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On Harmonic Functions Representable by Poisson's Integral

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Let R be a Riemann surface with positive boundary and let $\{R_n\}$ $(n=0, 1, 2, \cdots)$ be its exhaustion with compact relative boundaries ∂R_n. If an open set G has relative boundary consisting of at most enumerably infinite number of analytic curves which cluster nowhere in R, we call G a domain. Let $w_{n,n+i}(z)$ be a harmonic function in $R_{n+i}-(G\cap(R_{n+i}-R_n))$ such that $w_{n,n+i}(z)=0$ on $\partial R_{n+i}-G$ and $w_{n,n+i}(z)=1$ on $\partial(G\cap(R_{n+i}-R_n))$ and let $\omega_{n,n+i}(z)$ be a harmonic function in $R-R_0-(G\cap(R_{n+i}-R_n))$ such that $\omega_{n,n+i}(z)=0$ on ∂R_0, $\omega_{n,n+i}(z)=1$ on $\partial(G\cap(R_{n+i}-R_n))$ and $\frac{\partial}{\partial n}\omega_{n,n+i}(z)$ $=0$ on $\partial R_{n+i}-G$. We call $\lim\limits_{i}\lim\limits_{n} w_{n,n+i}(z)$ and $\lim\limits_{n}\lim\limits_{i} \omega_{n,n+i}(z)$[1] the *harmonic measure and the capacitary potential of the ideal boundary* $(G\cap B)$ *determined by G respectively.* We call a function $G(z)$ *a generalized Green's function,* if $G(z)$ is non negatively harmonic in R, the harmonic measure of $(B\cap E[z\in R\colon G(z)>\delta])$ is zero for $\delta>0$ and the Dirichlet integral $D(\min(M, G(z))\leqq kM$ for $M<\infty$.

We map the universal covering surface R^∞ of R onto $|\xi|<1$. Then

Theorem 1. *Let $W(z)$ be a positive harmonic in R and superharmonic in $\bar{R}$[2]. Then $W(z)=U(z)+V(z)$, where $U(z)$ is a harmonic function in R representable by Poisson's integral in $|\xi|<1$ and $V(z)$ is a generalized Green's function. If furthermore R has no irregular point of the Green's function, then $V(z)=0$, therefore $W(z)$ is representable by Poisson's integral.*

Let $W(z)$ be a function in Theorem 1. Then $W(z)-S(z)$ is also positively harmonic in $R-R_0$ and superharmonic in $\overline{R-R_0}$ and $W(z)-S(z)=W'(z)=0$ on ∂R_0, where $S(z)$ is harmonic in $R-R_0$ such that $S(z)=W(z)$ on ∂R_0 and $S(z)$ has M. D. I. (minimal Dirichlet integral).

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1) Z. Kuramochi: Harmonic measure and capacity of subsets of the ideal boundary, Proc. Japan Acad. 31, 1955.

2) Let $U(z)$ be a positively harmonic function which satisfies $D(\min(M, U(z))<\infty$. If $U(z)$ $>U_G(z)$ for every compact or noncompact domain G, we say $U(z)$ is superharmonic in $\overline{R}$, where $U_G(z)=\lim\limits_{M=\infty} U_G^M(z)$, $U_G^M(z)=\min(M, U(z))$ on ∂G and $U_G^M(z)$ has minimal Dirichlet integral over G.

Then $W'(z)$ is representable by a positive mass distribution as follows:[3]

$$W'(z)=\int_{B_1} N(z, p)d\mu(p),$$

where B_1[4] is the set of minimal points and the total mass μ_0 is given by $ds\int_{\partial R_0}\frac{\partial}{\partial n}W'(z)$ and $D(\min(M, W(z))\leqq 2\pi M\mu_0$.

First we shall prove for $N(z, p)$. Then

Theorem 2. *Let $N(z, p)$ be a minimal function*[5]. *Then $N(z, p)=U(z, p)+V(z, p)$, where $U(z, p)$ is a positive harmonic function representable by Poisson's integral and $V(z, p)$ is a generalized Green's function. $U(z, p)$ and $V(z, p)$ are functions of at most second class of Baire's function of p for fixed $z\in R-R_0$ with respect to Martin's topology.*[4]

If $\sup N(z, p)<\infty$, our assertion is trivial and in this case by the boundedness of $V(z, p)$, $V(z, p)$ reduces to constant zero. We shall suppose $\sup N(z, p)=\infty$. Put $G_M=E[z\in R: N(z, p)>M]$. Then G_M is a non compact domain. Consider a harmonic function $w_n(z)$ in $R_n-G_M-R_0$ such that $w_n(z)=0$ on $\partial R_0+\partial R_n-G_M$ and $w_n(z)=1$ on ∂G_M. Let $w_M(z)=\lim_n w_n(z)$. Since $N(z, p)$ has M. I. D. over $R-R_0-G_M$ among all functions with values 0 on ∂R_0 and M on ∂G_M respectively, $N(z, p)=\lim_n N_n(z, p)$, where $N_n(z, p)$ is harmonic in $R_n-R_0-G_M$ such that $N_n(z, p)=M$ on ∂G_M, $N_n(z, p)=0$ on ∂R_0 and $\frac{\partial}{\partial n}N_n(z, p)=0$ on ∂R_n-G_M. Hence by the maximum principle $N(z, p)\geqq Mw_M(z)$, whence $\lim_{M=\infty} w_M(z)=0$. Map the universal covering surface $(R-R_0)^\infty$ onto $|\zeta|<1$ and consider $w_M(z)$ in $|\zeta|<1$. Then $w_M(z)$ has angular limits $=0$ a. e. (almost everywhere) on a set E_M on $|\zeta|=1$ where $N(z, p)$ has angular limits $<M$. To the contrary, suppose that there exists a set E of positive measure such that $w_M(z)$ has angular limits >0 on E and $N(z, p)$ has angular limits $<M$. Then there exists a closed set $E'\subset E$ such that mes $(E-E')<\varepsilon$, $N(z, p)<M-\varepsilon$ in angular domain $D_\varepsilon=[\arg|\zeta-\zeta_0|<\frac{\pi}{2}-\varepsilon, \zeta_0\in E', |\zeta|>1-\varepsilon]$ for any given positive number ε. Let D' be one of components of D_ε. Then the image of

3) Z. Kuramochi: Mass distributions on the ideal boundaries, II. Osaka Math. Jour., 8, 1956.

4) See 3).

5) If $U(z)$ has no functions $V(z)$ such that both $V(z)>0$ and $U(z)-V(z)>0$ are harmonic and superharmonic in $\overline{R-R_0}$ except its own multiples, we say that $U(z)$ is a minimal function.

