Fox, Ralph H.
Osaka Math. J.
10 (1958), 31–35

On Knots whose Points are Fixed under a Periodic Transformation of the 3-Sphere

By R. H. Fox

Let Λ be a simple closed curve in the 3-sphere Σ and let g be a positive integer greater than 1. If the knot type of Λ is trivial there are transformations of Σ of period g that leave fixed every point of Λ. An example of such a transformation is one that is equivalent to a rotation of Σ, or if $g=2$, one that is equivalent to a reflection. Furthermore [1] there exist transformations of period g that leave fixed every point of a wild knot Λ. However it is generally conjectured that *no such transformation is possible if the knot type of* Λ *is tame and non-trivial.* Although a complete proof may well turn out to be extremely difficult, it is possible, assuming, as we shall from now on, that the transformation is semi-linear, to verify this conjecture for certain integers g and knots Λ of certain tame non-trivial types.

Thus Montgomery and Samelson [2] proved a theorem that, together with the recently proved [3] Dehn lemma, shows that, for $g=2$, Λ cannot be the boundary of any Möbius band. More recently Kinoshita proved [4], again for $g=2$, that, modulo the Poincaré conjecture, Λ cannot be any of the prime knots of fewer than 10 crossings except possibly 8_9 or 8_{20}.

In this note I shall extend part of the Montgomery-Samelson result to a rather larger class of knots. The method is the same as that of Kinoshita, and is based on a formula to be found in [6]. Along the way I shall show that, by referring to Blanchfield's theorem [9], Kinoshita's argument is seen to be valid without assumption of Poincaré's conjecture.

§1. Let Λ be a simple closed curve in the 3-sphere Σ whose knot type is tame and non-trivial, and suppose that T is a semi-linear transformation of Σ of period $g\geq 2$ such that $T(p)=p$ for each point p of Λ. Then it follows from theorems of P. A. Smith [7] that (1) the points of Λ are the only points of Σ that are fixed under T, and (2) T preserves the orientation of Σ. The orbit space S is a simply connected 3-dimensional manifold and Σ is a g-fold cyclic covering of S ramified over the simply closed curve L that lies under Λ. I do not assume that S is necessarily the 3-sphere.

Denote by $\Delta(\tau)$ and $D(t)$ the Alexander polynomials of the fundamental groups $\pi(\Sigma-\Lambda)$ and $\pi(S-L)$ respectively. Then [6, p. 418]

$$\Delta(\tau) = \Pi_{j=0}^{g-1} D(\omega^j t) ,$$

where ω is a primitive g-th root of unity and $t^g=\tau$. (As Kinoshita remarked [4], the derivation of this formula is entirely valid if the hypothesis "S is a 3-sphere" is weakened to the hypotheses "S is simply connected".) If the roots of $D(t)=0$ are $\alpha_1, \cdots, \alpha_n$ then

$$D(t) = c\, \Pi_{i=1}^{n} (t-\alpha_i) ;$$

hence

$$\begin{aligned}\Delta(\tau) &= c^g\, \Pi_{i=1}^{n} \Pi_{j=0}^{g-1} (\omega^j t-\alpha_i) \\ &= c^g\, \Pi_{i=1}^{n} (\tau-\alpha_i^g) .\end{aligned}$$

Thus

(1) *The leading coefficient of $\Delta(\tau)$ is the g-th power of the leading coefficient of $D(t)$.*

(2) *The roots of $\Delta(\tau)$ are the g-th powers $\alpha_1^g, \cdots, \alpha_n^g$ of the roots $\alpha_1, \cdots, \alpha_n$ of $D(t)$; in particular $\Delta(\tau)$ and $D(\tau)$ have the same degree.*

