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Congruence Classes of Knots

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Consider a solid torus V in 3-dimensional euclidean space S. An oriented simple closed curve on the boundary T of V is a *meridian* of V if it bounds in V but not on T, and it is called a *longitude* if it bounds in $S-V$ but not on T. The fundamental group of T is a free abelian group of rank 2, and it has a basis consisting of elements a and b represented respectively by a meridian and a longitude. Any orientation preserving autohomeomorphism τ of V induces an automorphism $\tau_* : a \to a^{\pm 1}$, $b \to a^m b^{\pm 1}$ of $\pi(T)$ and τ is described up to homotopy by the automorphism τ_* that it induces. I shall call an autohomeomorphism τ a *simple twist* if the automorphism that it induces is $\tau_* : a \to a$, $b \to ab$. The automorphism induced by τ^m is $\tau_*^m : a \to a$, $b \to a^m b$.

If k is any simple closed curve in the interior of V then $\tau^m(k)$ is also a simple closed curve in the interior of V. Moreover if k is oriented and $\tau^m(k)$ is given its inherited orientation then the linking numbers $L(k, a)$ and $L(\tau^m(k), a)$ are equal.

Let n and q be non-negative integers and κ and λ knot types. I shall say that κ and λ are *congruent modulo* n, q, and write $\kappa \equiv \lambda$ (mod n, q), if there are simple closed curves $k_0, k_1, \cdots, k_l$, integers $c_1, \cdots, c_l$, and solid tori $V_1, \cdots, V_l$ such that

(1) V_i contains $k_{i-1} \cup k_i$ in its interior;

(2) $\tau_i^{c_i n}(k_{i-1}) = k_i$, where τ_i is a simple twist of V_i;

(3) $L(k_{i-1}, a_i) = L(k_i, a_i) \equiv 0 \pmod q$, where a_i is represented by a meridian of V_i;

(4) k_0 represents κ and k_l represents λ.

Congruence modulo n, q is symmetric, reflexive and transitive. Congruence modulo 0, q is just equivalence, i.e. $\kappa \equiv \lambda$ (mod 0, q) iff $\kappa = \lambda$. The well-known fact [1] that any knot projection may be normed so as to be the diagram of a trivial knot shows that any two knot types are congruent modulo 1, 0 and also congruent modulo 1, 2. It is not difficult to find distinct knot types that are congruent modulo 1, 1; on the other hand, to find knot types that can be shown to be incongruent modulo 1, 1 does not seem to be easy.

It would be interesting to know whether equivalence could be replaced by a set of congruences. The question is: Do there exist distinct knot types κ, λ such that $\kappa \equiv \lambda \pmod{n, q}$ for every $n > 0$ and $q \geq 0$?

The object of this note is to give a necessary condition for congruence modulo n, q. The condition, which is rather effective if $n > 1$, involves the Alexander polynomial $\Delta_\kappa(t)$ of a knot type κ, i.e. the Alexander polynomial [2] of the fundamental group of the complement of a simple closed curve k that represents κ. An Alexander matrix of κ, i.e. an Alexander matrix [2] of $\pi(S-k)$, is denoted by $\boldsymbol{A}_\kappa(t)$, and $\sigma_n(t)$ denotes $(t^n-1)/(t-1) = 1+t+t^2+\cdots+t^{n-1}$.

Theorem[1]. *If* $\kappa \equiv \lambda \pmod{n, q}$ *then, for properly chosen* $\boldsymbol{A}_\kappa(t)$ *and* $\boldsymbol{A}_\lambda(t)$,

$$\boldsymbol{A}_\kappa(t) \equiv \boldsymbol{A}_\lambda(t) \quad \mod \sigma_n(t^q) ,$$

hence

$$\Delta_\kappa(t) \equiv \pm t^r \Delta_\lambda(t) \quad \mod \sigma_n(t^q) ,$$

(*and similarly for the elementary ideals* [2] *of deficiency greater than* 1).

Proof: We need consider only the case $l=1$, $c_1=1$, so that k represents κ and $\tau^n(k)$ represents λ. Presentations of the fundamental groups $\pi(S-k)$ and $\pi(S-\tau^n(k))$ may be obtained from the fundamental groups $\pi(S-V)$ and $\pi(V-k)$ and $\pi(V-\tau^n(k))$ by application of the van Kampen theorem [3]. This procedure yields presentations that are almost the same:

$$\begin{aligned} \pi(S-k) \quad &= (a, b, A, B, x_1, x_2, \cdots : a=A,\ b=B,\ r_1=1,\ r_2=1, \cdots) \\ \pi(S-\tau^n(k)) &= (a, b, A, B, x_1, x_2, \cdots : a=A,\ b=A^nB,\ r_1=1,\ r_2=1, \cdots) \end{aligned}$$

where a and b are represented by meridian and longitude of V, and A and B are represented by the same curves in (the closure of) $S-V$. The 1-dimensional homology groups of $S-k$ and $S-\tau^n(k)$ are infinite cyclic; we denote ambiguously by t a generator of either group (selected so that one is carried into the other by τ^n). Then abelianization of $\pi(S-k)$ or $\pi(S-\tau^n(k))$ maps A into t^q (and B into 1). Hence

1) The theory can be generalized to links (which must be ordered and oriented). For links of multiplicity μ we replace q by $(q_1, \cdots, q_\mu)$ where q_i is the linking number of the ith component with a. Instead of $\sigma_n(t^q)$ we have $\sigma_n(t_1^{q_1} t_2^{q_2} \cdots t_\mu^{q_\mu})$.

