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*On the Invariants of Finite Nilpotent Groups**

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1. If $x_1, \cdots, x_n$ are independent variables over a field k operated on by a group G of linear transformations of finite order, and if $x_1^\sigma, \cdots, x_n^\sigma$ are the linear functions into which the variables are changed by a linear transformation of the group, then a rational function $F(x_1, \cdots, x_n)$ of variables is called an invariant of the group, if $F(x_1, \cdots, x_n)=F(x_1^\sigma, x_2^\sigma, \cdots, x_n^\sigma)$ for each linear transformation σ of the group. Under what condition is the field of invariants birational[1]? This is a very difficult question for general group G and field k.

In the present note, calculating the complete systems of generators of the invariants inductively, we shall prove the birationality of the fields of invariants for the particular type of representations of nilpotent groups. Our result is the following:

Theorem. *Let G be a finite nilpotent group of exponent N and let k be a field containing N-th roots of unity, where we assume that the characteristic of k is coprime to N. Let $G=G_0\supset G_1\supset \cdots \supset G_{r+1}=\{e\}$ be the descending chain of normal subgroups of G such that G_i/G_{i+1} is a cyclic subgroup of G/G_{i+1} $(i=0, 1, \cdots, r)$. Let λ_i be the natural homomorphism of G onto G/G_i and let $\{M_i(\sigma)\}$ be the regular representation of G/G_i $(i=1, 2, \cdots, r+1)$. Put*

$$M(\sigma) = \overbrace{M_1(\lambda_1(\sigma))+ \cdots +M_1(\lambda_1(\sigma))}^{t_1} \oplus \overbrace{M_2(\lambda_2(\sigma))+ \cdots +M_2(\lambda_2(\sigma))}^{t_2} \oplus$$
$$\cdots \oplus \overbrace{M_{r+1}(\sigma)+ \cdots +M_{r+1}(\sigma)}^{t_{r+1}}, \quad \text{where } t_i \geqq 1 \ (i=1, 2, \cdots r+1).$$

Then the field of invariants of $\{M(\sigma)\,;\ \sigma\in G\}$ is birational.

2. The proof of the theorem.

We shall prove the theorem by the induction on r. For the case $r+1=1$ the theorem is evidently true. We assume that it is true for the case the length of the above descending chain of normal subgroups is less than $r+1$.

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1) We mean by birationality the purely transcendency over k.

We denote by $\alpha_1, \cdots, \alpha_{n_r}$ all the elements of G_r and by $\bar{\tau}_1, \cdots, \bar{\tau}_{m_r}$ all the elements of G/G_r. We denote by $[\bar{\tau}_l]$ a representative of $\bar{\tau}_l$ in G, where we assume that $[\bar{e}]=e$. Let $\{C_{\bar{\tau}_i, \bar{\tau}_j}\}$ be the factor set of G/G_r in G_r which defines G and let χ be a generator of the charactor group of G_r. Let $x_1, \cdots, x_m$ be the variables on which $\{M(\sigma) : \sigma \in G\}$ operates, where m is the degree of $\{M(\sigma)\}$. We divide the variables $\{x_1, \cdots, x_m\}$ into two parts $\{x_1, \cdots, x_{m'}\}$ and $\{x_{m'+1}, \cdots, x_m\}$ such that

$$(x_1^\sigma, \cdots, x_{m'}^\sigma) = (x_1, \cdots, x_{m'})(M_1(\lambda_1(\sigma)) + \cdots + M_r(\lambda_r(\sigma))$$

and

$$(x_{m'+1}^\sigma, \cdots, x_m^\sigma) = (x_{m'+2}, \cdots, x_m)(M_{r+1}(\sigma)) + \cdots + M_{r+1}(\sigma)) .$$