Condition (1) alone is sufficient to show that Λ cannot belong to any of the thirty four types
$5_2, 6_1, 7_2, 7_3, 7_5, 8_1, 8_4, 8_6, 8_8, 8_{11}, 8_{13}, 8_{14}, 8_{15}, 9_3, 9_4, 9_5, 9_6, 9_7, 9_8, 9_9, 9_{12}, 9_{14}, 9_{15}, 9_{16}, 9_{19}, 9_{21}, 9_{25}, 9_{35}, 9_{37}, 9_{38}, 9_{39}, 9_{41}, 9_{46}, 9_{49}$, for any g, or to any of the seven types $7_4, 8_3, 9_2, 9_{10}, 9_{13}, 9_{18}, 9_{23}$, for any $g \geq 3$. It is also sufficient to show, for example, that $\Delta(\tau)$ cannot, for any g, be a reducible quadratic. For $\Delta(1)=\pm 1$ and $\Delta(1/\tau)=\tau^{2r}\Delta(\tau)$ [8] show that a reducible quadratic $\Delta(\tau)$ must be of the form

$$\Delta(\tau) = ((h+1)\,\tau-h)\,(h\tau-(h+1)) ;$$

but, of course, for no integer $h \neq 0, -1$ is $h(h+1)$ ever a power.

§2. Since[1] $\pi(S-L)$ has a presentation in which the number of

---

1) *If M is any triangulated closed 3-dimensional manifold and K is a 1-dimensional subcomplex which has μ components and whose 1-dimensional betti number is p, then $\pi(M-K)$ has a presentation in which the number of generators exceeds the number of relators by $p-\mu+1$.*

Proof: In K select p (open) edges $\sigma_1, \cdots, \sigma_p$ such that $K-(\sigma_1+\cdots+\sigma_p)$ is a tree T^* and let T be a maximal tree that contains T^*. In T^* select $\mu-1$ edges $\tau_1, \cdots, \tau_{\mu-1}$ such that each of the μ components of $T'=T-(\tau_1+\cdots+\tau_{\mu-1})$ contains a component of K. Note that $\pi(M-K)\approx\pi(M-(K+T'))$.

Let C be a maximal cave in M, i.e. the dual of a maximal tree in the dual triangulation. Since the Euler characteristic of M is equal to zero, the number α_1 of edges of M that are not on T is equal to the number α_2 of faces of M that are not in C.

The group $\pi(M-(K+T'))$ has a presentation $(x_1, \cdots, x_n : r_1, \cdots, r_m)$ in which the generators x_j correspond to the faces of M that are not in C, and the relators r_i correspond to the edges that are not in $K+T'$. Thus $n=\alpha_2=\alpha_1$ and $m=\alpha_1+(\mu-1)-p=n+\mu-1-p$.

generators exceeds the number of relations by one, the first elementary ideal is the principal ideal [5] generated by $\pm t^r D(t)$. Since $H(S-L)$ is infinite cyclic, $D(1)=\pm 1$. Therefore it follows from a theorem of Blanchfield [9] that $t^u D(1/t)=\varepsilon D(t)$ where u is some integer and $\varepsilon=\pm 1$. Since $\Delta(t)$ is a knot polynomial its degree is even, say $2r$. But $D(t)$ must have the same degree. If we write $D(t)=c\,\Pi_{i=1}^{2r}(t-\alpha_i)$ we see that u must equal $2r$. Thus

$$c\,\Pi_{i=1}^{2r}(1-\alpha_i t)=\varepsilon c\,\Pi_{i=1}^{2r}(t-\alpha_i)\,.$$

Denoting by σ_k the k-th symmetric function of the roots $\alpha_1,\cdots,\alpha_{2r}$ we get (since $c\neq 0$)

$$1-\sigma_1 t+\sigma_2 t^2-\cdots+\sigma_{2r}t^{2r}=\varepsilon(\sigma_{2r}-\sigma_{2r-1}t+\cdots-\sigma_1 t^{2r-1}+t^{2r})\,,$$

whence

$$\sigma_{2r}=\varepsilon\,,\quad \sigma_{2r-1}=\varepsilon\sigma_1\,,\cdots,\quad \sigma_{r+1}=\varepsilon\sigma_{r-1}\,,\quad \sigma_r=\varepsilon\sigma_r\,.$$