G_M does not intersect the above D'. Let $H(z)$ be a harmonic function in D' with values 1 on $\partial D'-E[|\zeta|=1]$ and 0 on $\partial D'\cap E[|\zeta|=1]$. Since $\partial D'$ is rectifiable, $H(z)=0$ on a.e. $\partial D'\cap E[|\zeta|=1]$. But $w_M(z)\leqq H(z)$, whence $w_M(z)=0$ a.e. on E_M.

Let $N_n'(z, p)$ be a harmonic function in $R_n-R_0-G_L(=E[z\in R: N(z, p)>L])$ such that $N_n'(z, p)=0$ on ∂R_0, $N_n'(z, p)=L$ on $\partial G_L\cap R_n$, $N_n'(z, p)=N(z, p)$ on $\partial R_n-G_M(M<L)$ and $\frac{\partial}{\partial n}N_n'(z, p)=0$ on $\partial R_n\cap(G_M-G_L)$. Then

$$D_{R_n-G_L}(N_n'(z, p))<D_{R_n-G_L}(N(z, p))\,.$$

Since $N(z, p)$ has M. D. I. over $R-G_L$, $N_n'(z, p)\rightarrow N(z, p)$ in mean. Let $U_{M,n}(z, p)$ be a harmonic function in R_n-R_0 such that $U_{M,n}(z, p)=0$ on ∂R_0, $U_{M,n}(z, p)=N_n'(z, p)$ on ∂R_n-G_M and $U_{M,n}(z, p)=M$ on $\partial R_n\cap G_M$. In $R_n-R_0-G_M$, $0<N_n'(z, p)-U_{M,n}(z, p)\leqq Lw_n(z)$. Hence by letting $n\rightarrow\infty$, $0<N(z, p)-U_M(z, p)<Lw_M(z)$, where $U_M(z, p)$ is a limit function from a subsequence $(n_1, n_2, \cdots)$. Thus $U_M(z, p)$ has the same angular limits as $N(z, p)$ a.e. on a set E_M on $|\zeta|=1$ on which $N(z, p)$ has angular limits $<M$. Next let $U'_{M,n}(z, p)$ be a harmonic function in R_n-R_0 such that $U'_{M,n}(z, p)=0$ on ∂R_0 and $U'_{M,n}(z,p)=\min(M, N(z, p))$ on ∂R_n. Then we have clearly $\lim\limits_n U_{M,n}(z, p)=\lim\limits_n U'_{M,n}(z, p)$ and $U_{M_2,n}(z, p)>U_{M_1,n}(z, p)$ for $M_2>M_1$.

Choose a subsequence $(n_1', n_2', \cdots)$ from $(n_1, n_2, \cdots)$ such that $U_{M_2,n'}(z, p)$ converges to $U_{M_2}(z, p)$. Then $U_{M_2}(z, p)\geqq U_{M_1}(z, p)$. Let $U(z, p)=\lim\limits_{M=\infty} U_M(z, p)$. Then $U(z, p)$ is a function representable by Poisson's integral and $U(z, p)$ has the same angular limits as $N(z, p)$ a.e. on $|\zeta|=1$, because $\lim\limits_{M=\infty} w_M(z)=0$. Hence such $U(z, p)$ does not depend on the subsequences. This $U(z, p)$ is the function stated in the theorem.

Next we shall show that $N(z, p)-U(z, p)$ is a generalized Green's function. We proved that $\int\limits_{\partial G_L} N(z, p)ds=\lim\limits_{n=\infty}\int\limits_{\partial G_L} N_n(z, p)ds$[6] for almost all L (i.e. the set of L whose ∂G_L does not satisfy the above condition is of measure zero), where $N_n(z, p)$ is a harmonic function in $R_n-R_0-G_L$ such that $N_n(z, p)=0$ on ∂R_0, $N_n(z, p)=L$ on $\partial G_L\cap R_n$ and $\frac{\partial}{\partial n}N_n(z, p)=0$ on ∂R_n-G_L.

We call such G_L a regular domain. Hence we can suppose without loss of generality that G_L is regular. We see the following assertion from $\frac{\partial}{\partial n}N_n(z, p)>0$ on ∂G_L, it is necessary and sufficient condition for

6) sec 3). p. 151.

the regularity of G_L that *there exist* n_0 *and* m_0 *such that* $\int_{(R-R_m)\cap\partial G_L}\frac{\partial}{\partial n}N_n(z, p)ds$ $<\varepsilon$ *for* $n>n_0$ *and* $m>m_0$ *for any given positive number* $\varepsilon>0$.

Let $J_n(z)$ be a harmonic function in $R_n-R_0-(R_n\cap(G_M-G_L))$ such that $J_n(z)=0$ on ∂G_M, $J_n(z)=1$ on ∂G_L and $\frac{\partial}{\partial n}J_n(z)=0$ on $\partial R_n\cap(G_M-G_L)$. Then $(M+(L-M)J_n(z))\to N(z, p)$ in mean, because $N(z, p)$ has M.D.I. over G_M-G_L. Hence $\lim_n\int_{\partial G_L}\frac{\partial}{\partial n}J_n(z)ds=\int_{\partial G_L}\lim_n\frac{\partial}{\partial n}J_n(z)ds$ and there exist m_0 and n_0 such that $\int_{(R-R_m)\cap\partial G_L}\frac{\partial}{\partial n}J_n(z)ds<\varepsilon$ for $n>n_0$ and $m>m_0$ for any given positive number $\varepsilon>0$. But $N_n'(z, p)>(L-M)J_n(z)$ in G_M-G_L and $N_n'(z, p)=(M+(L-M)J_n(z))$ on ∂G_L implies

$$(L-M)\frac{\partial}{\partial n}J_n(z)>\frac{\partial}{\partial n}N_n'(z, p)>0 \text{ on } \partial G_L.$$

Hence $\int_{\partial G_L}\lim_n\frac{\partial}{\partial n}N_n'(z, p)ds=\lim_n\int_{\partial G_L}\frac{\partial}{\partial n}N_n'(z, p)ds.$

Thus ∂G_L *is also regular for* $N_n'(z, p)$.

Let $V_{M,n}(z, p)$ be a harmonic function $=N_n'(z, p)-U_{M,n}(z, p)$. Then $V_{M,n}(z, p)$ is harmonic in R_n-R_0, $V_{M,n}(z, p)=0$ on $\partial R_0+(\partial R_n-G_M)$ and $V_{M,n}(z, p)>L-M$ in G_L. By the regularity of ∂G_L, $\int_{\partial G_L\cap R_n}\frac{\partial}{\partial n}N_n'(z, p)ds$ $\to 2\pi$, as $n\to\infty$. Hence there exists a number n_0 for any given ε such that $\int_{\partial G_L\cap R_n}\frac{\partial}{\partial n}N_n'(z, p)ds\leqq 2\pi+\varepsilon$ for $n>n_0$.

