \quad \text{or simply} \quad f\begin{pmatrix} a_{11} \cdots a_{1r} \\ \cdots\cdots\cdots\cdots \\ a_{s1} \cdots a_{sr} \end{pmatrix}.$$

Again, the generator a_ρ is called a V-word of order 0. And the V-word $f(a_1, \cdots, a_r)$ which can be written in the form

$$v(g_1(a_1, \cdots, a_r), \cdots, g_{N(v)}(a_1, \cdots, a_r))$$

is called a V-word of order k, where $g_N(a_1, \cdots, a_r)$ $(N=1, \cdots, N(v))$ are V-words of order $k-1$ or less, and some $g_N(a_1, \cdots, a_r)$ is precisely of order $k-1$. Then each V-word $f(a_1, \cdots, a_r)$ is clearly of some order k. Let A_V be a system of composition-identities with respect to V. By an A_V-algebraic system, we shall mean a ϕ_V-algebraic system satisfying A_V. And we denote by $F(\{a_1, \cdots, a_r\}, A_V)$ the free A_V-algebraic system with a free generator system $\{a_1, \cdots, a_r\}$. If two V-words $f(a_1, \cdots, a_r)$ and $g(a_1, \cdots, a_r)$ are equivalent in $F(\{a_1, \cdots, a_r\}, A_V)$, then we say that $f(a_1, \cdots, a_r)$ and $g(a_1, \cdots, a_r)$ are A_V-congruent, and denote it by $f(a_1, \cdots, a_r) \overset{A_V}{=} g(a_1, \cdots, a_r)$. Moreover let R_V be a set of relations, i.e. identities between elements of $F(\{a_1, \cdots, a_r\}, \phi_V)$. The free A_V-algebraic system with a generator system $\{a_1, \cdots, a_r\}$ satisfying R_V is denoted by $F(\{a_1, \cdots, a_r\}, A_V, R_V)$.

Let $V=\{v, \cdots\}$ and $W=\{w, \cdots\}$ be two sets of finitary compositions. And let $\{\xi_1, \cdots, \xi_m\}$[6] and $\{x_1, \cdots, x_{N(v)}\}$ be sets of formal variables, to be replaced by mappings and elements of a ϕ_V-algebraic system respectively. An identity of the form

$$\begin{aligned} &\xi_\mu(v(x_1, \cdots, x_{N(v)})) \\ &= P_{\xi_\mu v}(\xi_1(x_1), \cdots, \xi_1(x_{N(v)}), \cdots, \xi_m(x_1), \cdots, \xi_m(x_{N(v)})) \end{aligned}$$

4) Cf. [6; §3], [7; Chapter II, §1] and [8; §4].

5) For convenience, we use the same notation as in the case of a finite set, but the generator system $\{a_1, \cdots, a_r\}$ does not necessarily mean a finite set.

6) The set $\{\xi_1, \cdots, \xi_m\}$ may be any non-empty finite or infinite set. But, for convenience, we use the same notation as in the case of a finite set.

is called a basic mapping-formula of ξ_μ concerning v, where $P_{\xi_\mu v}(\xi_1(x_1), \cdots, \xi_1(x_{N(v)}), \cdots, \xi_m(x_1), \cdots, \xi_m(x_{N(v)}))$ is a W-polynomial in $F(\{\xi_1(x_1), \cdots, \xi_1(x_{N(v)}), \cdots, \xi_m(x_1), \cdots, \xi_m(x_{N(v)})\}, \phi_W)$. A set of basic mapping-formulas which is of the form

$$\left\{\xi_\mu(v(x_1, \cdots, x_{N(v)})) = P_{\xi_\mu v}\begin{pmatrix}\xi_1(x_1), \cdots, \xi_1(x_{N(v)}) \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ \xi_m(x_1), \cdots, \xi_m(x_{N(v)})\end{pmatrix}; \mu = 1, \cdots, m, v \in V\right\}$$

is simply denoted by $\boldsymbol{P}_{V,W}\{\xi_1, \cdots, \xi_m\}$, and called a family of basic mapping-formulas. Now let $\boldsymbol{P}$ be a family $\boldsymbol{P}_{V,W}\{\xi_1, \cdots, \xi_m\}$ of basic mapping-formulas, and let $\varphi_1, \cdots, \varphi_m$ be single-valued mappings from a ϕ_V-algebraic system $\mathfrak{A}$ into a ϕ_W-algebraic system $\mathfrak{B}$. If, for any elements $a_1, \cdots, a_{N(v)}$ in $\mathfrak{A}$, all the identities obtained by the substitution of $\varphi_1, \cdots, \varphi_m$ and $a_1, \cdots, a_{N(v)}$ for $\xi_1, \cdots, \xi_m$ and $x_1, \cdots, x_{N(v)}$ of all basic mapping-formulas of $\boldsymbol{P}$ are true in $\mathfrak{B}$, then we say that $\{\varphi_1, \cdots, \varphi_m\}$ is a system of $\boldsymbol{P}$-mappings, or simply that $\varphi_1, \cdots, \varphi_m$ are $\boldsymbol{P}$-mappings.

Let $\boldsymbol{P}$ be a family $\boldsymbol{P}_{V,W}\{\xi_1, \cdots, \xi_m\}$ of basic mapping-formulas. And let $\mathfrak{B}$ be a ϕ_W-algebraic system, and $\mathfrak{B}^m$ the set of all ordered m-tuples $[b_1, \cdots, b_m]$ each of which consists of elements of $\mathfrak{B}$. Now we define the compositions $v \in V$ in $\mathfrak{B}^m$ as follows:

$$\begin{aligned} & v([b_1^1, \cdots, b_m^1], \cdots, [b_1^{N(v)}, \cdots, b_m^{N(v)}]) \\ = & \left[P_{\xi_1 v}\begin{pmatrix} b_1^1, \cdots, b_1^{N(v)} \\ \cdots\cdots\cdots\cdots \\ b_m^1, \cdots, b_m^{N(v)}\end{pmatrix}, \cdots, P_{\xi_m v}\begin{pmatrix} b_1^1, \cdots, b_1^{N(v)} \\ \cdots\cdots\cdots\cdots \\ b_m^1, \cdots, b_m^{N(v)}\end{pmatrix}\right]. \end{aligned}$$

Then it is clear that $\mathfrak{B}^m$ forms a ϕ_V-algebraic system. Such a ϕ_V-algebraic system is called a $\boldsymbol{P}_{V,W}\{\xi_1, \cdots, \xi_m\}$-product system over $\mathfrak{B}$ or simply a $\boldsymbol{P}$-product system over $\mathfrak{B}$, and denoted by $\boldsymbol{P}_{V,W}^{\xi_1 \cdots \xi_m}(\mathfrak{B})$ or simply by $\boldsymbol{P}(\mathfrak{B})$.

Examples: (1) Let $\boldsymbol{P}$ be a family $\boldsymbol{P}_{V,V}\{\xi_1, \cdots, \xi_m\}$ of basic mapping-formulas each of which is of the form $\xi_\mu(v(x_1, \cdots, x_{N(v)})) = v(\xi_\mu(x_1), \cdots, \xi_\mu(x_{N(v)}))$. Then the $\boldsymbol{P}$-product system $\boldsymbol{P}(\mathfrak{B})$ over a ϕ_V-algebraic system $\mathfrak{B}$ is the direct product $\underbrace{\mathfrak{B} \times \cdots \times \mathfrak{B}}_{m}$. (2) Let $V = \{+, -, \cdot\}$, and let $\boldsymbol{P}$ be a family $\boldsymbol{P}_{V,V}\{\xi_0, \xi_1, \cdots\}$ of basic mapping-formulas of the form $\xi_\mu(x \pm y) = \xi_\mu(x) \pm \xi_\mu(y)$, $\xi_\mu(xy) = \sum_{i=0}^{\mu} \xi_{\mu-i}(x)\xi_i(y)$[7]. Then the $\boldsymbol{P}$-product system $\boldsymbol{P}(\mathfrak{B})$ over a ring $\mathfrak{B}$ can be considered as the ring of formal power series over $\mathfrak{B}$. (3) Let $V = \{+, -, \cdot\}$ and $\boldsymbol{P} = \boldsymbol{P}_{V,V}\{\xi_{11}, \cdots, \xi_{1m}, \cdots, \xi_{m1}, \cdots, \xi_{mm}\}$.

7) In this example, $\sum_{i=1}^{n} X_i$ denotes the V-polynomial $((\cdots((X_1+X_2)+X_3)\cdots)+X_n)$. But, $\sum_{i=1}^{n} X_i$ may be considered as the sum in the usual sense, because $\mathfrak{B}$ satisfies the associative law $(x+y)+z = x+(y+z)$.

If the mapping-formulas of $\boldsymbol{P}$ are of the form $\xi_{\mu\nu}(x\pm y)=\xi_{\mu\nu}(x)\pm\xi_{\mu\nu}(y)$, $\xi_{\mu\nu}(xy)=\sum_{\lambda=1}^{m}\xi_{\mu\lambda}(x)\xi_{\lambda\nu}(y)$[7], then the $\boldsymbol{P}$-product system $\boldsymbol{P}(\mathfrak{B})$ over a ring $\mathfrak{B}$ forms the total matric algebra of degree m over $\mathfrak{B}$. (4) Let $V=\{\oplus, \ominus, *\}$ and $W=\{+, -, \cdot\}$. If $\boldsymbol{P}$ is the family which consists of basic mapping-formulas $\xi(x\oplus y)=\xi(x)+\xi(y)$, $\xi(x\ominus y)=\xi(x)-\xi(y)$ and $\xi(x*y)=\xi(x)\xi(y)-\xi(y)\xi(x)$, then the $\boldsymbol{P}$-product system $\boldsymbol{P}(\mathfrak{B})$ over a ring $\mathfrak{B}$ forms a Lie ring.