We denote by $(\sigma_1, \cdots, \sigma_n)$ the system of elements of G such that $(\sigma_1\sigma, \cdots, \sigma_n\sigma) = (\sigma_1, \cdots, \sigma_n)M_{r+1}(\sigma)$. Put $x_{1,\sigma_1}=x_{m'+2}, x_{1,\sigma_2}=x_{m'+2}, \cdots, x_{1,\sigma_n}=x_{m'+n}$, $x_{2,\sigma_1}=x_{m'+n+1}, \cdots, x_{2,\sigma_n}=x_{m'+2n}, \cdots, x_{h,\sigma_1}=x_{m-n+1}, \cdots, x_{h,\sigma_n}=x_m$. Then we have $x_{i,\sigma_j}^\sigma=x_{i,\sigma_j\sigma}$ $(i=1, 2, \cdots, h ; \sigma \in G)$. We denote by $k(x_1, \cdots, x_{m'})^{G/Gr}$ the subfield of $k(x_1, \cdots, x_{m'})$ consisting of elements which are fixed by all automorphisms induced by $M_1(\lambda_1(\sigma)) \oplus \cdots \oplus M_r\lambda_r(\sigma))$ $(\sigma \in G)$. Then the factor system $\chi(C_{\bar{\tau}_i, \bar{\tau}_j})$, $k(x_1, \cdots, x_{m'})$; $\bar{\tau}_i, \bar{\tau}_j \in G/G_r$ defines a normal simple algebra over $k(x_1, \cdots, x_{m'})^{G/Gr}$. Since this algebra splits in $k(x_1, \cdots, x_{m'})$, there exists a system $\{a(\bar{\tau}_l) ; \bar{\tau}_l \in G/G_r\}$ of elements of $k(x_1, \cdots, x_{m'})$ such that

$$(1) \qquad \chi(C_{\bar{\tau}_j, \bar{\tau}_l}) = \frac{a(\bar{\tau}_j)^{\bar{\tau}_l} a(\bar{\tau}_l)}{a(\bar{\tau}_j \bar{\tau}_l)}$$

and $a(\bar{e})=1$.

We put

$$(2) \qquad t_{\gamma,i,\sigma} = \sum_{\alpha \in G_r} \chi^i(\alpha^{-1}) x_{\gamma,\sigma\alpha} ,$$

$$(3) \qquad u_{\gamma,i,\bar{\tau}_j} = \frac{t_{\gamma,i,[\bar{\tau}_j]}}{t_{1,i,\sigma}}$$

and

$$(4) \qquad w_{\gamma,i,\bar{\tau}_j} = \frac{\dfrac{1}{a(\tau_j)^i} t_{\gamma,i,[\bar{\tau}_j]}}{\displaystyle\sum_l \frac{1}{a(\tau_l)^i} t_{1,i,[\bar{\tau}_l]}}$$

Then we have

$$(5) \qquad (t_{\gamma,i,\sigma})^{\sigma'} = t_{\gamma,i,\sigma\sigma'} ,$$

$$(6) \qquad (t_{\gamma,i,\sigma})^{\alpha_l} = \chi^i(\alpha_l) t_{\gamma,i,\sigma} ,$$

(7) $\quad (u_{\gamma,i,\bar{\tau}_j})^{\alpha_l} = u_{\gamma,i,\bar{\tau}_j}\,,$

(8) $\quad (w_{\gamma,i,\bar{\tau}_j})^{\tau_l} = w_{\gamma,i,\bar{\tau}_j\bar{\tau}_l}\,,$

(9) $$w_{1,i,\bar{e}} = \frac{1}{\sum \frac{1}{a(\bar{\tau}_l)^i} u_{1,i,\bar{\tau}_l}}$$

(10) $$w_{\gamma,i,\bar{\tau}_j} = \frac{\frac{1}{a(\bar{\tau}_j)^i} u_{\gamma,i,\bar{\tau}_j}}{\sum_{\bar{a}} \frac{1}{a(\bar{\tau}_l)_i} u_{1,i,\bar{\tau}_l}}$$

and

(11) $$u_{\gamma,i,\bar{\tau}_j} = a(\bar{\tau}_j)^i \frac{w_{\gamma,i,\bar{\tau}_j}}{w_{1,i,\bar{e}}}$$

Next, denoting by $k(x_1, \cdots, x_{m'}, x_{1,\sigma_1}, \cdots, x_{h,\sigma_n})^{G_r}$ the subfield to $k(x_1, \cdots, x_{m'}, x_{1,\sigma_1}, \cdots, x_{h,\sigma_n})$ consisting of elements ξ such that $\xi^\alpha = \xi (\alpha \in G_r)$, we observe that $k(x_1, \cdots, x_{m'}, x_{1,\sigma_1}, x_{h,\sigma_n})^{G_r}$ is generated by monomials

$$t_{\gamma_1, i_1, \sigma_{j_1}} \cdots t_{\gamma_e, i, \sigma_{j_l}}$$

satisfying $i_1 + \cdots + i_l \equiv 0 \pmod{n_r}$ over $k(x_1, \cdots, x_{m'})$ and $k(x_1, \cdots, x_{m'}, x_{1,\alpha_1}, \cdots, x_{h,\alpha_{n_r}})^{G_r}$ is generated by