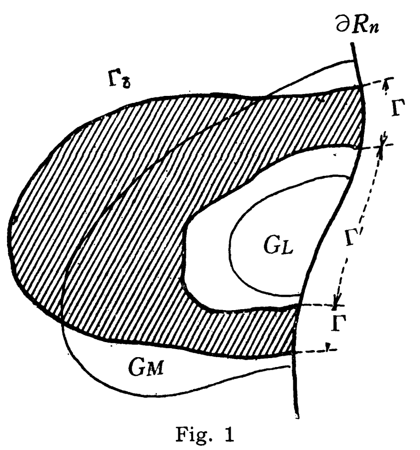

Fig. 1

Put $D=E[z\in R:\ \delta<V_{M,n}(z, p)<M'<M]$. $\Gamma_{M'}=E[z\in R:\ V_{M,n}(z, p)=M']$, $\Gamma_\delta=E[z\in R:\ V_{M,n}(z, p)=\delta]$, $\Gamma=\partial R_m\cap D$ and $\Gamma'=\partial R_n\cap E[z\in R:\ V_{M,n}(z, p)\geqq M']$. Then D intersects only $\partial R_n\cap(G_M-G_L)$, because $N_n'(z, p)-U_{M,n}(z, p)=0$ on ∂R_n-G_M and $N_n'(z, p)-U_{M,n}(z, p)>M'$ on ∂G_L for $L>2M'$. Hence $\Gamma\subset\partial R_n\cap(G_M-G_L)$. Now $\frac{\partial}{\partial n}N_n'(z, p)=0$ on ∂R_n-D_L. Since $U_{M,n}(z, p)=\max U_{M,n}(z, p)=M$ on Γ, $\frac{\partial}{\partial n}U_{M,n}(z, p)>0$ on Γ and

$$\int_{\Gamma} \frac{\partial}{\partial n} U_{M,n}(z, p) ds < \int_{\partial R_n \cap (G_N - G_L)} \frac{\partial}{\partial n} U_{M,n}(z, p) ds \leqq \int_{\partial G_L \cap R_n} \frac{\partial}{\partial n} N_n'(z, p) ds \leqq 2\pi + \varepsilon .$$

$$\int_{\Gamma_{M'}} \frac{\partial}{\partial n} N_n'(z, p) ds = \int_{\partial G_L \cap R_n} \frac{\partial}{\partial n} N_n'(z, p) ds < 2\pi + \varepsilon .$$

$$0 < \int_{\Gamma_{M'}} \frac{\partial}{\partial n} U_{M,n}(z, p) ds = \int_{\Gamma'} \frac{\partial}{\partial n} U_{M,n}(z, p) ds \leqq \int_{\partial R_n \cap G_M} \frac{\partial}{\partial n} U_{M,n}(z, p) ds < 2\pi + \varepsilon .$$

$$\int_{\Gamma_\delta} \frac{\partial}{\partial n} N_n'(z, p) ds = - \int_{\partial G_L \cap R_n} \frac{\partial}{\partial n} N_n'(z, p) ds \geqq -2\pi - \varepsilon .$$

$$\int_{\Gamma_\delta} \frac{\partial}{\partial n} U_{M,n}(z, p) ds = \int_{\Gamma + \Gamma'} \frac{\partial}{\partial n} U_{M,n}(z, p) ds$$

$$\geqq - \int_{\partial R_n \cap G_M} \frac{\partial}{\partial n} U_{M,n}(z, p) ds > -2\pi - \varepsilon .$$

Hence $D(\min V_{M,n}(z, p), M') = D_D(V_{M,n}(z, p)) = \int_{\Gamma_\delta + \Gamma + \Gamma_{M'}} (N_n'(z, p) - U_{M,n}(z, p)) \frac{\partial}{\partial n} (N_n'(z, p) - U_{M,n}(z, p)) ds \leqq M'(4\pi + 2\varepsilon) + \delta(2\pi + \varepsilon)$ and $D_{R_m - R_0}(\min (V_{M,n}(z, p), M') \leqq M'(4\pi + 2\varepsilon) + \delta(2\pi + \varepsilon)$, for every m (for every $n > 1$).

Let $n \to \infty$, then $N_n'(z, p) \to N(z, p)$ in $R - R_0 - G_L$, $U_{M,n}(z, p) \to U_M(z, p)$, $V_{M,n}(z, p) \to V(z, p)$ and derivatives of $V_{M,n}(z, p) \to$ derivatives of $V_M(z, p)$. By letting $n \to \infty$ and then $\delta \to 0$ and $\varepsilon \to 0$, $D_{R-R_m}(\min (V_M(z, p), M') \leqq 4\pi M'$.

Let $L \to \infty$ and then $M \to \infty$. Then $U_M(z, p) \uparrow U(z, p)$ and $V_M(z, p) \downarrow V(z, p)$ and then by letting $m \to \infty$, we have

$$D_{R-R_0}(V(z, p), M')) \leqq 4\pi M' .$$

On the other hand, clearly $V(z, p) = N(z, p) - U(z, p)$ has angular limits $= 0$ a.e. on $|\zeta| = 1$. Hence $V(z, p)$ *is a generalized Green's function.*

Since $U_M(z, p) = \lim_n U_{M,n}(z, p) = \lim_n U'_{M,n}(z, p)$, where $U'_{M,n}(z, p)$ is a harmonic function in $R_n - R_0$ such that $U'_{M,n}(z, p) = \min (M, N(z, p))$ on $\partial R_0 + \partial R_n$. Hence $U'_{M,n+i}(z, p) \leqq U'_{M,n}(z, p)$ on ∂R_n, whence $U'_{M,n}(z, p) \downarrow U_M(z, p)$. Therefore there exists a number n_0 such that $U'_M(z, p) \leqq U'_{M,n}(z, p) - \varepsilon$ for $n > n_0$ for any given positive number ε. Next since $N(z, p)$ is a continuous function of p for any point $z \in R - R_0$, there exists a number δ_0 such that

$$|N(z, p) - N(z, p_j)| < \varepsilon \text{ on } \partial R_n \text{ for } \delta(p, p_j) < \delta_0 .$$

Hence $U_M(z, p) \geqq U'_{M,n}(z, p) - \varepsilon \geqq U'_{M,n}(z, p_j) - 2\varepsilon \geqq U_M(z, p_j) - 2\varepsilon$. *Thus $U_M(z, p)$ is an upper semicontinuous function of p,* whence $V_M(z, p)$ is a lower semicontinuous function of p by the continuity of $N(z, p)$. $U_M(z, p) \uparrow U(z, p)$ and $V_M(z, p) \downarrow V(z, p)$ imply that *$U(z, p)$ and $V(z, p)$ are at most second class of Baire's functions.*

Properties of generalized Green's functions.