Let $\boldsymbol{P}$ be a family $\boldsymbol{P}_{V,W}\{\xi_1, \cdots, \xi_m\}$ of basic mapping-formulas. And let $\mathfrak{A}$ be a ϕ_V-algebraic system, and $\mathfrak{B}$ a ϕ_W-algebraic system. A homomorphism from $\mathfrak{A}$ into $\boldsymbol{P}(\mathfrak{B})$ may be called a representation of $\mathfrak{A}$ into $\boldsymbol{P}(\mathfrak{B})$. Now we shall show the fundamental theorem as follows:

Theorem 1.1. (The algebraic Taylor's expansion theorem)[8] *Let $\boldsymbol{P}$ be a family $\boldsymbol{P}_{V,W}\{\xi_1, \cdots, \xi_m\}$ of basic mapping-formulas. And let $\mathfrak{A}$ be a ϕ_V-algebraic system, and $\mathfrak{B}$ a ϕ_W-algebraic system. Then $\{\varphi_1, \cdots, \varphi_m\}$ is a system of $\boldsymbol{P}$-mappings from $\mathfrak{A}$ into $\mathfrak{B}$, if and only if the mapping θ:*

$$\mathfrak{A}\ni a\rightarrow\theta(a)=[\varphi_1(a), \cdots, \varphi_m(a)]\in\boldsymbol{P}(\mathfrak{B})$$

is a representation of $\mathfrak{A}$ into $\boldsymbol{P}(\mathfrak{B})$.

Proof. Let v be any composition of V, and let $a_1, \cdots, a_{N(v)}$ be any elements of $\mathfrak{A}$. Then, by the definitions of θ and $\boldsymbol{P}(\mathfrak{B})$, we have

$$\begin{aligned}(1.1)\qquad &\theta(v(a_1, \cdots, a_{N(v)}))\\ &=[\varphi_1(v(a_1, \cdots, a_{N(v)})), \cdots, \varphi_m(v(a_1, \cdots, a_{N(v)}))],\end{aligned}$$

and

$$\begin{aligned}(1.2)\qquad &v(\theta(a_1), \cdots, \theta(a_{N(v)}))\\ &=v([\varphi_1(a_1), \cdots, \varphi_m(a_1)], \cdots, [\varphi_1(a_{N(v)}), \cdots, \varphi_m(a_{N(v)})])\\ &=\left[P_{\xi_1 v}\begin{pmatrix}\varphi_1(a_1), \cdots, \varphi_1(a_{N(v)})\\ \cdots\cdots\cdots\cdots\\ \varphi_m(a_1), \cdots, \varphi_m(a_{N(v)})\end{pmatrix}, \cdots, P_{\xi_m v}\begin{pmatrix}\varphi_1(a_1), \cdots, \varphi_1(a_{N(v)})\\ \cdots\cdots\cdots\cdots\\ \varphi_m(a_1), \cdots, \varphi_m(a_{N(v)})\end{pmatrix}\right].\end{aligned}$$

Now suppose that θ is a representation of $\mathfrak{A}$ into $\boldsymbol{P}(\mathfrak{B})$. Then we have

$$(1.3)\qquad \theta(v(a_1, \cdots, a_{N(v)}))=v(\theta(a_1), \cdots, \theta(a_{N(v)})).$$

Hence, by (1.1) and (1.2), we have

$$\begin{aligned}(1.4)\qquad &[\varphi_1(v(a_1, \cdots, a_{N(v)})), \cdots, \varphi_m(v(a_1, \cdots, a_{N(v)}))]\\ &=\left[P_{\xi_1 v}\begin{pmatrix}\varphi_1(a_1), \cdots, \varphi_1(a_{N(v)})\\ \cdots\cdots\cdots\cdots\\ \varphi_m(a_1), \cdots, \varphi_m(a_{N(v)})\end{pmatrix}, \cdots, P_{\xi_m v}\begin{pmatrix}\varphi_1(a_1), \cdots, \varphi_1(a_{N(v)})\\ \cdots\cdots\cdots\cdots\\ \varphi_m(a_1), \cdots, \varphi_m(a_{N(v)})\end{pmatrix}\right].\end{aligned}$$

8) Cf. [5; P. 100].

Hence we have

$$(1.5)\quad \varphi_\mu(v(a_1, \cdots, a_{N(v)})) = P_{\xi_\mu v}\begin{pmatrix}\varphi_1(a_1), \cdots, \varphi_1(a_{N(v)}) \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ \varphi_m(a_1), \cdots, \varphi_m(a_{N(v)})\end{pmatrix} \quad (\mu = 1, \cdots, m),$$

i.e., $\{\varphi_1, \cdots, \varphi_m\}$ is a system of $\boldsymbol{P}$-mappings. Conversely we assume that $\{\varphi_1, \cdots, \varphi_m\}$ is a system of $\boldsymbol{P}$-mappings from $\mathfrak{A}$ into $\mathfrak{B}$. Then we have the identities (1.5). Hence we have the identity (1.4). Therefore it is clear from (1.1) and (1.2) that the identity (1.3) is true, i.e., θ is a representation of $\mathfrak{A}$ into $\boldsymbol{P}(\mathfrak{B})$. This completes the proof.

The homomorphism θ in the above theorem is called a homomorphism deduced from the $\boldsymbol{P}$-mappings $\varphi_1, \cdots, \varphi_m$. Now we shall show the following two theorems as the simple applications of the above theorem.

Theorem 1.2. *Let $\boldsymbol{P}$ be a family $\boldsymbol{P}_{V,W}\{\xi_1, \cdots, \xi_m\}$ of basic mapping-formulas. And let $\mathfrak{A}$ be a ϕ_V-algebraic system generated by $a_1, \cdots, a_r$, and $\mathfrak{B}$ a ϕ_W-algebraic system. If $\{\varphi_1, \cdots, \varphi_m\}$ and $\{\varphi_1', \cdots, \varphi_m'\}$ are two systems of $\boldsymbol{P}$-mappings from $\mathfrak{A}$ into $\mathfrak{B}$ such that $\varphi_\mu(a_\rho) = \varphi_\mu'(a_\rho)$ $(\mu=1, \cdots, m; \rho=1, \cdots, r)$, then $\varphi_\mu = \varphi_\mu'$ $(\mu=1, \cdots, m)$.*

Proof. Let θ and θ' be the homomorphisms deduced from $\varphi_1, \cdots, \varphi_m$ and $\varphi_1', \cdots, \varphi_m'$ respectively. Then we have

$$\theta(a_\rho) = [\varphi_1(a_\rho), \cdots, \varphi_m(a_\rho)] = [\varphi_1'(a_\rho), \cdots, \varphi_m'(a_\rho)] = \theta'(a_\rho)$$

for all a_ρ in the generator system $\{a_1, \cdots, a_r\}$ of $\mathfrak{A}$. Hence we have $\theta = \theta'$. Therefore it is easily verified by Theorem 1.1 that $\varphi_\mu(a) = \varphi_\mu'(a)$ $(\mu=1, \cdots, m)$ for all elements a in $\mathfrak{A}$. This completes the proof.

Theorem 1.3. *Let $\boldsymbol{P}$ be any family $\boldsymbol{P}_{V,W}\{\xi_1, \cdots, \xi_m\}$ of basic mapping-formulas. And let $\mathfrak{A}$ be any free ϕ_V-algebraic system $F(\{a_1, \cdots, a_r\}, \phi_V)$, and $\mathfrak{B}$ any ϕ_W-algebraic system. Then, for any elements $b_{\mu\rho}$ $(\mu=1, \cdots, m; \rho=1, \cdots, r)$ of $\mathfrak{B}$, there exists a system $\{\varphi_1, \cdots, \varphi_m\}$ of $\boldsymbol{P}$-mappings from $\mathfrak{A}$ into $\mathfrak{B}$ such that $\varphi_\mu(a_\rho) = b_{\mu\rho}$ $(\mu=1, \cdots, m; \rho=1, \cdots, r)$.*

Proof. Since $\mathfrak{A}$ is a free ϕ_V-algebraic system freely generated by $a_1, \cdots, a_r$, it is clear that the mapping $a_\rho \to [b_{1\rho}, \cdots, b_{m\rho}]$ can be extended to a homomorphism from $\mathfrak{A}$ into $\boldsymbol{P}(\mathfrak{B})$. Hence it is obvious by Theorem 1.1 that there exists a system $\{\varphi_1, \cdots, \varphi_m\}$ of $\boldsymbol{P}$-mappings from $\mathfrak{A}$ into $\mathfrak{B}$ such that $\varphi_\mu(a_\rho) = b_{\mu\rho}$ $(\mu=1, \cdots, m; \rho=1, \cdots, r)$.

§ 2. The notation $\boldsymbol{F}_{\xi_\mu f(x_1 \cdots x_r)}(\xi_1(\boldsymbol{x}_1), \cdots, \xi_1(\boldsymbol{x}_r), \cdots, \xi_m(\boldsymbol{x}_1), \cdots, \xi_m(\boldsymbol{x}_r))$.

Let $\boldsymbol{P}$ be a family $\boldsymbol{P}_{V,W}\{\xi_1, \cdots, \xi_m\}$ of basic mapping-formulas. And let $\mathfrak{A}$ be any free ϕ_V-algebraic system $F(\{x_1, \cdots, x_r\}, \phi_V)$, and $\mathfrak{B}$ the free ϕ_W-algebraic system $F(\{\xi_1(x_1), \cdots, \xi_1(x_r), \cdots, \xi_m(x_1), \cdots, \xi_m(x_r)\}, \phi_W)$. Then it is clear from Theorems 1.2 and 1.3 that there exists one and only one system $\{\varphi_1, \cdots, \varphi_m\}$ of $\boldsymbol{P}$-mappings from $\mathfrak{A}$ into $\mathfrak{B}$ such that

$$(2.1)\qquad \varphi_\mu(x_\rho) = \xi_\mu(x_\rho) \quad (\mu = 1, \cdots, m; \rho=1, \cdots, r).$$