(12) $\quad f_i = t_{1,i,e}\, t_{1,1,e}^{n_r - i} \quad (i = 1, 2, \cdots, h)$

over $k(x_1, \cdots, x_{m'})$. Hence we have

(13) $$\begin{aligned} &k(x_1, \cdots, x_{m'}, x_{1,\sigma_1}, \cdots, x_{h,\sigma_n})^{G_r} \\ &\quad = k(x_1, \cdots, x_{m'}, f_1, \cdots, f_{n_r}, u_{1,\bar{\tau}_1}, \cdots, u_{h,n_r,\bar{\tau}_{m_r}})\,. \end{aligned}$$

Putting

(14) $$v_i = \sum_{j=1}^{m_r} f_i^{\bar{\tau}_j}$$

we observe that $v_i = (\sum_{j=1}^{m_r} u_{1,i\bar{\tau},j}\, u_{1,1,\bar{\tau}_j}^{n_r - i}) f_i$ and

(15) $$\begin{aligned} &k(x_1, \cdots, x_{m'}, x_{1,\sigma_1}, \cdots, x_{h,\sigma_n}) \\ &\quad = k(x_1, \cdots, x_{m'}, v_1, \cdots, v_{n_r}, u_{1,1,\bar{\tau}_1}, \cdots, u_{h,n_r,\bar{\tau}_m})\,. \end{aligned}$$

If we put

(16) $$y_{\gamma,i,\bar{\tau}_j} = w_{\gamma,i,\bar{\tau}_j} \qquad \begin{pmatrix} i = 1, 2, \cdots, n_r \\ j = 1, 2, \cdots, m_r \\ \gamma = 1, 2, \cdots, h \end{pmatrix}$$

and

$$y_{1,i,\bar{\tau}_j} = w_{1,i,\bar{\tau}_j}v_i \qquad \begin{pmatrix} i = 1, 2, \cdots, n_r \\ j = 1, 2, \cdots, m_r \end{pmatrix} \tag{17}$$

then we have

$$(y_{\gamma,i,\bar{\tau}_j})^{\bar{\tau}_l} = y_{\gamma,i,\bar{\tau}_j\bar{\tau}_l} \tag{18}$$

and

$$y_{1,i,\bar{e}} = \frac{v_i}{\sum_l \frac{1}{a(\bar{\tau}_l)^i} u_{1,i,\bar{\tau}_l}} \tag{19}$$

Hence we get

$$\sum_l \frac{1}{a(\bar{\tau}_l)_i} u_{1,i,\bar{\tau}_l} = \frac{v_i}{y_{1,i,l}} \tag{20}$$

By virtue of (11) we observe that

$$\frac{1}{a(\bar{\tau}_j)^i} u_{\gamma,i,\bar{\tau}_j} = \frac{y_{\gamma,i,\bar{\tau}_j}v_i}{y_{1,i,\bar{e}}} \qquad (\gamma = 2, 3, \cdots, h) \tag{21}$$

$$\frac{1}{a(\bar{\tau}_j)^i} u_{1,i,\bar{\tau}_j} = \frac{y_{1,i,\bar{\tau}_j}}{y_{1,i,\bar{e}}} \tag{22}$$

and

$$v_i = \sum_l y_{1,i,\bar{\tau}_l} \tag{23}$$

Therefore we can observe that $y_{1,l,\bar{\tau}_1}, \cdots, y_{h,n_r,\bar{\tau}_{m_r}}$ are independent variables over $k(x_1, \cdots, x_{m'})$ and

$$\begin{aligned} k(x_1, \cdots, x_{m'}, x_{1,\sigma_1}, \cdots, x_{h,\sigma_n})^{G_r} &= k(x_1, \cdots, x_{m'}, v_1, \cdots, v_{n_r}, u_{1,l,\bar{\tau}_1}, \cdots, u_{h,n_r,\bar{\tau}_{m_r}}) \\ &= k(x_1, \cdots, x_{m'}, y_{1,l,\bar{\tau}_1}, \cdots, y_{h,n_r,\bar{\tau}_{m_r}}) . \end{aligned} \tag{24}$$

Hence, by virtue of (18), G/G_r opervtes on $k(x_1, \cdots, x_m)^{G_r}$ similarly as G on $k(x_1, \cdots, x_m)$. By virtue of the assumption of the induction, we observe that the subfield $k((x_1, \cdots, x_m)^{G_r})^{G/G_r}$ in $k(x_1, \cdots, x_m)^{G_r}$ consisting of element ξ such that $\xi^\sigma = \xi$ $(\sigma \in G/G_r)$ is birational. On the other hand $k(x_1, \cdots, x_m)^{G_e})^{G/G_e}$ is the field of G. This proves the theorem.

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