Theorem 3. *Let $V(z)$ be a generalized Green's function such that $D(\min(V(z), M)) \leqq \pi M$. Let $V'(z)$ be a non negative harmonic function such that $V'(z) \leqq V(z)$. Then $V'(z)$ is also a generalized Green's function such that $D(\min(V(z), M)) \leqq \pi M$.*

Put $D = E[z \in R: V'(z) < M$ and $V(z) > M]$. Let $V'_{n,n+i}(z)$ be a harmonic function in $R_{n+i} - R_0 - E[z \in R: V'(z) > M] - (D \cap (R_{n+i} - R_n))$ such that $V'_{n,n+i}(z) = V'(z)$ on $\partial R_0 + (E[z \in R: V'(z) \leqq M] \cap R_n)$, $\frac{\partial}{\partial n} V'_{n,n+i}(z) = 0$ on $\partial R_n \cap D$ and $V'_{n,n+i}(z) = V(z)$ on $\partial R_{n+i} - E[z \in R: V(z) > M]$. Then by the Dirichlet principle

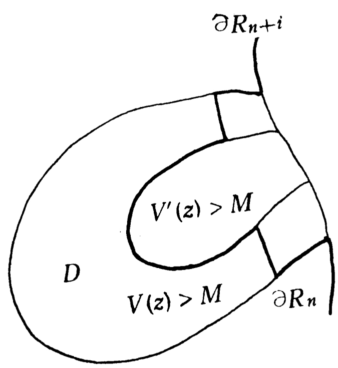

Fig. 2

$$D(\min M, V'_{n,n+i}(z)) \leqq D(\min(M, V(z)))$$

for every i and n.

Next clearly $\lim_n \lim_i V'_{n,n+i}(z) = \tilde{V}(z)$ exists and $\tilde{V}(z)$ has angular limits $\leqq V(z)$ a.e. where $V(z)$ has angular limits $< M$. But $V(z)$ has angular limits $= 0$ a.e. on $|\zeta| = 1$, whence $\tilde{V}(z) = V'(z)$ and

$$D(\min(M, V'(z)) \leqq D(\min(M, V(z)))\,.$$

Hence $V'(z)$ is a generalized Green's function.

Theorem 4. Let *$V(z)$ be a generalized Green's function and put $R_\delta = E[z \in R: V(z) > \delta]$ and $D_M = E[z \in R: V(z) > M]$. Then D_M determines a set of the ideal boundary of capacity zero.*

Let $V_{n,n+i}(z)$ be a harmonic function in $(R_\delta \cap R_{n+i}) - ((R_{n+i} - R_n) \cap D_M)$ such that $V_{n,n+i}(z) = 0$ on $\partial R_\delta \cap R_{n+i}$, $V_{n,n+i}(z) = 1$ on $\partial(D_M \cap (R_{n+i} - R_n))$ and $\frac{\partial}{\partial n} V_{n,n+i}(z) = 0$ and $\partial R_{n+i} \cap (R_\delta - D_M)$. Then by the Dirichlet principle

$$\int_{\partial(D_M \cap (R_{n+i} - R_n))} \frac{\partial}{\partial n} V_{n,n+i}(z) ds = D(V_{n,n+i}(z)) \leqq \frac{1}{(M-\delta)^2} D(V(z)) \leqq \frac{2\pi M}{(M-\delta)^2}$$

for every i and n,

and clearly $V_{n,n+i}(z)$ converges to $V_n(z)$ in mean as $i\to\infty$.

$$\int\frac{\partial}{\partial n}(V_{n,n+i}(z)-V_{m,n+i}(z))V_{n,n+i}(z)=\int\limits_{\partial(D_M\cap(R_{n+i}-R_n))}\frac{\partial}{\partial n}V_{n,n+i}(z)ds$$

$$-\frac{\partial}{\partial n}V_{m,n+i}(z)ds=D(V_{n,n+i}(z))-D(V_{m,n+i}(z)),\quad \text{for } n<m<n+i.$$

Since $V_{m,n+i}(z)\to V_n(z)$ in mean, we have

$D(V_n(z)-V_m(z),\ V_n(z))=D(V_n(z))-D(V_m(z))$ and $D(V_n(z)-V_m(z))=D(V_n(z))-D(V_m(z))$.

Hence $V_n(z)$ converges to a function $V^*(z)$ in mean as $n\to\infty$.

Map the universal couvring surface R_δ^∞ of R_δ onto $|\zeta|<1$. Then $V^*(z)$ has angular limits $=0$ a.e. on $|\xi|=1$ by that $V(z)$ has angular limits $=\delta$ a.e. on $|\zeta|=1$. Hence $V^*(z)=0$. Let F be a closed arc on ∂R_δ. Let $\omega_{n,n+i}(z)$ be a harmonic function in $R_\delta\cap R_{n+i}-((R_{n+i}-R_n)\cap D_M)$ such that $\omega_{n,n+i}(z)=0$ on F, $\omega_{n,n+i}(z)=1$ on $\partial(D_M\cap(R_{n+i}-R_n))$ and $\frac{\partial\omega_{n,n+i}}{\partial n}(z)=0$ on $\partial R_{n+i}-D_M$. Then by the Dirichlet principle

$$D(\omega_{n,n+i}(z))\leqq D(V_{n,n+i}(z)).$$

We see as above that $\omega_{n,n+i}(z)\to\omega_n(z)$ in mean and $\omega_n(z)\to\omega(z)$ in mean and by $V_n(z)\to V^*(z)$ in mean. We have $D(\omega(z))\leqq D(V^*(z))\leqq 0$.
Thus D_M determines a set of the boundary of capacity zero.

Theorem 5. *Let $V(z)$ be a generalized Green's function. Then $\int\limits_c\frac{\partial}{\partial n}V(z)ds=k$ on every niveau curve, where k is a constant such that $D(\min(M,\ V(z))=Mk$.*

Let $\omega_n(z)$ and D_M be in Theorem 4. Let $\omega_n'(z)$ be a harmonic function in $D_M\cap(R_n-R_{n_0})+(R_\delta\cap R_{n_0})$ such that $\omega_n'(z)=0$ on $F\cap R_{n_0}$, $\omega_n'(z)=1$ on $D_M\cap\partial R_n$ and $\frac{\partial\omega_n'}{\partial n}(z)=0$ on $(\partial R_\delta\cap R_{n_0})+(\partial R_{n_0}-D_M)+(\partial D_M\cap(R_n-R_{n_0}))-F$. Then clearly

$$D(\omega_n'(z))\leqq D(\omega_n(z)),$$

whence by Theorem 4 $\omega_n(z)\to 0$ as $n\to\infty$. Hence there exists for any given large number T, a number n and a harmonic function $\omega_n^*(z)$ in $(R_\delta\cap R_{n_0})+(\partial R_{n_0}-D_M)+(\partial D_M\cap(R_n-R_{n_0}))$ such that $\omega_n^*(z)=0$ on F, $\frac{\partial\omega_n^*}{\partial n}(z)=0$ on $(\partial R_\delta\cap R_{n_0})-F+(\partial D_M\cap(R_n-R_{n_0}))+(\partial R_{n_0}-D_M)$, $\omega_n^*(z)=T$ on $\partial R_n\cap D_M$ and $\int\limits_{F\cap R_{n_0}}\frac{\partial\omega_n^*}{\partial n}(z)=2\pi$.