Hence

$$\begin{aligned} D(1)&=c(1-\sigma_1+\sigma_2-\cdots+\sigma_{2r})\\ &=c(1+\varepsilon)(1-\sigma_1+\sigma_2-\cdots+(-1)^{r-1}\sigma_{r-1})-(-1)^r c\sigma_r=\pm 1\,,\end{aligned}$$

so that

$$(-1)^r c\sigma_r\equiv 1 \pmod 2\quad \text{hence } \sigma_r\neq 0\,.$$

It follows now from $\sigma_r=\varepsilon\sigma_r$ that $\varepsilon=1$. Therefore $t^{2r}D(1/t)=D(t)$. Since $D(1)=\pm 1$, it follows [8] that there must be some knot in ordinary 3-space whose Alexander polynomial is $D(t)$. Thus $D(t)$ *is a knot polynomial.*

In the proof of Kinoshita's theorem [4] the Poincaré conjecture is invoked solely for the purpose of showing that the polynomial $D(t)$ is symmetric, i.e. $t^{2r}D(1/t)=D(t)$. Accordingly the above shows that it is not necessary to assume Poincaré's conjecture in that argument.

§ 3. Let the d-th cyclotomic polynomial be denoted by $\Phi_d(t)$. Its degree is $\phi(d)$ and its roots are the primitive d-th roots of unity. $\Delta(\tau)$ has a unique factorization of the form

$$\Delta(\tau)=\Phi_{a_1}^{m(a_1)}(\tau)\cdots\Phi_{a_r}^{m(a_r)}(\tau)\cdot\Psi(\tau)\,,$$

where $a_1,\cdots,a_r$ are distinct from one another, and no root of $\Psi(\tau)$ is a root of unity.

Theorem. *If p is any prime divisor of (g, a_i) then $m(a_i)$ must be divisible by the highest power of p that divides g.*

Proof. Suppose that $\Phi_d(t)=\Pi_{j=1}^{\phi(d)}(t-\omega_j)$ divides $D(t)$. Then the corresponding factor of $\Delta(\tau)$ is $\Pi_{j=1}^{\phi(d)}(\tau-\omega_j^g)$. Since each ω_j is a primitive d-th root of unity, each ω_j^g is a primitive d'-th root of unity, where $d'=d/(g,d)$. (For $\omega_j^{gx}=1$ iff $d|gx$, i.e. iff $d'|g'x$, where $g'=g/(g,d)$; since $(d',g')=1$ this holds iff $d'|x$.) Therefore

$$\Pi_{j=1}^{\phi(d)}(\tau-\omega_j^g)=\Phi_a^{n(d)}(\tau)$$

Furthermore, since $\Pi_{j=1}^{\phi(d)}(\tau-\omega_j^g)$ is of degree $\phi(d)$ and $\Phi_{d'}(\tau)$ is of degree $\phi(d')$, we have

$$n(d)=\phi(d)/\phi(d')\,.$$

Clearly the exponent $m(a)$ of the factor $\Phi_a^{m(a)}(\tau)$ of $\Delta(\tau)$ must therefore be of the form $\sum v_d n(d)$, where v_d are integers and the sum is extended over those integers d for which $d'=a$.

Let p range over the prime divisors of a, and let q range over the remaining primes. Write $a=\Pi p^\alpha$, $d=\Pi p^\delta\,\Pi q^\beta$ and $g=\Pi p^\gamma\,\Pi q^\theta$. If $d'=a$ then we must have $0<\alpha=\gamma<\delta$ and $\beta\leq\theta$. It follows that $d/d'=(g,d)=\Pi p^{\min(\delta,\gamma)}\,\Pi q^{\min(\beta,\theta)}$ must be divisible by p^γ, and hence that p^γ must divide $n(d)=\phi(d)/\phi(d')=d\,\Pi\left(1-\frac{1}{p}\right)\Pi_{\beta>0}\left(1-\frac{1}{q}\right)\Big/ d'\,\Pi\left(1-\frac{1}{p}\right)$. Since p^γ devides $n(d)$ for each d for which $d'=a$, it follows that p^γ divides $m(a)$.