For a V-polynomial $f(x_1, \cdots, x_r)$ in $\mathfrak{A}$, we denote by

$$(2.2)\qquad F_{\xi_\mu f(x_1\cdots x_r)}(\xi_1(x_1), \cdots, \xi_1(x_r), \cdots, \xi_m(x_1), \cdots, \xi_m(x_r))$$ [9]

the W-polynomial $\varphi_\mu(f(x_1, \cdots, x_r))$ in $\mathfrak{B}$. Then, since

$$\begin{aligned}&\varphi_\mu(v(f_1(x_1, \cdots, x_r), \cdots, f_{N(v)}(x_1, \cdots, x_r)))\\ &= P_{\xi_\mu v}\begin{pmatrix}\varphi_1(f_1(x_1, \cdots, x_r)), \cdots, \varphi_1(f_{N(v)}(x_1, \cdots, x_r))\\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\ \varphi_m(f_1(x_1, \cdots, x_r)), \cdots, \varphi_m(f_{N(v)}(x_1, \cdots, x_r))\end{pmatrix},\end{aligned}$$

we have

$$\begin{aligned}(2.3)\quad &F_{\xi_\mu v(f_1(x_1\cdots x_r), \cdots, f_{N(v)}(x_1\cdots x_r))}(\xi_1(x_1), \cdots, \xi_1(x_r), \cdots, \xi_m(x_1), \cdots, \xi_m(x_r))\\ &= P_{\xi_\mu v}\begin{pmatrix} F_{\xi_1 f_1(x_1\cdots x_r)}\begin{pmatrix}\xi_1(x_1)\cdots\xi_1(x_r)\\ \cdots\cdots\cdots\\ \xi_m(x_1)\cdots\xi_m(x_r)\end{pmatrix}, \cdots, F_{\xi_1 f_{N(v)}(x_1\cdots x_r)}\begin{pmatrix}\xi_1(x_1)\cdots\xi_1(x_r)\\ \cdots\cdots\cdots\\ \xi_m(x_1)\cdots\xi_m(x_r)\end{pmatrix}\\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\ F_{\xi_m f_1(x_1\cdots x_r)}\begin{pmatrix}\xi_1(x_1)\cdots\xi_1(x_r)\\ \cdots\cdots\cdots\\ \xi_m(x_1)\cdots\xi_m(x_r)\end{pmatrix}, \cdots, F_{\xi_m f_{N(v)}(x_1\cdots x_r)}\begin{pmatrix}\xi_1(x_1)\cdots\xi_1(x_r)\\ \cdots\cdots\cdots\\ \xi_m(x_1)\cdots\xi_m(x_r)\end{pmatrix}\end{pmatrix}.\end{aligned}$$

Now it is easy to see from (2.1) and (2.3) that the W-polynomial (2.2) coincides with the decomposition of $\xi_\mu(f(x_1, \cdots, x_r))$ which is obtained by making formal use of the family $\boldsymbol{P}$. Therefore we can easily obtain the following:

Theorem 2.1. *Let* $\boldsymbol{P}$ *be any family* $\boldsymbol{P}_{V,W}\{\xi_1, \cdots, \xi_m\}$ *of basic mapping-formulas. Then*

$$\begin{aligned}&F_{\xi_\mu f(g_1(x_1\cdots x_r), \cdots, g_s(x_1\cdots x_r))}(\xi_1(x_1), \cdots, \xi_1(x_r), \cdots, \xi_m(x_1), \cdots, \xi_m(x_r))\\ &= F_{\xi_\mu f(x_1\cdots x_s)}\begin{pmatrix} F_{\xi_1 g_1(x_1\cdots x_r)}\begin{pmatrix}\xi_1(x_1)\cdots\xi_1(x_r)\\ \cdots\cdots\cdots\\ \xi_m(x_1)\cdots\xi_m(x_r)\end{pmatrix}, \cdots, F_{\xi_1 g_s(x_1\cdots x_r)}\begin{pmatrix}\xi_1(x_1)\cdots\xi_1(x_r)\\ \cdots\cdots\cdots\\ \xi_m(x_1)\cdots\xi_m(x_r)\end{pmatrix}\\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\ F_{\xi_m g_1(x_1\cdots x_r)}\begin{pmatrix}\xi_1(x_1)\cdots\xi_1(x_r)\\ \cdots\cdots\cdots\\ \xi_m(x_1)\cdots\xi_m(x_r)\end{pmatrix}, \cdots, F_{\xi_m g_s(x_1\cdots x_r)}\begin{pmatrix}\xi_1(x_1)\cdots\xi_1(x_r)\\ \cdots\cdots\cdots\\ \xi_m(x_1)\cdots\xi_m(x_r)\end{pmatrix}\end{pmatrix}\end{aligned}$$

for any V-polynomial $f(g_1(x_1, \cdots, x_r), \cdots, g_s(x_1, \cdots, x_r))$.

Theorem 2.2. *Let* $\mathfrak{A}$ *be a* ϕ_V*-algebraic system, and* $\mathfrak{B}$ *a* ϕ_W*-algebraic system. And let* a_ρ $(\rho=1, \cdots, r)$ *be elements of* $\mathfrak{A}$, *and* $b_{\mu\rho}$ $(\mu=1, \cdots, m;$

9) This may be considered as a generalization of the symbol $\sum_{i=1}^{r}\frac{\partial f(x_1\cdots x_r)}{\partial x_i}D(x_i)$ in the theory of derivations.

$\rho=1, \cdots, r$) elements of $\mathfrak{B}$. If $\{\varphi_1, \cdots, \varphi_m\}$ is a system of $\boldsymbol{P}_{V,W}\{\xi_1, \cdots, \xi_m\}$-mappings from $\mathfrak{A}$ into $\mathfrak{B}$ such that $\varphi_\mu(a_\rho)=b_{\mu\rho}$ ($\mu=1, \cdots, m$; $\rho=1, \cdots, r$), then

$$\varphi_\mu(f(a_1, \cdots, a_r)) = F_{\xi_\mu f(x_1\cdots x_r)}\begin{pmatrix} b_{11}, \cdots, b_{1r} \\ \cdots\cdots\cdots\cdots \\ b_{m1}, \cdots, b_{mr} \end{pmatrix} \quad (\mu = 1, \cdots, m) \tag{2.4}$$

for every V-polymonial $f(x_1, \cdots, x_r)$.

Proof. We shall prove this theorem by induction on order of V-polynomials $f(x_1, \cdots, x_r)$. For every V-polynomial of order 0, the identities (2.4) are clearly true. Now assume that the identities (2.4) are valid for every V-polynomial of order $k-1$ or less. Let $f(x_1, \cdots, x_r)$ be any V-polynomial of order k. Then we have

$$f(x_1, \cdots, x_r) = v(g_1(x_1, \cdots, x_r), \cdots, g_{N(v)}(x_1, \cdots, x_r)),$$

and hence

$$f(a_1, \cdots, a_r) = v(g_1(a_1, \cdots, a_r), \cdots, g_{N(v)}(a_1, \cdots, a_r)).$$

Since $g_N(x_1, \cdots, x_r)$ ($N=1, \cdots, N(v)$) are of order $k-1$ or less, it follows from the assumption of induction that

$$\varphi_\mu(g_N(a_1, \cdots, a_r)) = F_{\xi_\mu g_N(x_1\cdots x_r)}(b_{11}, \cdots, b_{1r}, \cdots, b_{m1}, \cdots, b_{mr}).$$

Hence we have

$$\begin{aligned}
&\varphi_\mu(f(a_1, \cdots, a_r)) = \varphi_\mu(v(g_1(a_1, \cdots, a_r), \cdots, g_{N(v)}(a_1, \cdots, a_r))) \\
&= P_{\xi_\mu v}\begin{pmatrix} \varphi_1(g_1(a_1, \cdots, a_r)), \cdots, \varphi_1(g_{N(v)}(a_1, \cdots, a_r)) \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ \varphi_m(g_1(a_1, \cdots, a_r)), \cdots, \varphi_m(g_{N(v)}(a_1, \cdots, a_r)) \end{pmatrix} \\
&= P_{\xi_\mu v}\begin{pmatrix} F_{\xi_1 g_1(x_1\cdots x_r)}\begin{pmatrix} b_{11}\cdots b_{1r} \\ \cdots\cdots \\ b_{m1}\cdots b_{mr} \end{pmatrix}, \cdots, F_{\xi_1 g_{N(v)}(x_1\cdots x_r)}\begin{pmatrix} b_{11}\cdots b_{1r} \\ \cdots\cdots \\ b_{m1}\cdots b_{mr} \end{pmatrix} \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ F_{\xi_m g_1(x_1\cdots x_r)}\begin{pmatrix} b_{11}\cdots b_{1r} \\ \cdots\cdots \\ b_{m1}\cdots b_{mr} \end{pmatrix}, \cdots, F_{\xi_m g_{N(v)}(x_1\cdots x_r)}\begin{pmatrix} b_{11}\cdots b_{1r} \\ \cdots\cdots \\ b_{m1}\cdots b_{mr} \end{pmatrix} \end{pmatrix}.
\end{aligned}$$

On the other hand, using the identity (2.3), we have

$$\begin{aligned}
&F_{\xi_\mu f(x_1\cdots x_r)}(b_{11}, \cdots, b_{1r}, \cdots, b_{m1}, \cdots, b_{mr}) \\
&= P_{\xi_\mu v}\begin{pmatrix} F_{\xi_1 g_1(x_1\cdots x_r)}\begin{pmatrix} b_{11}\cdots b_{1r} \\ \cdots\cdots \\ b_{m1}\cdots b_{mr} \end{pmatrix}, \cdots, F_{\xi_1 g_{N(v)}(x_1\cdots x_r)}\begin{pmatrix} b_{11}\cdots b_{1r} \\ \cdots\cdots \\ b_{m1}\cdots b_{mr} \end{pmatrix} \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ F_{\xi_m g_1(x_1\cdots x_r)}\begin{pmatrix} b_{11}\cdots b_{1r} \\ \cdots\cdots \\ b_{m1}\cdots b_{mr} \end{pmatrix}, \cdots, F_{\xi_m g_{N(v)}(x_1\cdots x_r)}\begin{pmatrix} b_{11}\cdots b_{1r} \\ \cdots\cdots \\ b_{m1}\cdots b_{mr} \end{pmatrix} \end{pmatrix}.
\end{aligned}$$

Hence we have

$$\varphi_\mu(f(a_1, \cdots, a_r)) = F_{\xi_\mu f(x_1 \cdots x_r)}(b_{11}, \cdots, b_{1r}, \cdots, b_{m1}, \cdots, b_{mr}).$$

Therefore it is clear by induction that the identities (2.4) are valid for every V-polynomial.