Put $re^{i\theta}=\exp(\omega_n^*(z)+i\tilde{\omega}_n^*(z))$, where $\tilde{\omega}_n^*(z)$ is the conjugate function of $\omega_n^*(z)$. Put $L(r)=\int\left|\frac{\partial}{\partial n}V(z)\right|rd\theta$, where the integration is taken over $((R_\delta\frown R_{n_0})+(D_M-D_{M_2}))\frown(E[z\in R;\ \omega_n^*(z)=\log r])(M<M_2)$.

Suppose $L(r)>\varepsilon_0$ for every r. Then

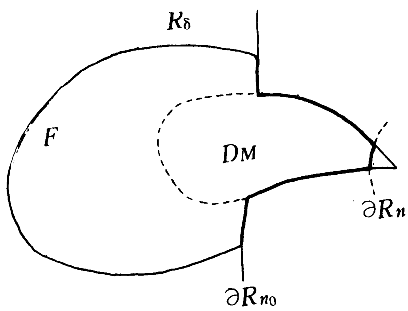

Fig. 3

$$\varepsilon_0^2\int_1^T\frac{1}{r}dr\leqq\int_1^T\frac{L^2(r)}{r}dr$$

$$\leqq\iint_{D_M-D_{M_2}}\left\{\left(\frac{\partial V(z)}{\partial r}\right)^2+r^2\left(\frac{\partial V(z)}{\partial\theta}\right)^2\right\}rdrd\theta$$

$$\leqq D_{R_\delta-D_{M_2}}(V(z))<\infty\,.$$

Let $T\to\infty$. Then $D(V(z))\to\infty$. This is a contradiction. Hence there exists a sequence $\{r_i\}$ such that $L(r_i)\to 0$. Since $\frac{\partial V}{\partial n}(z)<0$ on ∂D_M and $\frac{\partial}{\partial n}V(z)>0$ on ∂D_{M_2}. Hence $k=\int_{\partial D_M}\frac{\partial}{\partial n}V(z)ds=\int_{\partial D_{M_2}}\frac{\partial}{\partial n}V(z)ds$ and $D_{D_M-D_{M_2}}(V(z))=k(M_2-M)$. Hence we have the theorem.

Lemma 3. *Let $V(z)$ be a positive harmonic function (not necessarily a generalized Green's function) in $R-R_0$. Let G and G' be non compact domains such that $R-R_0=\bar{G}+G'$.*[7] *Let ${}_nV_G^\alpha(z)({}_nV_G^\beta(z))$ be the lower (upper) envelope of super (sub) harmonic functions larger (smaller) than $V(z)$ in $G\frown(R-R_n)$. Put $V_G^\alpha(z)=\lim_n{}_nV_G^\alpha(z)$ and $V_G^\beta(z)=\lim_n{}_nV_G^\beta(z)$. Then*

$${}^{\alpha}_{G}(V_G^\alpha(z))=V_G^\alpha(z)\ \text{and}\ {}^{\alpha}_{G}(V_{G'}^\beta(z))=0\,.$$

Let $V_{n,n+i}(z)$ be a harmonic function in $R_n+((R_{n+i}-R_n)\frown G)-R_0$ such that $V_{n,n+i}(z)=0$ on $\partial R_0+(\partial R_{n+i}-G)$ and $V_{n,n+i}(z)=V(z)$ on $\partial G\frown(R_{n+i}-R_n)+G\frown(R-R_n)$. Then for every $G\frown(R-R_n)$ by $V_{n,n+i}(z)\uparrow V_n(z)$ and by $G_i(\zeta,z)\uparrow G(\zeta,z)$

$$\lim_i V_{n,n+i}(z)=V_n(z)=\int_{\partial(G\frown(R-R_n))+(G\frown\partial R_n)}V_n(\zeta)\frac{\partial}{\partial n}G(\zeta,z)ds,$$

where $G_i(\zeta,z)$ and $G(\zeta,z)$ are the Green's function of $R_{n+i}-R_0-(G\frown(R_{n+i}-R_n)$ and $R-R_0-(G\frown(R-R_n))$ respectively.

---

7) $\bar{G}$ means the closure of G.

Since $V_n(z) \downarrow V_G^{\alpha}(z)$, $\quad V_G^{\alpha}(z) = \displaystyle\int_{\partial(G \cap (R-R_n)) + (G \cap \partial R_n)} V_G^{\alpha}(\zeta) \frac{\partial}{\partial n} G(\zeta, z) ds .$

Next let $V'_{n,n+i}(z)$ be a harmonic function in $R_n + ((R_{n+i}-R_n) \cap G) - R_0$ such that $V'_{n,n+i}(z) = 0$ on $\partial R_0 + (\partial R_{n+i} - G)$ and $V'_{n,n+i}(z) = V_G^{\alpha}(z)$ on $(\partial G \cap (R_{n+i}-R_n)) + (G \cap \partial R_n)$. Then

$$\lim_i V'_{nn,+i}(z) = \int V_G^{\alpha}(\zeta) \frac{\partial}{\partial n} G(\zeta, z) ds ,$$

i. e. $\lim\limits_i V'_{n,n+i}(z) = V_G^{\alpha}(z)$ for every n, hence

$$ {}_{G}^{\alpha}(V_G^{\alpha}(z)) = V_G^{\alpha}(z) . \tag{1}$$

Let $\tilde{V}_{n,n+i}(z)$ be a harmonic function in $R_{n+i} - ((R_{n+i}-R_n) \cap G) - R_0$ such that $\tilde{V}_{n,n+i}(z) = 0$ on $\partial R_0 + (\partial R_n \cap G) + (\partial G \cap (R_{n+i}-R_n))$ and $\tilde{V}_{n,n+i}(z) = V(z)$ on $\partial R_{n+i} \subset G'$. Then

$$V(z) = V_{n,n+i}(z) + \tilde{V}_{n,n+i}(z), \quad \text{which implies}$$

$$V(z) = V_G^{\alpha}(z) + V_{G'}^{\beta}(z) . \tag{2}$$

From (1) we have
$$\begin{aligned} V(z) &= {}_{G}^{\alpha}(V_G^{\alpha}(z) + V_{G'}^{\beta}(z)) + V_{G'}^{\beta}(z) \\ &= {}_{G}^{\alpha}(V_G^{\alpha}(z)) + {}_{G}^{\alpha}(V_{G'}^{\beta}(z)) + V_{G'}^{\beta}(z) , \end{aligned}$$

whence by (1) and (2) we have

$$ {}_{G}^{\alpha}(V_{G'}^{\beta}(z)) = 0 . \tag{3}$$

Let $V(z)$ be a generalized Green's function. Let $G(z, q)$ be the Green's function of $R-R_0$ with pole at q. Put $G = E[z \in R : G(z, q) > k]$ and $G' = E[z \ni R : G(z, q) < k]$. Then $V(z) = V_G^{\alpha}(z) + V_{G'}^{\beta}(z)$. We shall study the properties of $V_{G'}^{\beta}(z)$.