§4. **Corollary 1.** *If g and AB are not relatively prime then Λ cannot be a torus knot of type A, B.*

Proof. If Λ is a torus knot of type A, B then

$$\Delta(\tau)=\frac{(\tau^{AB}-1)(\tau-1)}{(\tau^A-1)(\tau^B-1)}=\Phi_{AB}(\tau)\cdots$$

so that $m(AB)=1$. It follows from the theorem that if p is any prime divisor of (g, AB) then $p\not| g$; consequently $(g, AB)=1$.

Corollary 2. (Montgomery-Samelson) *If $g=2$, Λ cannot be the boundary of a twisted*[2] *Möbius band.*

Proof. If κ is the knot type of a Möbius band then [10] $\Delta(\tau)=\Delta_\kappa(\tau^2)\cdot\frac{(\tau^2-1)(\tau^B-1)}{(\tau^{2B}-1)(\tau-1)}$, where Δ_κ is the polynomial of κ. Let q be any prime divisor of B. Since $(2,B)=1$, $q>2$. Thus

2) The *twist* ρ of Möbius band in 3-space was defined in [11] to be $\varepsilon\cdot v(k,l)$, where k is the boundary of the Möbius band, l is its meridian, v denotes linking number, and $\varepsilon=\pm1$. By a *twisted* Möbius band I mean one for which $|\rho|>1$.

$$\frac{(\tau^{2B}-1)(\tau-1)}{(\tau^2-1)(\tau^B-1)} = \Phi_{2q}(\tau) \cdots$$

Now if $\Phi_d(\tau)$ divides $\Delta_\kappa(\tau)$ the corresponding factor of $\Delta_\kappa(\tau^2)$ must be

$$\begin{aligned} \Phi_d(\tau^2) &= \Phi_{2d}(\tau) && \text{if } d \text{ is even} \\ &= \Phi_{2d}(\tau)\,\Phi_d(\tau) && \text{if } d \text{ is odd.} \end{aligned}$$

Hence $\Phi_{2q}(\tau) \mid \Delta_\kappa(\tau^2)$ only if $\Phi_q(\tau) \mid \Delta_\kappa(\tau)$.

But this last is impossible because $\Delta_\kappa(1) = \pm 1$ and $\Phi_q(1) = q$. Thus we conclude that

$$\Delta(\tau) = \Phi_{2q}(\tau) \cdots,$$

i.e. that $m(2q) = 1$. Therefore, by the theorem, 1 must be divisable by the highest power of $p=2$ that divides 2. This is impossible.

(Received February 5, 1958)

References

[1] Bing, R. H.: Decomposition of E^3 into points and tame arcs. Summary of lectures and seminars at the summer institute on set-theoretic topology Madison (1955), 40–47.

[2] Montgomery, D. and Samelson, H.: A theorem on fixed points of involutions in S^3. Canadian J. Math. **7** (1955), 208–220.

[3] Papakyriakopoulos, C. D.: On Dehn's lemma and the asphericity of knots. Annals of Math. **66** (1957), 1–26.

[4] Kinoshita, S.: Notes on knots and periodic transformations. Proc. Japan Acad. **33** (1957), 358–362.

[5] Fox, R. H.: Free differential calculus II. Annals of Math. **59** (1954), 196–210.

[6] Fox, R. H.: Free differential calculus III. Annals of Math. **64** (1956), 407–419.

[7] Smith, P. A.: Transformations of finite period II. Annals of Math. **40** (1939), 690–711.

[8] Seifert, H.: Über das Geschlecht von Knoten. Math. Ann. **110** (1934), 571–592.

[9] Blanchfield, R. C.: Intersection theory of manifolds with operators with applications to knot theory. Annals of Math. **65** (1957), 340–356.

[10] Reidemeister, K.: Knotentheorie, Berlin (1932).

[11] Kyle, R. H.: Embeddings of Möbius bands in 3-dimensional space. Proc. Royal Irish Acad. **57** section A No. 7 (1935), 131–136.