Theorem 2.3. *Let $\boldsymbol{P}(\mathfrak{B})$ be any $\boldsymbol{P}_{V,W}\{\xi_1, \cdots, \xi_m\}$-product system over any ϕ_W-algebraic system $\mathfrak{B}$. Then*

$$f([b_1^1, \cdots, b_m^1], \cdots, [b_1^r, \cdots, b_m^r]) = \left[F_{\xi_1 f(x_1 \cdots x_r)}\begin{pmatrix} b_1^1, \cdots, b_1^r \\ \cdots\cdots\cdots\cdots \\ b_m^1, \cdots, b_m^r \end{pmatrix}, \cdots, F_{\xi_m f(x_1 \cdots x_r)}\begin{pmatrix} b_1^1, \cdots, b_1^r \\ \cdots\cdots\cdots\cdots \\ b_m^1, \cdots, b_m^r \end{pmatrix}\right]$$

for every V-polynomial $f(x_1, \cdots, x_r)$, and for every set of elements $[b_1^\rho, \cdots, b_m^\rho]$ $(\rho=1, \cdots, r)$ in $\boldsymbol{P}(\mathfrak{B})$.

Proof. Let φ_μ $(\mu=1, \cdots, m)$ be mappings from $\boldsymbol{P}(\mathfrak{B})$ onto $\mathfrak{B}$ each of which is defined by

$$\varphi_\mu : [b_1, \cdots, b_m] \to b_\mu \quad \text{for all} \quad [b_1, \cdots, b_m] \in \boldsymbol{P}(\mathfrak{B}).$$

Then it is easy to see that $\{\varphi_1, \cdots, \varphi_m\}$ is a system of $\boldsymbol{P}$-mappings from $\boldsymbol{P}(\mathfrak{B})$ onto $\mathfrak{B}$. Since $\varphi_\mu([b_1^\rho, \cdots, b_m^\rho]) = b_\mu^\rho$, it is easily verified by Theorem 2.2 that

$$\varphi_\mu(f([b_1^1, \cdots, b_m^1], \cdots, [b_1^r, \cdots, b_m^r])) = F_{\xi_\mu f(x_1 \cdots x_r)}(b_1^1, \cdots, b_1^r, \cdots, b_m^1, \cdots, b_m^r).$$

Hence we have

$$\begin{aligned} & f([b_1^1, \cdots, b_m^1], \cdots, [b_1^r, \cdots, b_m^r]) \\ &= [\varphi_1(f([b_1^1, \cdots, b_m^1], \cdots, [b_1^r, \cdots, b_m^r])), \cdots \\ &\qquad \cdots, \varphi_m(f([b_1^1, \cdots, b_m^1], \cdots, [b_1^r, \cdots, b_m^r]))] \\ &= \left[F_{\xi_1 f(x_1 \cdots x_r)}\begin{pmatrix} b_1^1, \cdots, b_1^r \\ \cdots\cdots\cdots\cdots \\ b_m^1, \cdots, b_m^r \end{pmatrix}, \cdots, F_{\xi_m f(x_1 \cdots x_r)}\begin{pmatrix} b_1^1, \cdots, b_1^r \\ \cdots\cdots\cdots\cdots \\ b_m^1, \cdots, b_m^r \end{pmatrix}\right]. \end{aligned}$$

This completes the proof.

§ 3. $(\boldsymbol{A}_V, \boldsymbol{B}_W)$-universality and existence of $\boldsymbol{P}$-mappings.

Let $\boldsymbol{P}$ be a family $\boldsymbol{P}_{V,W}\{\xi_1, \cdots, \xi_m\}$ of basic mapping-formulas. And let A_V, B_W be systems of composition-identities with respect to V, W respectively. If, for any free A_V-algebraic system $\mathfrak{A} = F(\{a_1, \cdots, a_r\}, A_V)$ and for any elements $b_{\mu\rho}$ $(\mu=1, \cdots, m; \rho=1, \cdots, r)$ of any B_W-algebraic system $\mathfrak{B}$, there exists a system $\{\varphi_1, \cdots, \varphi_m\}$ of $\boldsymbol{P}$-mappings from $\mathfrak{A}$

into $\mathfrak{B}$ such that $\varphi_\mu(a_\rho)=b_{\mu\rho}$ ($\mu=1, \cdots, m; \rho=1, \cdots, r$), then we say that $\boldsymbol{P}$ is (A_V, B_W)-universal. And also, if, for any B_W-algebraic system $\mathfrak{B}$, the $\boldsymbol{P}$-product system $\boldsymbol{P}(\mathfrak{B})$ over $\mathfrak{B}$ forms an A_V-algebraic system, then we say that $\boldsymbol{P}$ is a constructor of an A_V-algebraic system from a B_W-algebaic system, or simply that $\boldsymbol{P}$ is an (A_V, B_W)-constructor.

REMARK: The $\boldsymbol{P}$-product system $\boldsymbol{P}(\mathfrak{B})$, defined by an ($A_V$, B_W)-constructor $\boldsymbol{P}$, over a B_W-algebraic system $\mathfrak{B}$ can be considered as a generalization of the concept of an algebra over a field.

Theorem 3.1. *Let $\boldsymbol{P}$ be a family $\boldsymbol{P}_{V,W}\{\xi_1, \cdots, \xi_m\}$ of basic mapping-formulas. And let A_V, B_W be systems of composition-identities with respect to V, W respectively. Then, $\boldsymbol{P}$ is (A_V, B_W)-universal if and only if $\boldsymbol{P}$ is an (A_V, B_W)-constructor.*

Proof of "only if" part. Let $f(x_1, \cdots, x_s)=g(x_1, \cdots, x_s)$ be any composition-identity in A_V. And let $\mathfrak{B}$ be any B_W-algebraic system, and $[b_1^\sigma, \cdots, b_m^\sigma]$ ($\sigma=1, \cdots, s$) any elements of $\boldsymbol{P}(\mathfrak{B})$. Now suppose that $\boldsymbol{P}$ is (A_V, B_W)-universal. Then there exists a system $\{\varphi_1, \cdots, \varphi_m\}$ of $\boldsymbol{P}$-mappings from $\mathfrak{A}=F(\{a_1, \cdots, a_s\}, A_V)$ into $\mathfrak{B}$ such that $\varphi_\mu(a_\sigma)=b_\mu^\sigma$ ($\mu=1, \cdots, m; \sigma=1, \cdots, s$). Hence it is clear by Theorem 1.1 that there exists a homomorphism θ from $\mathfrak{A}$ into $\boldsymbol{P}(\mathfrak{B})$ which satisfies

$$\theta(a_\sigma) = [\varphi_1(a_\sigma), \cdots, \varphi_m(a_\sigma)] = [b_1^\sigma, \cdots, b_m^\sigma] \quad (\sigma = 1, \cdots, s).$$

Hence we have

$$\begin{aligned} & f([b_1^1, \cdots, b_m^1], \cdots, [b_1^s, \cdots, b_m^s]) \\ &= f(\theta(a_1), \cdots, \theta(a_s)) = \theta(f(a_1, \cdots, a_s)) \\ &= \theta(g(a_1, \cdots, a_s)) = g(\theta(a_1), \cdots, \theta(a_s)) \\ &= g([b_1^1, \cdots, b_m^1], \cdots, [b_1^s, \cdots, b_m^s]). \end{aligned}$$

Therefore $\boldsymbol{P}(\mathfrak{B})$ is an A_V-algebraic system. Hence $\boldsymbol{P}$ is an (A_V, B_W)-constructor.

Proof of "if" part. Let $\mathfrak{A}$ be any free A_V-algebraic system freely generated by $a_1, \cdots, a_r$. And let $\mathfrak{B}$ be any B_W-algebraic system, and $b_{\mu\rho}$ ($\mu=1, \cdots, m; \rho=1, \cdots, r$) any elements of $\mathfrak{B}$. Now, suppose that $\boldsymbol{P}$ is an (A_V, B_W)-constructor. Then $\boldsymbol{P}(\mathfrak{B})$ forms an A_V-algebraic system. Hence there exists a homomorphism θ from $\mathfrak{A}$ into $\boldsymbol{P}(\mathfrak{B})$ such that $\theta(a_\rho)=[b_{1\rho}, \cdots, b_{m\rho}]$ ($\rho=1, \cdots, r$). Hence it is obvious from Theorem 1.1 that there exists a system $\{\varphi_1, \cdots, \varphi_m\}$ of $\boldsymbol{P}$-mappings from $\mathfrak{A}$ into $\mathfrak{B}$ such that $\varphi_\mu(a_\rho)=b_{\mu\rho}$ ($\mu=1, \cdots, m; \rho=1, \cdots, r$). Therefore $\boldsymbol{P}$ is (A_V, B_W)-universal.