Lemma 4. *Let $V(z)$ be a generalized Green's function and put $G = E[z \in R : G(z, q) > k]$ and $G' = E[z \in R : G(z, q) < k]$ and $D_M = E[z \in R : V_{G'}^{\beta}(z) > M]$. Let $H_{G'}^M(z)$ be the lower envelope of superharmonic function larger than* $\min(M, V_{G'}^{\beta}(z))$ *on $G' \cap D_M$. Then*

$$\lim_{M=\infty} H_{G'}^M(z) = V_{G'}^{\alpha}(z) .$$

For simplicity denote $V_{G'}^{\beta}(z)$ by $H(z)$. Let ${}_nH_{G'}^M(z)$ be a harmonic function in $R_n - R_0 - (D_M \cap G')$ such that ${}_nH_{G'}^M(z) = 0$ on

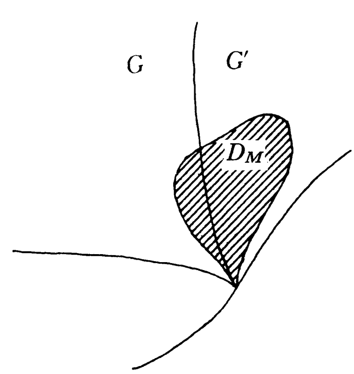

Fig. 4

$\partial R_0+\partial R_n-(D_M\cap G')$ and ${}_nH_{G'}^M(z)=M$ on $\partial(D_M\cap G')$. Let ${}_n\check{H}_G^M(z)$ be a harmonic function in $R_n-R_0-(D\cap G)$ such that ${}_n\check{H}_G^M(z)=0$ on $\partial R_0+(\partial R_n-(D_M\cap G))$, ${}_n\check{H}_G^M(z)=M$ on $\partial D_M\cap G$ and ${}_n\check{H}_G^M(z)=H(z)-M$ on $\partial G\cap D_M$. Then cleary

$$\lim_n {}_nH_G^M(z) \leqq H(z) \leqq \lim_n {}_nH_{G'}^M(z)+\lim_n {}_n\check{H}_G^M(z)$$

and

$$\lim_{M\to\infty}(\lim_n {}_n\check{H}_G^M(z)) \leqq {}_G^{\alpha}(H(z)) = {}_G^{\alpha}(V_{G'}^{\beta}(z)) = 0\,.$$

Hence

$$V_{G'}^{\beta}(z) = H(z) = \lim_{M=\infty}\lim_n {}_nH_{G'}^M(z) = \lim_{M\to\infty} H_{G'}^M(z)\,.$$

Theorem 6. *Let $V(z)$ be a generalized Green's function such that $D(\min M, V(z))\leqq M\pi$. Then by Lemma 3, $V(z)=V_G^{\alpha}(z)+V_{G'}^{\beta}(z)$, where $G'=E[z\in R:\ G(z, q)<k]$.*

$$\textit{Then } V_{G'}^{\beta}(q) \leqq \frac{k}{2}\,.$$

Clearly $V(z)\geqq V_G^{\alpha}(z)$ and $V(z)\geqq V_{G'}^{\beta}(z)\geqq 0$. If $V_{G'}^{\beta}(z)=0$, our assertion is trivial. Suppose $V_{G'}^{\beta}(z)>0$. Then by Theorem 3, $V_{G'}^{\beta}(z)$ is also a generalized Green's function such that $D(\min((M, V_{G'}^{\beta}(z))\leqq M\pi$.
Next by Lemma 3

$$V_{G'}^{\beta}(z) = H(z) = \lim_{M=\infty}\lim_n\ {}_nH_{G'}^M(z) \text{ and } H(z)\geqq H_{G'}^M(z) = \lim_n {}_nH_{G'}^M(z)\,.$$

Hence by Theorem 5

$$\int_{\partial(D_M\cap G')} \frac{\partial}{\partial n} H_{G'}^M(z)ds \leqq \int_{\partial D_M} \frac{\partial}{\partial n} H(z)ds \leqq \pi \tag{4}$$

where $D_M=E[z\in R:\ H(z)>M]$.

Since $g_\delta=E[z\in R:\ H_{G'}^M(z)>\delta]\subset E[z\in R:\ H(z)>\delta]$, $(E[z\in R:\ H_{G'}^M(z)>L]=)D_L\cap G'$ determines a set of the boundary of capacity zero for $L>\delta$ by Theorem 4. Hence by $D({}_{G'}^{M}H(z))<\infty$ over $R-R_0-(D_M\cap G')$, we can prove as in Theorem 5

$$\int_{\Gamma_\delta} \frac{\partial}{\partial n} H_{G'}^M(z)ds = -\int_{\partial(D_M\cap G')} \frac{\partial}{\partial n} H_{G'}^M(z)ds\,,$$

where $\Gamma_\delta=E[z\in R:\ H_{G'}^M(z)=\delta]$.