$$\boldsymbol{A}_\kappa(t) = \begin{array}{c|cccccc} & a & b & A & B & \cdot & \cdots \\ \hline a=A & -1 & 0 & 1 & 0 & 0 & \cdots \\ b=B & 0 & -1 & 0 & 1 & 0 & \cdots \\ \cdot & * & * & * & * & * & \cdots \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdots \\ \cdot\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdots \end{array}$$

$$\boldsymbol{A}_\lambda(t) = \begin{array}{c|cccccc} & a & b & A & B & \cdot & \cdots \\ \hline a=A & -1 & 0 & 1 & 0 & 0 & \cdots \\ b=A^nB & 0 & -1 & \dfrac{t^{nq}-1}{t^q-1} & t^{nq} & 0 & \cdots \\ \cdot & * & * & * & * & * & \cdots \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdots \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdots \end{array}$$

Therefore

$$\boldsymbol{A}_\lambda(t)-\boldsymbol{A}_\kappa(t) = \begin{array}{|cccccc} \hline 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & \dfrac{t^{nq}-1}{t^q-1} & t^{nq}-1 & 0 & \cdots \\ 0 & 0 & 0 & 0 & 0 & \cdots \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdots \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdots \end{array}$$

$$\equiv 0 \bmod \sigma_n(t^q)\,.$$

The only congruence classes that are fairly easy to deal with experimentally are those for which $n\equiv 0 \pmod 2$ and $q=0$ or 2. Let us consider the repartition of the fifteen prime knots of not more than seven crossings (cf. the knot table [4]) into congruence classes modulo 2, 0 and modulo 2, 2.

The polynomial character of κ mod 2, 0 is $\Delta_\kappa(t)$ mod 2. Each residue class has as principal representative a polynomial whose coefficients are all either 0 or 1. By experiment I find the following congruence classes mod 2, 0:

$$\begin{array}{lll} 0\equiv 5_2\equiv 6_1\equiv 7_4\equiv 7_5, & \Delta(t)\equiv 1 & \pmod 2 \\ 3_1\equiv 4_1\equiv 7_2\equiv 7_3, & \Delta(t)\equiv 1+t+t^2 & \pmod 2 \\ 5_1\equiv 6_2\equiv 6_3\equiv 7_6\equiv 7_7, & \Delta(t)\equiv 1+t+t^2+t^3+t^4 & \pmod 2 \\ 7_1, & \Delta(t)\equiv 1+t+t^2+t^3+t^4+t^5+t^6 & \pmod 2 \end{array}$$

It was observed by Kinoshita that the non-amphicheiral knot 3_1 must be congruent mod 2, 0 to its reflexion, since it is congruent mod 2, 0 to the amphicheiral knot 4_1.

The polynomial character of κ mod 2, 2 is $\Delta_\kappa(t)$ mod $(1+t^2)$; its principal representative is a positive odd integer (since $\Delta(t)$ is symmetric and $\Delta(1)=1$). By experiment I find the following residue classes mod 2, 2.

$$0 \equiv 3_1 \equiv 5_1 \equiv 6_2 \equiv 7_1 \equiv 7_3 \equiv 7_5, \qquad \Delta(t) \equiv 1$$
$$4_1 \equiv 5_2 \equiv 6_3, \qquad \Delta(t) \equiv 3$$
$$6_1 \equiv 7_2 \equiv 7_6, \qquad \Delta(t) \equiv 5$$
$$7_4 \equiv 7_7, \qquad \Delta(t) \equiv 7$$

The congucence mod 2, 2 of 7_4 and 7_7 was discovered by F. Hosokawa. I conclude with some examples of the experimental work involved.

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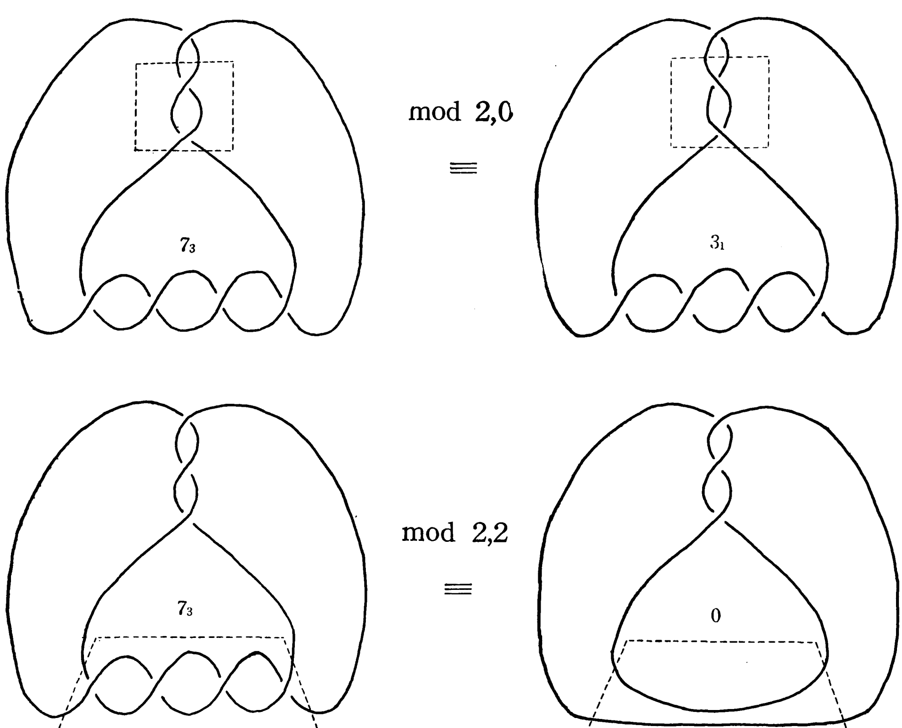

Fig. 1

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