Theorem 3.2. *Under the same assumptions as in Theorem* 3.1, *the necessary and sufficient condition that* $\boldsymbol{P}$ *is an* (A_V, B_W)*-constructor, i.e., an* (A_V, B_W)*-universal family, is that*

$$(3.1)\quad \begin{cases} F_{\xi_\mu f(x_1\cdots x_s)}\begin{pmatrix}\xi_1(x_1), \cdots, \xi_1(x_s)\\ \cdots\cdots\cdots\cdots\\ \xi_m(x_1), \cdots, \xi_m(x_s)\end{pmatrix} \overset{B_W}{=} F_{\xi_\mu g(x_1\cdots x_s)}\begin{pmatrix}\xi_1(x_1), \cdots, \xi_1(x_s)\\ \cdots\cdots\cdots\cdots\\ \xi_m(x_1), \cdots, \xi_m(x_s)\end{pmatrix} (\mu=1, \cdots, m) \\ \textit{for every composition-identity } f(x_1, \cdots, x_s)=g(x_1, \cdots, x_s) \textit{ in } A_V. \end{cases}$$

Proof of necessity. Let $f(x_1, \cdots, x_s)=g(x_1, \cdots, x_s)$ be any composition-identity of A_V. And let $\mathfrak{B}$ be the free B_W-algebraic system freely generated by $\xi_1(x_1), \cdots, \xi_1(x_s), \cdots, \xi_m(x_1), \cdots, \xi_m(x_s)$. Now suppose that $\boldsymbol{P}$ is an (A_V, B_W)-constructor. Then the $\boldsymbol{P}$-product system $\boldsymbol{P}(\mathfrak{B})$ over $\mathfrak{B}$ is an A_V-algebraic system. Hence we have

$$(3.2)\quad \begin{aligned} &f([\xi_1(x_1), \cdots, \xi_m(x_1)], \cdots, [\xi_1(x_s), \cdots, \xi_m(x_s)]) \\ &= g([\xi_1(x_1), \cdots, \xi_m(x_1)], \cdots, [\xi_1(x_s), \cdots, \xi_m(x_s)]), \end{aligned}$$

where $[\xi_1(x_\sigma), \cdots, \xi_m(x_\sigma)]$ $(\sigma=1, \cdots, s)$ are elements of $\boldsymbol{P}(\mathfrak{B})$. On the other hand, by Theorem 2.3, we have

$$\begin{aligned} &f([\xi_1(x_1), \cdots, \xi_m(x_1)], \cdots, [\xi_1(x_s), \cdots, \xi_m(x_s)]) \\ &= \left[F_{\xi_1 f(x_1\cdots x_s)}\begin{pmatrix}\xi_1(x_1), \cdots, \xi_1(x_s)\\ \cdots\cdots\cdots\cdots\\ \xi_m(x_1), \cdots, \xi_m(x_s)\end{pmatrix}, \cdots, F_{\xi_m f(x_1\cdots x_s)}\begin{pmatrix}\xi_1(x_1), \cdots, \xi_1(x_s)\\ \cdots\cdots\cdots\cdots\\ \xi_m(x_1), \cdots, \xi_m(x_s)\end{pmatrix}\right], \end{aligned}$$

and

$$\begin{aligned} &g([\xi_1(x_1), \cdots, \xi_m(x_1)], \cdots, [\xi_1(x_s), \cdots, \xi_m(x_s)]) \\ &= \left[F_{\xi_1 g(x_1\cdots x_s)}\begin{pmatrix}\xi_1(x_1), \cdots, \xi_1(x_s)\\ \cdots\cdots\cdots\cdots\\ \xi_m(x_1), \cdots, \xi_m(x_s)\end{pmatrix}, \cdots, F_{\xi_m g(x_1\cdots x_s)}\begin{pmatrix}\xi_1(x_1), \cdots, \xi_1(x_s)\\ \cdots\cdots\cdots\cdots\\ \xi_m(x_1), \cdots, \xi_m(x_s)\end{pmatrix}\right]. \end{aligned}$$

Hence it is clear from (3.2) that

$$F_{\xi_\mu f(x_1\cdots x_s)}\begin{pmatrix}\xi_1(x_1), \cdots, \xi_1(x_s)\\ \cdots\cdots\cdots\cdots\\ \xi_m(x_1), \cdots, \xi_m(x_s)\end{pmatrix} = F_{\xi_\mu g(x_1\cdots x_s)}\begin{pmatrix}\xi_1(x_1), \cdots, \xi_1(x_s)\\ \cdots\cdots\cdots\cdots\\ \xi_m(x_1), \cdots, \xi_m(x_s)\end{pmatrix} \quad (\mu=1, \cdots, m)$$

are true in $\mathfrak{B}$. This completes the proof of necessity.

Proof of sufficiency. Let $f(x_1, \cdots, x_s)=g(x_1, \cdots, x_s)$ be any composition-identity of A_V. And let $\mathfrak{B}$ be any B_W-algebraic system, and $[b_1^\sigma, \cdots, b_m^\sigma]$ $(\sigma=1, \cdots, s)$ any elements of $\boldsymbol{P}(\mathfrak{B})$. Then it follows from the condition (3.1) that

$$F_{\xi_\mu f(x_1\cdots x_s)}\begin{pmatrix}b_1^1, \cdots, b_1^s\\ \cdots\cdots\cdots\\ b_m^1, \cdots, b_m^s\end{pmatrix} = F_{\xi_\mu g(x_1\cdots x_s)}\begin{pmatrix}b_1^1, \cdots, b_1^s\\ \cdots\cdots\cdots\\ b_m^1, \cdots, b_m^s\end{pmatrix} \quad (\mu=1, \cdots, m).$$

Hence, by Theorem 2. 3, we have

$$\begin{aligned} & f([b_1^1, \cdots, b_m^1], \cdots, [b_1^s, \cdots, b_m^s]) \\ &= \left[F_{\xi_1 f(x_1\cdots x_s)}\begin{pmatrix} b_1^1, \cdots, b_1^s \\ \cdots\cdots\cdots \\ b_m^1, \cdots, b_m^s \end{pmatrix}, \cdots, F_{\xi_m f(x_1\cdots x_s)}\begin{pmatrix} b_1^1, \cdots, b_1^s \\ \cdots\cdots\cdots \\ b_m^1, \cdots, b_m^s \end{pmatrix}\right] \\ &= \left[F_{\xi_1 g(x_1\cdots x_s)}\begin{pmatrix} b_1^1, \cdots, b_1^s \\ \cdots\cdots\cdots \\ b_m^1, \cdots, b_m^s \end{pmatrix}, \cdots, F_{\xi_m g(x_1\cdots x_s)}\begin{pmatrix} b_1^1, \cdots, b_1^s \\ \cdots\cdots\cdots \\ b_m^1, \cdots, b_m^s \end{pmatrix}\right] \\ &= g([b_1^1, \cdots, b_m^1], \cdots, [b_1^s, \cdots, b_m^s]). \end{aligned}$$

Therefore $\boldsymbol{P}(\mathfrak{B})$ is an A_V-algebraic system. Hence $\boldsymbol{P}$ is an (A_V, B_W)-constructor.

Theorem 3. 3. *Let* $\boldsymbol{P}$ *be an* (A_V, B_W)*-universal family* $\boldsymbol{P}_{V,W}$ $\{\xi_1, \cdots, \xi_m\}$ *of basic mapping-formulas. And let* $\mathfrak{A}$ *be an* A_V*-algebraic system* $F(\{a_1, \cdots, a_r\}, A_V, R_V)$, *and* $\mathfrak{B}$ *a* B_W*-algebraic system. Moreover let* $b_{\mu\rho}$ $(\mu=1, \cdots, m; \rho=1, \cdots, r)$ *be elements of* $\mathfrak{B}$. *Then, in order that there exists a system* $\{\varphi_1, \cdots, \varphi_m\}$ *of* $\boldsymbol{P}$*-mappings from* $\mathfrak{A}$ *into* $\mathfrak{B}$ *such that* $\varphi_\mu(a_\rho)=b_{\mu\rho}$ $(\mu=1, \cdots, m; \rho=1, \cdots, r)$, *it is necessary and sufficient that*

$$(3.3)\quad \begin{cases} F_{\xi_\mu f(x_1\cdots x_r)}\begin{pmatrix} b_{11}, \cdots, b_{1r} \\ \cdots\cdots\cdots \\ b_{m1}, \cdots, b_{mr} \end{pmatrix} = F_{\xi_\mu g(x_1\cdots x_r)}\begin{pmatrix} b_{11}, \cdots, b_{1r} \\ \cdots\cdots\cdots \\ b_{m1}, \cdots, b_{mr} \end{pmatrix} \quad (\mu = 1, \cdots, m) \\ \textit{for every relation } f(a_1, \cdots, a_r)=g(a_1, \cdots, a_r) \textit{ in } R_V. \end{cases}$$

Proof. Suppose that there exists a system $\{\varphi_1, \cdots, \varphi_m\}$ of $\boldsymbol{P}$-mappings from $\mathfrak{A}$ into $\mathfrak{B}$ such that $\varphi_\mu(a_\rho)=b_{\mu\rho}$ $(\mu=1, \cdots, m; \rho=1, \cdots, r)$. Then it is obvious by Theorem 1. 1 that the mapping

$$\theta: a \to [\varphi_1(a), \cdots, \varphi_m(a)] \quad (a \in \mathfrak{A})$$

is a homomorphism from $\mathfrak{A}$ into the $\boldsymbol{P}$-product system $\boldsymbol{P}(\mathfrak{B})$. Now let $f(a_1, \cdots, a_r)=g(a_1, \cdots, a_r)$ be any relation of R_V. Then it is, of course, clear that $\theta(f(a_1, \cdots, a_r))=\theta(g(a_1, \cdots, a_r))$, i.e.,

$$\varphi_\mu(f(a_1, \cdots, a_r)) = \varphi_\mu(g(a_1, \cdots, a_r)) \quad (\mu = 1, \cdots, m).$$

Hence it is easy to see from Theorem 2. 2 that

$$F_{\xi_\mu f(x_1\cdots x_r)}\begin{pmatrix} b_{11}, \cdots, b_{1r} \\ \cdots\cdots\cdots \\ b_{m1}, \cdots, b_{mr} \end{pmatrix} = F_{\xi_\mu g(x_1\cdots x_r)}\begin{pmatrix} b_{11}, \cdots, b_{1r} \\ \cdots\cdots\cdots \\ b_{m1}, \cdots, b_{mr} \end{pmatrix} \quad (\mu = 1, \cdots, m).$$

Conversely, suppose the condition (3. 3). Then it follows from Theorem 2. 3 that, for any relation $f(a_1, \cdots, a_r)=g(a_1, \cdots, a_r)$ in R_V,

$$
\begin{aligned}
&f([b_{11},\cdots,b_{m1}],\cdots,[b_{1r},\cdots,b_{mr}]) \\
&=\left[F_{\xi_1 f(x_1\cdots x_r)}\begin{pmatrix} b_{11}\ ,\cdots,b_{1r} \\ \cdots\cdots\cdots\cdots \\ b_{m1}\ ,\cdots,b_{mr}\end{pmatrix},\cdots,F_{\xi_m f(x_1\cdots x_r)}\begin{pmatrix} b_{11}\ ,\cdots,b_{1r} \\ \cdots\cdots\cdots\cdots \\ b_{m1}\ ,\cdots,b_{mr}\end{pmatrix}\right] \\
&=\left[F_{\xi_1 g(x_1\cdots x_r)}\begin{pmatrix} b_{11}\ ,\cdots,b_{1r} \\ \cdots\cdots\cdots\cdots \\ b_{m1}\ ,\cdots,b_{mr}\end{pmatrix},\cdots,F_{\xi_m g(x_1\cdots x_r)}\begin{pmatrix} b_{11}\ ,\cdots,b_{1r} \\ \cdots\cdots\cdots\cdots \\ b_{m1}\ ,\cdots,b_{mr}\end{pmatrix}\right] \\
&=g([b_{11},\cdots,b_{m1}],\cdots,[b_{1r},\cdots,b_{mr}])\,.
\end{aligned}
$$

Hence it is easy to see by the fundamental theorem for free algebraic systems[10] or directly that there exists a homomorphism from $\mathfrak{A}$ into $\boldsymbol{P}(\mathfrak{B})$ which is an extension of the mapping

$$a_\rho \to [b_{1\rho},\cdots,b_{m\rho}] \quad (\rho=1,\cdots,r)\,.$$

Hence it is clear by Theorem 1.1 that there exists a system $\{\varphi_1,\cdots,\varphi_m\}$ of $\boldsymbol{P}$-mappings from $\mathfrak{A}$ into $\mathfrak{B}$ which satisfy $\varphi_\mu(a_\rho)=b_{\mu\rho}$ ($\mu=1,\cdots,m$; $\rho=1,\cdots,r$).