Let $G_\delta(z, q)$ be the Green's function of $g_\delta\cap(R-R_0)$. Then $D(G_\delta(z, q))<\infty$ over a neighbourhood of the ideal boundary. Hence there exists a sequence of curves $\{\Gamma_i\}$ such that $\int_{\Gamma_i \cap D_M} \left|\frac{\partial}{\partial n} G_\delta(z, q)\right| ds \to 0$ as $i \to \infty$ and $\{\Gamma_i\}$ clusters at the ideal boundary as $i \to \infty$ and every Γ_i separates the boundary determined by D_M from q. Let $C = \partial(D_M \cap G')$ and C_i be the part of C_i contained in the domain $\ni q$ separated by Γ_i and $C_i' = C - C_i$. Then

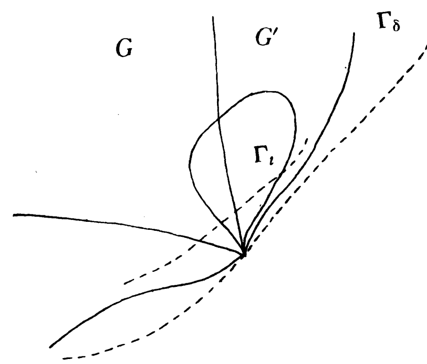

Fig. 5

$$\int_{C_i + C_i' + q + \Gamma_\delta} H_{G'}^M(z) \frac{\partial}{\partial n} G_\delta(z, q) ds =$$

$$\int_{C + q + \Gamma_\delta} G_\delta(z, q) \frac{\partial}{\partial n} H_{G'}^M(z) ds,$$

$$M \int_{C_i + C_i'} \frac{\partial}{\partial n} G_\delta(z, q) ds + 2\pi H_{G'}^M(q) + \delta \int_{\Gamma_\delta} \frac{\partial}{\partial n} G_\delta(z, q) ds = \int_C G_\delta(z, q) \frac{\partial}{\partial n} H_{G'}^M(z) ds .$$

But the first term of the left hand side $\to 0$ as $i \to \infty$ and the remaining terms don't depend on i. Hence by letting $\delta \to 0$ and by $G_\delta(z, q) \uparrow G(z, q)$, we have

$$2\pi H_{G'}^M(q) = \int_C G(z, q) \frac{\partial}{\partial n} H_{G'}^M(z) ds \leqq k\pi ,$$

because $G(z, q) \leqq k$ in G'. Then by letting $M \to \infty$

$$H(q) = V_{G'}^\beta(q) \leqq \frac{k}{2} .$$

Put $V_G^\alpha(z) = V^{*k}(z)$ and $V_{G'}^\beta(z) = V'^k(z)$. Then by Theorem 6, $V'^k(z) \to 0$ as $k \to 0$. Then we have

Theorem 7. *Every generalized Green's function $V(z)$ is divided into two parts such that*

$$V(z) = V^{*k}(z) + V'^k(z) \text{ and } V(z) = \lim_{k \to 0} V^{*k}(z) .$$

Remark. $K(z, p_i) = \frac{G(z, p_i)}{G(p_0, p_i)}$ (p_0 is a fiexed point) is a positive harmonic function. Martin[8] defined *ideal boundary points* by using above functions and prove that every positive harmonic function is representable

8) R. S. Martin: Minimal positive harmonic functions. Trans. Amer. Math. Soc. 39, 1941.

by a unique mass distribution ν as follows: $\int_{B_1} K(z, p)d\nu(p)$, where B_1 is the set of minimal points. If $\overline{\lim_{i\to\infty}} G(p_i, q) > 0$ as p_i tends to a boundary point p and $K(z, p_i) \to K(z, p)$, we call p an irregular boundary point. In this case, $K(z, p)$ is a constant multiple of $G(z, p) = \lim_i G(z, p_i)$. We denote by I_k the set of Martin's boundary point p such that $\lim_{z\to p} G(z, q) \geqq k$. Then $V^{*k}(z)$ is represented by a mass distribution ν on I_k. Hence by Theorem 8 a generalized Green's function is represented by a mass distribution ν on $I = \bigcup_{k>0} I_k$.

Theorem 8. *Let $W(z)$ be a positive harmonic in $R-R_0$ and superharmonic function in $\overline{R-R_0}$ vanishing on ∂R_0. Then*

$$W(z) = \int N(z, p)d\mu(p) = \int U(z, p)d\mu(p) + \int V(z, p)d\mu(p) = U(z) + V(z),$$

where $U(z) = \int U(z, p)d\mu(p)$ is a harmonic function representable by Poisson's integral and $V(z) = \int V(z, p)d\mu(p)$ is a generalized Green's function.

Since $0 < U(z, p) \leqq N(z, p)$, family $\{U(z, p)\}$ is uniformly bounded in every compact domain in $R-R_0$ and the partial derivatives of them are equicontinuous and $\Delta U(z, p) = 0$, hence $U(z)$ and $V(z)$ are harmonic in $R-R_0$.

For a harmonic function $H(z)$ define $H^M(z) = \lim_n H_n^M(z)$, where $H_n^M(z)$ is a harmonic function in $R_n - R_0$ such that $H_n^M(z) = \min(M, H(z))$ on $\partial R_0 + \partial R_n$. Then clealry ${}^M(H^M(z)) = H^M(z)$. Since $0 < U(z, p) \leqq N(z, p)$ and $U^M(z) \uparrow U(z, p)$ as $M \uparrow \infty$, whe have

$$U(z) = \int U(z, p)d\mu(p) = \lim_{M=\infty} \int U^M(z, p)d\mu(p) \leqq \lim_{M=\infty} {}^M\Big[\int N(z, p)d\mu(p)\Big]$$
$$= \lim_{M=\infty} \lim_n W_n^M(z),$$

where $W_n^M(z)$ is a harmonic function in $R-R_0$ such that $W_n^M(z) = \min(M, W(z))$ on $\partial R_0 + \partial R_n$. Now $\lim_{M=\infty} \lim_n W_n^M(z) = W^p(z)$ *is representable by Poisson's integral.* $0 < U(z) \leqq W^p(z)$ implies the *Poisson's integrability of* $U(z)$.

By the Remark $V(z, p) = \int_I K(z, q)d\nu(q)$, whence $V(z) = \int V(z, p)d\mu(p) = \int_I K(z, q)d\lambda(q)$. Hence there exist n_0 and k_0 such that

$$\int V(z, p)d\mu(p) < \int_{I_k} K(z, q)d\lambda(q) + \varepsilon \tag{5}$$

for $z \in R_n - R_0$, $n < n_0$ and $k < k_0$ for any given positive number ε, where λ' is the restriction of λ on I_k.