REMARK: The above theorem can be considered as a generalization of the criterion for the existence of a derivation, and also that of the criterion for the existence of the extension of a derivation[11].

§ 4. The product of families of basic mapping-formulas.

Let $\boldsymbol{P}$ and $\boldsymbol{Q}$ be families $\boldsymbol{P}_{V,W}\{\xi_1,\cdots,\xi_m\}$ and $\boldsymbol{Q}_{W,U}\{\eta_1,\cdots,\eta_n\}$ of basic mapping-formulas respectively. The set of the form

$$
\left\{
\begin{aligned}
&\eta_\nu\xi_\mu(v(x_1,\cdots,x_{N(v)})) \\
&=F_{\eta_\nu P_{\xi_\mu v}\begin{pmatrix} x_{11}\cdots x_{1N(v)} \\ \cdots\cdots\cdots \\ x_{m1}\cdots x_{mN(v)}\end{pmatrix}}\begin{pmatrix}\eta_1\xi_1(x_1),\cdots,\eta_1\xi_1(x_{N(v)}),\cdots,\eta_1\xi_m(x_1),\cdots,\eta_1\xi_m(x_{N(v)}) \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ \eta_n\xi_1(x_1),\cdots,\eta_n\xi_1(x_{N(v)}),\cdots,\eta_n\xi_m(x_1),\cdots,\eta_n\xi_m(x_{N(v)})\end{pmatrix}; \\
&\qquad\qquad \mu=1,\cdots,m,\quad \nu=1,\cdots,n,\quad v\in V
\end{aligned}
\right\}
$$

can be considered as a family of basic mapping-formulas of $\eta_\nu\xi_\mu$ ($\mu=1,\cdots,m$; $\nu=1,\cdots,n$). Such a family of basic mapping-formulas is called a product of $\boldsymbol{P}$ by $\boldsymbol{Q}$, and denoted by $\boldsymbol{QP}$. If $\{\varphi_1,\cdots,\varphi_m\}$ is a system of $\boldsymbol{P}$-mappings from a ϕ_V-algebraic system $\mathfrak{A}$ into a ϕ_W-algebraic system $\mathfrak{B}$, and if $\{\psi_1,\cdots,\psi_n\}$ is a system of $\boldsymbol{Q}$-mappings from $\mathfrak{B}$ into a ϕ_U-algebraic system $\mathfrak{C}$, then $\{\psi_1\varphi_1,\cdots,\psi_1\varphi_m,\cdots,\psi_n\varphi_1,\cdots,\psi_n\varphi_m\}$ is clearly a system of $\boldsymbol{QP}$-mappings from $\mathfrak{A}$ into $\mathfrak{C}$. Moreover the $\boldsymbol{QP}$-

10) Cf. [2], [6; § 3], [7; Chapter II, § 1] and [8; § 4].

11) Cf. [9; P. 12].

product system over a ϕ_U-algebraic system $\mathfrak{C}$ is, of course, denoted by $\boldsymbol{QP}(\mathfrak{C})$.

Theorem 4.1. *Let $\boldsymbol{P}$ and $\boldsymbol{Q}$ be families $\boldsymbol{P}_{V,W}\{\xi_1, \cdots, \xi_m\}$ and $\boldsymbol{Q}_{W,U}$ $\{\eta_1, \cdots, \eta_n\}$ of basic mapping-formulas respectively. If $\mathfrak{C}$ is a ϕ_U-algebraic system, then the $\boldsymbol{QP}$-product system $\boldsymbol{QP}(\mathfrak{C})$ is isomorphic to $\boldsymbol{P}(\boldsymbol{Q}(\mathfrak{C}))$.*

Proof. The mapping

$$\theta : [c_{11}, \cdots, c_{1m}, \cdots, c_{n1}, \cdots, c_{nm}] \to [[c_{11}, \cdots, c_{n1}], \cdots, [c_{1m}, \cdots, c_{nm}]]$$

is clearly a one to one mapping from $\boldsymbol{QP}(\mathfrak{C})$ onto $\boldsymbol{P}(\boldsymbol{Q}(\mathfrak{C}))$. Hereafter we shall prove that the mapping θ is an isomorphism. Let v be any composition of V, and let

$$(4.1) \qquad [c_{11}^N, \cdots, c_{1m}^N, \cdots, c_{n1}^N, \cdots, c_{nm}^N] \quad (N = 1, \cdots, N(v))$$

be any elements of $\boldsymbol{QP}(\mathfrak{C})$. If we put

$$\begin{aligned} &[c_{11}, \cdots, c_{1m}, \cdots, c_{n1}, \cdots, c_{nm}] \\ &= v([c_{11}^1, \cdots, c_{1m}^1, \cdots, c_{n1}^1, \cdots, c_{nm}^1], \cdots \\ &\qquad \cdots, [c_{11}^{N(v)}, \cdots, c_{1m}^{N(v)}, \cdots, c_{n1}^{N(v)}, \cdots, c_{nm}^{N(v)}]), \end{aligned}$$

then, by the definition of the $\boldsymbol{QP}$-product system $\boldsymbol{QP}(\mathfrak{C})$, we have

$$(4.2) \qquad c_{\nu\mu} = F_{\eta_\nu P_{\xi_\mu v}}\begin{pmatrix} x_{11} \cdots x_{1N(v)} \\ \cdots\cdots\cdots \\ x_{m1} \cdots x_{mN(v)} \end{pmatrix}\begin{pmatrix} c_{11}^1, \cdots, c_{11}^{N(v)}, \cdots, c_{1m}^1, \cdots, c_{1m}^{N(v)} \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ c_{n1}^1, \cdots, c_{n1}^{N(v)}, \cdots, c_{nm}^1, \cdots, c_{nm}^{N(v)} \end{pmatrix}.$$

On the other hand, by the mapping θ, the elements (4.1) correspond to $[[c_{11}^N, \cdots, c_{n1}^N], \cdots, [c_{1m}^N, \cdots, c_{nm}^N]]$ $(N=1, \cdots, N(v))$. If we put

$$\begin{aligned} &[[b_{11}, \cdots, b_{n1}], \cdots, [b_{1m}, \cdots, b_{nm}]] \\ &= v([[c_{11}^1, \cdots, c_{n1}^1], \cdots, [c_{1m}^1, \cdots, c_{nm}^1]], \cdots \\ &\qquad \cdots, [[c_{11}^{N(v)}, \cdots, c_{n1}^{N(v)}], \cdots, [c_{1m}^{N(v)}, \cdots, c_{nm}^{N(v)}]]), \end{aligned}$$

then, by the definition of the $\boldsymbol{P}$-product system $\boldsymbol{P}(\boldsymbol{Q}(\mathfrak{C}))$, we have

$$[b_{1\mu}, \cdots, b_{n\mu}] = P_{\xi_\mu v}\begin{pmatrix} [c_{11}^1, \cdots, c_{n1}^1], \cdots, [c_{11}^{N(v)}, \cdots, c_{n1}^{N(v)}] \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ [c_{1m}^1, \cdots, c_{nm}^1], \cdots, [c_{1m}^{N(v)}, \cdots, c_{nm}^{N(v)}] \end{pmatrix}.$$

Moreover, by Theorem 2.3, we have

$$(4.3) \qquad b_{\nu\mu} = F_{\eta_\nu P_{\xi_\mu v}}\begin{pmatrix} x_{11} \cdots x_{1N(v)} \\ \cdots\cdots\cdots \\ x_{m1} \cdots x_{mN(v)} \end{pmatrix}\begin{pmatrix} c_{11}^1, \cdots, c_{11}^{N(v)}, \cdots, c_{1m}^1, \cdots, c_{1m}^{N(v)} \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ c_{n1}^1, \cdots, c_{n1}^{N(v)}, \cdots, c_{nm}^1, \cdots, c_{nm}^{N(v)} \end{pmatrix}.$$

Now it follows from (4.2) and (4.3) that $c_{\nu\mu} = b_{\nu\mu}$. Hence $[c_{11}, \cdots,$ $c_{1m}, \cdots, c_{n1}, \cdots, c_{nm}]$ corresponds to $[[b_{11}, \cdots, b_{n1}], \cdots, [b_{1m}, \cdots, b_{nm}]]$ by the mapping θ. Therefore θ is an isomorphism, completing the proof.

The following theorem can be easily obtained from Theorems 3.1 and 4.1.

Theorem 4.2. *If $\boldsymbol{P}$ is an (A_V, B_W)-universal family of basic mapping-formulas, and if $\boldsymbol{Q}$ is a (B_W, C_U)-universal family of basic mapping-formulas, then the product $\boldsymbol{QP}$ is an (A_V, C_U)-universal family of basic mapping-formulas.*

§ 5. Families of homomorphism type.

Let $\boldsymbol{P}$ be a family $\boldsymbol{P}_{V,V}\{\xi_1, \cdots, \xi_m\}$ of basic mapping-formulas. If $\boldsymbol{P}$ is (A_V, A_V)-universal, then we simply say that $\boldsymbol{P}$ is A_V-universal. If $\boldsymbol{P}$ is A_V-universal for every system A_V of composition-identities with respect to V, then we say that $\boldsymbol{P}$ is absolutely universal. If each basic mapping-formula of $\boldsymbol{P}$ is of the form

$$\xi_\mu(v(x_1, \cdots, x_{N(v)})) = v(\xi_\mu(x_1), \cdots, \xi_\mu(x_{N(v)})),$$

then $\boldsymbol{P}$ is called a family of homomorphism type. Now if $\boldsymbol{P}$ is a family of homomorphism type, then it is obvious from Theorem 3.2 that $\boldsymbol{P}$ is absolutely universal. In this section, we shall prove the following:

Theorem 5.1. *Let V be a set of finitary compositions which contains at least one non-unary composition. And let $\boldsymbol{P}$ be a family $\boldsymbol{P}_{V,V}\{\xi_1, \cdots, \xi_m\}$ of basic mapping-formulas. If $\boldsymbol{P}$ is absolutely universal, then $\boldsymbol{P}$ is a family of homomorphism type.*

To prove this theorem, we shall first show the following facts (I), (II) and (III) with respect to free algebraic systems.

(I). Let V be a set of finitary compositions. And let L_V be the system of all the composition-identities $f(x_1, \cdots, x_s) = g(x_1, \cdots, x_s)$ with respect to V such that every x_σ which appears in the expression of $f(x_1, \cdots, x_s)$ also appears in that of $g(x_1, \cdots, x_s)$, and conversely. Moreover, let $\{a_1, \cdots, a_r\}$ be any non-empty set, and $\mathfrak{L}$ the set of all finite non-empty subsets of $\{a_1, \cdots, a_r\}$. We now define the compositions $v \in V$ in $\mathfrak{L}$ as follows:

$$v(\{a_{\rho_1}, \cdots\}, \cdots, \{a_{\rho_{N(v)}}, \cdots\}) = \{a_{\rho_1}, \cdots\} \cup \cdots \cup \{a_{\rho_{N(v)}}, \cdots\},$$

where $\cup$ denotes the set-sum. Then it is clear that $\mathfrak{L}$ forms an L_V-algebraic system generated by $\{a_1\}, \cdots, \{a_r\}$. Now let us denote by $h[a_{\rho_1}, \cdots, a_{\rho_t}]$ the V-word $h(a_1, \cdots, a_r)$ such that every a_ρ $(\rho = \rho_1, \cdots, \rho_t)$ appears in the expression of $h(a_1, \cdots, a_r)$, and that any a_ρ $(\rho \neq \rho_1, \cdots, \rho_t)$ does not appear in that of $h(a_1, \cdots, a_r)$. Then it is easy to see that the mapping

$$\theta : h[a_{\rho_1}, \cdots, a_{\rho_t}] \to \{a_{\rho_1}, \cdots, a_{\rho_t}\}$$

is an isomorphism from $F(\{a_1, \cdots, a_r\}, L_V)$ onto $\mathfrak{L}$. Hence we have that a V-word $f(a_1, \cdots, a_r)$ is L_V-congruent to a_1 if and only if any a_ρ $(\rho \neq 1)$ does not appear in the expression of $f(a_1, \cdots, a_r)$.

(II). For any V-polynomial $f(x_1, \cdots, x_s)$, we denote by $M_f(x_\sigma)$ the number M such that x_σ appears in the expression of $f(x_1, \cdots, x_s)$ for M times, but not for $M+1$ times. Now, let M_V be the system of all the composition-identities $f(x_1, \cdots, x_s) = g(x_1, \cdots, x_s)$ with respect to V such that $M_f(x_\sigma) = M_g(x_\sigma)$ for all $\sigma = 1, \cdots, s$. Moreover, let $\{a_1, \cdots, a_r\}$ be any finite non-empty set, and let $\mathfrak{M}$ be the set of all such symbols $(a_1^{p_1}, \cdots, a_r^{p_r})$ that p_ρ are non-negative integers which satisfy

$$p_1 + \cdots + p_r = 1 + k_1(N(v_1)-1) + \cdots + k_n(N(v_n)-1)$$

for some non-negative integers k_ν and some compositions $v_\nu \in V$. We now define the compositions $v \in V$ in $\mathfrak{M}$ as follows:

$$v((a_1^{p_{11}}, \cdots, a_r^{p_{1r}}), \cdots, (a_1^{p_{N(v)1}}, \cdots, a_r^{p_{N(v)r}})) = (a_1^{p_1}, \cdots, a_r^{p_r}),$$

where $p_\rho = p_{1\rho} + \cdots + p_{N(v)\rho}$. Then it is clear that $\mathfrak{M}$ forms an M_V-algebraic system with a generator system $\{(a_1^0, \cdots, a_{\rho-1}^0, a_\rho^1, a_{\rho+1}^0, \cdots, a_r^0);\ \rho = 1, \cdots, r\}$. Moreover it is easily verified that the mapping

$$\theta : f(a_1, \cdots, a_r) \to (a_1^{M_f(a_1)}, \cdots, a_r^{M_f(a_r)})$$

is an isomorphism from $F(\{a_1, \cdots, a_r\}, M_V)$ onto $\mathfrak{M}$. A V-word in $F(\{a_1, \cdots, a_r\}, \phi_V)$ which corresponds to $(a_1^{p_1}, \cdots, a_r^{p_r})$ by this isomorphism θ is simply called a V-word corresponding to $(a_1^{p_1}, \cdots, a_r^{p_r})$.

(III). Let V_1 be a non-empty set of unary compositions, and V_2 a non-empty set of non-unary compositions, and we define a set V of compositions as the set-sum of V_1 and V_2. Moreover, let N_V be the system consisting of all the composition-identities of the form $u(v(x_1, \cdots, x_{N(v)})) = x_1$ $(u \in V_1; v \in V_2)$ and all the composition-identities of the form $u(x) = v(x)$ $(u, v \in V_1)$. Now we shall show that, for any $v \in V_1$, and for any $s \geqq 1$, $\underbrace{v \cdots v}_{s}(x)$ is not N_V-congruent to x. Let $\mathfrak{N}$ be a free ϕ_{V_2}-algebraic system $F(\{a_0, a_1, a_2, \cdots\}, \phi_{V_2})$. Moreover, we define the compositions $u \in V_1$ in $\mathfrak{N}$ as follows: If f is a V_2-word of order 0 (we assume that $f = a_i$), then we define $u(f) = u(a_i) = a_{i+1}$. If f is a V_2-word of order $k \geqq 1$, then we define $u(f) = f_1$, where $f = v(f_1, \cdots, f_{N(v)})$ $(v \in V_2)$. Then it is clear that $\mathfrak{N}$ forms an N_V-algebaic system generated by the single element a_0. Hence there exists a homomorphism θ from a free N_V-algebraic system $F(\{x\}, N_V)$ generated by only one element x onto $\mathfrak{N}$ which is an extension of the mapping $x \to a_0$. Therefore we have that

$$\theta(\underbrace{v \cdots v}_{s}(x)) = a_s + a_0 = \theta(x)$$

for any $v \in V_1$ and for any $s \geqq 1$. Hence we have that $\underbrace{v \cdots v}_{s}(x)$ is not N_V-congruent to x for any $s \geqq 1$.

Proof of Theorem 5.1. Let

$$(5.1) \qquad \begin{aligned} &\xi_\mu(v(x_1, \cdots, x_{N(v)})) \\ &= P_{\xi_\mu v}(\xi_1(x_1), \cdots, \xi_1(x_{N(v)}), \cdots, \xi_m(x_1), \cdots, \xi_m(x_{N(v)})) \end{aligned}$$

be any basic mapping-formula of the family $\boldsymbol{P}$. First we shall prove that any $\xi_\lambda(x_N)$ $(\lambda \neq \mu; N=1, \cdots, N(v))$ does not appear in the expression of the V-polynomial

$$(5.2) \qquad P_{\xi_\mu v}(\xi_1(x_1), \cdots, \xi_1(x_{N(v)}), \cdots, \xi_m(x_1), \cdots, \xi_m(x_{N(v)}))$$

in the formula (5.1). Since $\boldsymbol{P}$ is, of course, L_V-universal (L_V was defined in (I)), and since the composition-identity $v(x, \cdots, x)=x$ is contained in L_V, it is easily obtained by Theorem 3.2 that

$$P_{\xi_\mu v}(\xi_1(x), \cdots, \xi_1(x), \cdots, \xi_m(x), \cdots, \xi_m(x)) \overset{L_V}{=} \xi_\mu(x).$$

Moreover, it is clear from (I) that any $\xi_\lambda(x)$ $(\lambda \neq \mu)$ does not appear in the expression of the V-polynomial $P_{\xi_\mu v}(\xi_1(x), \cdots, \xi_1(x), \cdots, \xi_m(x), \cdots, \xi_m(x))$ in the above identity. Hence any $\xi_\lambda(x_N)$ $(\lambda \neq \mu; N=1, \cdots, N(v))$ does not appear in the expression of the V-polynomial (5.2). Hence we simply denote the formula (5.1) by

$$(5.3) \qquad \xi_\mu(v(x_1, \cdots, x_{N(v)})) = P_{\xi_\mu v}(\xi_\mu(x_1), \cdots, \xi_\mu(x_{N(v)})).$$

Next we shall show that

$$(5.4) \qquad P_{\xi_\mu v}(\xi_\mu(x_1), \cdots, \xi_\mu(x_{N(v)}))$$

in (5.3) is a V-polynomial corresponding to $(\xi_\mu(x_1)^1, \cdots, \xi_\mu(x_{N(v)})^1)$ in the sense of (II). Assume that the V-polynomial (5.4) corresponds to $(\xi_\mu(x_1)^{p_1}, \cdots, \xi_\mu(x_{N(v)})^{p_{N(v)}})$. We shall first discuss in the case where v is non-unary. Since the composition-identity

$$v(x_1, x_2, x_3, \cdots, x_{N(v)}) = v(x_2, x_1, x_3, \cdots, x_{N(v)})$$

is contained in M_V of (II), and since $\boldsymbol{P}$ is, of course, M_V-universal, it is clear from Theorem 3.2 that

$$\begin{aligned} &P_{\xi_\mu v}(\xi_\mu(x_1), \xi_\mu(x_2), \xi_\mu(x_3), \cdots, \xi_\mu(x_{N(v)})) \\ &\overset{M_V}{=} P_{\xi_\mu v}(\xi_\mu(x_2), \xi_\mu(x_1), \xi_\mu(x_3), \cdots, \xi_\mu(x_{N(v)})). \end{aligned}$$