Denote by $(\int_{I_k} K(z, q) d\lambda'(q))^n_{I_k}$ the lower envelope of superharmonic functions larger than $\int_{I_k} K(z, q) d\lambda'(q)$ in $G \cap (R - R_0)$. Put $(\int_{I_k} K(z, q) d\lambda'(q))_{I_k} = \lim_n (\int_{I_k} K(z, q) d\lambda'(q))^n_{I_k}$. Then as in Lemma 3 and Theorem 2 it is proved that $(\int_{I_k} K(z, q) d\lambda'(q)) = (\int_{I_k} K(z, q) d\lambda'(q))_{I_k}$ and $(\int_{I_k} K(z, q) d\lambda'(q))$ has angular limits $=0$ a. e. on the ideal boundary[9]. In (5) let $\varepsilon \to 0$. Then $\int K(z, q) d\lambda(q) = \int V(z, p) d\mu(p)$ has angular limits $=0$ a. e. on the ideal bonndary. Hence $U(z) = \int U(z, p) d\mu(p)$ has the same angular limits as $\int N(z, p) d\mu(p)$ a. e. on the ideal bounary. Thus by Poisson's integrability of $U(z)$ and $W^p(z)$, we have $U(z) \equiv W^p(z)$ and $W(z) - W^p(z) \equiv \int V(z, p) d\mu(p)$. Now $W(z) - W^p(z) = \lim_{M' = \infty} \lim_n W'^{M'}_n(z)$, where $W'^{M'}_n(z)$ is a harmonic function in $R_n - R_0$ such that $W'^{M'}_n(z) = 0$ on ∂R_0 and $W'^{M'}_n(z) = W(z) - W^{M'}_n(z)$ on ∂R_n. Since $N(z, p)$ is a continuous function of p for $z \in R$, there exists a sequence $\{W_m(z)\}$ $(m = 1, 2, \cdots)$ of the form $W_m(z) = \sum^m c_i N(z, p_i)$ $(c_i > 0$, $\sum c_i = \mu_0 = \int d\mu(p))$ such that $W_m(z) \to W(z)$ in $R - R_0$. On the other hand, let $V^{M'}_{n,m}(z)$ be a harmonic function in $R_n - R_0$ such that $V^{M'}_{n,m}(z) = 0$ on ∂R_0 and $V^{M'}_{n,m}(z) = \min(W^m(z) - M', 0)$ on ∂R_n. Then there exists a sequence $\{V^{M'}_{n,m}(z)\}$ which converges to $\lim_n W'^{M'}_n(z)$ as $n \to \infty$ and $m \to \infty$.

Since $V^{M'}_{n,m}(z)$ is constructed from $W_m(z) = \sum^m c_i N(z, p)$, we can prove by the method used for $V(z, p)$ and $N(z, p)$ that $D(\min(M, V^{M'}_{n,m}(z)) \leqq 4\pi(\sum c_i) M'$ for $M' < M$. Hence by letting $n \to \infty$, $m \to \infty$ and $M \to \infty$ we have

$$D(\min(M', V(z)) = D(\min(M', \lim_n \lim_m V^{M'}_{n,m}(z))$$
$$\leqq \lim_{M=\infty} \lim_{m,n} D(\min(M', V^M_{n,m}(z)) \leqq 4\pi(\sum c_i) M'.$$

Hence $\int V(z, p) d\mu(p)$ is a generalized Green's function. We have Theorem 8.

Lemma 5. *Let $V(z)$ be a generalized Green's function in $R - R_0$ such*

9) We map the universal covering surface of $(R - R_0)$ onto $|\zeta| < 1$. If the function $U(z)$ has angular limits$=0$ a.e. on the image of the ideal boundary on $|\zeta| = 1$. We say simply $U(z)$ has angular limits$=0$ a.e. on the ideal boundary.

that $D(\min(M, V(z)) \leqq M\pi$. Then there exists a uniquely determined generalized Green's function $V^*(z)$ in R such that $D(\min(M, V^*(z)) \leqq M\pi$ and sup $(V^*(z)) - V(z)) < \infty$.

Since ∂R_0 is compact, there exists a contant L such that $0 < \frac{\partial}{\partial n} V(z) \leqq L$ on ∂R_0. Let $\omega(z)$ be a positive bounded harmonic function in $R-R_0$ such that $\omega(z)=1$ on ∂R_0 and $\omega(z)$ has angular limits $=0$. a. e. on the ideal boundary of $R-R_0$. Put $\tilde{\omega}(z)\equiv 1$ in R_0 and $\tilde{\omega}(z) \equiv \omega(z)$ in $R-R_0$. Then $V(z)+K\tilde{\omega}(z)(K>L)$ is a superharmonic function in R. Let $V_n^*(z)$ be a harmonic function in R_n such that $V_n^*(z)=V(z)$ on ∂R_n. Then $V(z) < V_n^*(z) \leqq V(z)+K\omega(z)$. Choose a subsequence $(n_1, n_2, , \cdots)$ so as $V_n^*(z)$ converges to $V^*(z)$. Then

$$V(z) \leqq V^*(z) \leqq V(z)+K\tilde{\omega}(z) .$$

Hence $V^*(z)$ has angular limits $=0$ a. e. on the boundary of R and by sup $(V^*(z) - V(z)) < \infty$, we see that such $V^*(z)$ does not depend on the above subsequence and $V^*(z)$ is uniquely determined.

Clearly $D(\min(M, V(z)) \leqq D(\min(M+K, V(z)+K\omega(z))$, hence
$D(\min(M, V(z)) \leqq 2D(\min(2M, V(z))+2D(\omega(z)) \leqq 10\pi M$, for large M.

But both $E[z \in R-R_0, V^*(z) > \delta]$ and $E[z \ni R-R_0, \omega(z) > \delta]$ determine sets of the boundary of capacity zero,[10] whence as in Theorem, we have

$$\int_C \frac{\partial}{\partial n} V^*(z) ds = k \leqq 10\pi ,$$

for every niveau curve C of $V(z)$ and $D(\min(M, V^*(z)) \leqq 10\pi M$ for every M. Thus $V^*(z)$ is a generalized Green's function.

Proof of Theorem 1. Let $W^*(z)$ be a harmonic and superharmonic function in $\bar{R}$. Let $S(z)$ be a harmonic function in $R-R_0$ such that $S(z)=W^*(z)$ on ∂R_0 and $S(z)$ has M. D. I. over $R-R_0$. Then $S(z)$ is bounded and $W^*(z)-S(z)=W(z)=U(z)+V(z)$ in $R-R_0$ in Theorem 9. Let $U_n^*(z)$ be a harmonic function in R_n such that $U_n^*(z)=U(z)+S(z)$ on ∂R_n. Let $V_n^*(z)$ be a harmonic function in Lemma 5. Then $W^*(z)= U_n^*(z)+V_n^*(z)$. Choose a subsequence $(n_1, n_2, \cdots)$ such that both $U_n^*(z)$ and $V_n^*(z)$ converge to $U^*(z)$ and $V^*(z)$ respectively. Then $U^*(z)$ is representable by Poisson's integral and $U^*(z)$ has angular limits as $U(z)+S(z)$ a. e. on the boundary of $R-R_0$, whence $U^*(z)$ does not depend on the above subseqence. Thus $W^*(z)=U^*(z)+V^*(z)$.

10) See 3) or Mass distributions. III (in this volume) (Properties of functiontheoretic equilibrium potential).

Apply our result to a unit-circle $|z|<1$. Then we have the following

Proposition. Let $U(z)$ be a logarithmic potential such that the total mass is bounded and whose mass does not exist in $|z|<1$. Then the potential $U(z)$ is representable by Poisson's integral in $|z|<1$, because in this case $|z|=1$ consists of only regular points of the Green's function and $V(z)=0$.

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