Hence it is obtained from (II) that

$$(\xi_\mu(x_1)^{p_1},\ \xi_\mu(x_2)^{p_2},\ \xi_\mu(x_3)^{p_3},\ \cdots,\ \xi_\mu(x_{N(v)})^{p_{N(v)}})$$
$$= (\xi_\mu(x_1)^{p_2},\ \xi_\mu(x_2)^{p_1},\ \xi_\mu(x_3)^{p_3},\ \cdots,\ \xi_\mu(x_{N(v)})^{p_{N(v)}})\,.$$

Hence we have that $p_1=p_2$, and similarly $p_2=p_3=\cdots=p_{N(v)}$. Since the composition-identity

$$v(v(x_1,\ x_2,\ \cdots,\ x_{N(v)}),\ x_2,\ x_3,\ \cdots,\ x_{N(v)})$$
$$= v(x_1,\ v(x_2,\ \cdots,\ x_{N(v)},\ x_2),\ x_3,\ \cdots,\ x_{N(v)})$$

is clearly contained in M_V, the identity

$$(\xi_\mu(x_1)^{p_1^2},\ \cdots) = (\xi_\mu(x_1)^{p_1},\ \cdots)$$

is similarly obtained as above. Hence we have that $p_1^2=p_1$, i.e., $p_1=0$ or $p_1=1$. Since there is no element of the form $(a_1^0,\ \cdots,\ a_r^0)$ in $\mathfrak{M}$, the V-polynomial (5.4) corresponds to $(\xi_\mu(x_1)^1,\ \cdots,\ \xi_\mu(x_{N(v)})^1)$. In the case where v is unary, the identity $(\xi_\mu(x_1)^{p_1})=(\xi_\mu(x_1)^1)$ is similarly obtained as above, since $v(x_1)=x_1$ is contained in M_V. Hence $P_{\xi_\mu v}(\xi_\mu(x_1))$ corresponds to $(\xi_\mu(x_1)^1)$.

In the following, we shall prove that the V-polynomial (5.4) is of the form $v(\xi_\mu(x_1),\ \cdots,\ \xi_\mu(x_{N(v)}))$.

(i). The case where the composition v is non-unary. For any V-polynomial $f(x_1,\ \cdots,\ x_r)$, we denote by $L[f(x_1,\ \cdots,\ x_r)]$ the element x_{ρ_0} which appears in the leftmost position of the arrangement of x_ρ in the expression of $f(x_1,\ \cdots,\ x_r)$, when we omit the parentheses and the compositions appearing in the expression of $f(x_1,\ \cdots,\ x_r)$. First we shall show that

$$L[P_{\xi_\mu v}(\xi_\mu(x_1),\ \cdots,\ \xi_\mu(x_{N(v)}))] = \xi_\mu(x_1)\,. \tag{5.5}$$

Let A_V be the system of all the composition-identities each of which is of the form $u(x_1,\ \cdots,\ x_{N(u)})=x_1$ $(u\in V)$. Then it is clear that

(5.6) $\begin{cases} F(\{\xi_\mu(x_1),\ \cdots,\ \xi_\mu(x_{N(v)})\},\ A_V) \text{ consists of } N(v) \text{ elements} \\ \xi_\mu(x_1),\ \cdots,\ \xi_\mu(x_{N(v)})\,. \end{cases}$

Since $\boldsymbol{P}$ is, of course, A_V-universal, and since the composition-identity $v(x_1,\ \cdots,\ x_{N(v)})=x_1$ is contained in A_V, the identity

$$P_{\xi_\mu v}(\xi_\mu(x_1),\ \cdots,\ \xi_\mu(x_{N(v)})) \overset{A_V}{=} \xi_\mu(x_1)$$

is obtained by Theorem 3.2. Now suppose that

$$L[P_{\xi_\mu v}(\xi_\mu(x_1),\ \cdots,\ \xi_\mu(x_{N(v)}))] = \xi_\mu(x_N)$$

for some N $(2\leqq N\leqq N(v))$. Then the identity

$$P_{\xi_\mu v}(\xi_\mu(x_1),\ \cdots,\ \xi_\mu(x_{N(v)})) \overset{A_V}{=} \xi_\mu(x_N)$$

is derived from A_V. Hence we have

$$\xi_\mu(x_1) \overset{A_V}{=} \xi_\mu(x_N) .$$

This contradicts the fact (5.6). Hence we have the identity (5.5).

Next we shall show that the composition v appears in the expression of the V-polynomial (5.4). Let B_V be the system consisting of all the composition-identities of the form $u(x_1, \cdots, x_{N(u)}) = x_1$ $(u \in V; u \neq v)$ and the composition-identity $v(x_1, \cdots, x_{N(v)}) = x_2$. Then it is clear that

$$(5.7) \qquad \begin{cases} F(\{\xi_\mu(x_1), \cdots, \xi_\mu(x_{N(v)})\}, B_V) \text{ consists of } N(v) \text{ elements} \\ \xi_\mu(x_1), \cdots, \xi_\mu(x_{N(v)}) . \end{cases}$$

Since $\boldsymbol{P}$ is B_V-universal, and since the composition-identity $v(x_1, \cdots, x_{N(v)}) = x_2$ is contained in B_V, the identity

$$P_{\xi_\mu v}(\xi_\mu(x_1), \cdots, \xi_\mu(x_{N(v)})) \overset{B_V}{=} \xi_\mu(x_2)$$

is obtained by Theorem 3.2. Now suppose that the composition v does not appear in the expression of the V-polynomial (5.4). Then the identity

$$P_{\xi_\mu v}(\xi_\mu(x_1), \cdots, \xi_\mu(x_{N(v)})) \overset{B_V}{=} \xi_\mu(x_1)$$

is derived from (5.5). Hence we have

$$\xi_\mu(x_1) \overset{B_V}{=} \xi_\mu(x_2) .$$

This contradicts the fact (5.7). Hence the composition v appears in the expression of the V-polynomial (5.4).

Thus, we can easily obtain that

$$(5.8) \qquad \begin{aligned} & P_{\xi_\mu v}(\xi_\mu(x_1), \cdots, \xi_\mu(x_{N(v)})) \\ & = u_1 \cdots u_s v(u_{11} \cdots u_{1s_1} \xi_\mu(x_{\pi(1)}), \cdots, u_{N(v)1} \cdots u_{N(v)s_{N(v)}} \xi_\mu(x_{\pi(N(v))})) \end{aligned}$$

for some permutation π of $1, \cdots, N(v)$, and for some (empty or non-empty) set of unary compositions u_σ and $u_{N\sigma}$ in V, because the V-polynomial (5.4) corresponds to $(\xi_\mu(x_1)^1, \cdots, \xi_\mu(x_{N(v)})^1)$, and the composition v appears in the expression of the V-polynomial (5.4). Now let C_V be a system consisting of only one composition-identity $v(x_1, \cdots, x_{N(v)}) = x_N$ $(1 \leqq N \leqq N(v))$. Since $\boldsymbol{P}$ is, of course, C_V-universal, the identity

$$u_1 \cdots u_s u_{N1} \cdots u_{Ns_N} \xi_\mu(x_{\pi(N)}) \overset{C_V}{=} \xi_\mu(x_N)$$

is obtained by (5.8) and Theorem 3.2. Hence it is easily verified that $s=0$, $s_N=0$ and $\pi(N)=N$. Therefore we have

$$P_{\xi_\mu v}(\xi_\mu(x_1), \cdots, \xi_\mu(x_{N(v)})) \equiv v(\xi_\mu(x_1), \cdots, \xi_\mu(x_{N(v)})) .$$

(ii). The case where the composition v is unary. Since $P_{\xi_\mu v}(\xi_\mu(x))$ is a V-polynomial corresponding to $(\xi_\mu(x)^1)$, we have that

$$P_{\xi_\mu v}(\xi_\mu(x)) = u_1 \cdots u_s \xi_\mu(x)$$

for some (empty or non-empty) set of unary compositions u_σ in V. Now let D_V be the system consisting of only one composition-identity $v(x)=x$. Since $\boldsymbol{P}$ is D_V-universal, the identity

$$u_1 \cdots u_s \xi_\mu(x) \overset{D_V}{=} \xi_\mu(x)$$

is obtained by Theorem 3.2. Hence it is easily verified that $u_1 = u_2 = \cdots = u_s = v$. Therefore we have that

$$P_{\xi_\mu v}(\xi_\mu(x)) = \underbrace{v \cdots v}_{s} \xi_\mu(x) \tag{5.9}$$

for some non-negative integer s. Since $\boldsymbol{P}$ is, of course, N_V-universal (N_V was defined in (III)), and since the composition-identity $v(u(x_1, \cdots, x_{N(u)})) = x_1$ (u : non-unary composition) is contained in N_V, it is easily obtained by (i), (5.9) and Theorem 3.2 that

$$\underbrace{v \cdots v}_{s} u(\xi_\mu(x_1), \cdots, \xi_\mu(x_{N(u)})) \overset{N_V}{=} \xi_\mu(x_1).$$

If $s=0$, then we have

$$u(\xi_\mu(x_1), \cdots, \xi_\mu(x_{N(u)})) \overset{N_V}{=} \xi_\mu(x_1).$$

It follows from the fact (III) that this identity is not true. Hence $s \geqq 1$, and hence

$$\underbrace{v \cdots v}_{s-1} \xi_\mu(x_1) \overset{N_V}{=} \xi_\mu(x_1).$$

Therefore it is clear from (III) that $s-1 \ngeqq 1$, and therefore $s=1$. Hence we have

$$P_{\xi_\mu v}(\xi_\mu(x)) \equiv v(\xi_\mu(x)).$$

This completes the proof.

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