Fujikawa, E. and Matsuzaki, K.
Osaka J. Math.
44 (2007), 173–185

NON-STATIONARY AND DISCONTINUOUS QUASICONFORMAL MAPPING CLASS GROUPS

EGE FUJIKAWA and KATSUHIKO MATSUZAKI

(Received October 20, 2005, revised April 26, 2006)

Abstract

Every stationary subgroup of the quasiconformal mapping class group of a Riemann surface acts on the Teichmüller space discontinuously if the surface satisfies a certain geometric condition. In this paper, we construct such a Riemann surface that the quasiconformal mapping class group is non-stationary but it still acts on the Teichmüller space discontinuously.

1. Introduction and statement of results

For a Riemann surface R of analytically infinite type whose Teichmüller space $T(R)$ is infinite dimensional, the action of the quasiconformal mapping class group $\mathrm{MCG}(R)$ on $T(R)$ is not discontinuous in general. However, we have shown in [9] that certain subgroups of $\mathrm{MCG}(R)$ act on $T(R)$ discontinuously. For example, under certain geometric conditions on R, a subgroup $G_c(R)$ of all quasiconformal mapping classes that preserve the free homotopy class of a simple close geodesic c acts on $T(R)$ discontinuously. Also we have shown in [8] that the eventually trivial mapping class group $E(R)$ acts on $T(R)$ discontinuously as well as the pure mapping class group $P(R)$. These subgroups have a common property: they are stationary.

DEFINITION 1.1. A subgroup G of $\mathrm{MCG}(R)$ is said to be *stationary* if there exists a compact subsurface W of R such that $g(W) \cap W \neq \emptyset$ for every representative g of every element of G. An element $[g] \in \mathrm{MCG}(R)$ is said to be stationary if the cyclic group generated by $[g]$ is stationary.

REMARK 1.2. There exists a subgroup $G \subset \mathrm{MCG}(R)$ such that each element of G is stationary but G is not stationary. Indeed, there exists an abstract countable infinite group Γ such that every element of Γ is of finite order, and for any countable group Γ, there exists a Riemann surface R such that the group $\mathrm{Conf}(R)$ of all conformal automorphisms of R contains a subgroup G isomorphic to Γ. Then we may regard G as a subgroup of $\mathrm{MCG}(R)$. Every element $[g] \in G$ is stationary since it is of finite order. On the other hand, G is not stationary since $\mathrm{Conf}(R)$ acts on R properly discontinuously.

2000 Mathematics Subject Classification. Primary 30F60; Secondary 37F30.

Actually, for stationary subgroups in general, we know the following result. The lower and upper bound conditions are defined later in Section 2.

Proposition 1.3. [6, Theorem 4.8] *Let R be a hyperbolic Riemann surface satisfying the lower and upper bound conditions and having no ideal boundary at infinity. Then every stationary subgroup of* $\mathrm{MCG}(R)$ *acts on* $T(R)$ *discontinuously.*

On the other hand, we did not know any example of a non-stationary subgroup that acts discontinuously, not to say the whole quasiconformal mapping class group $\mathrm{MCG}(R)$. In fact, if the genus of R is positive finite or the number of the punctures of R is positive finite, then $\mathrm{MCG}(R)$ must be stationary (see [9, Theorem 2]). Furthermore, a countable quasiconformal mapping class group constructed in [10] acts discontinuously but it is also stationary as is seen in Section 5.

In Section 3, we first give an easy example of a Riemann surface R such that a non-stationary cyclic subgroup G of $\mathrm{MCG}(R)$ acts on $T(R)$ discontinuously. Actually, this argument tells us certain obstruction for making our desired Riemann surfaces. Then in Section 4, we prove the following, which is the main result of this paper.

Theorem 1.4. *There exists a Riemann surface R such that the whole quasiconformal mapping class group* $\mathrm{MCG}(R)$ *is non-stationary but acts on* $T(R)$ *discontinuously.*

The existence of non-stationary and discontinuous quasiconformal mapping class groups is crucial for the theory of dynamics on infinite dimensional Teichmüller spaces because it requires further investigations completely different from those in the finite dimensional cases.

2. Preliminaries

Throughout this paper, we assume that a Riemann surface R is hyperbolic. Namely, the universal covering surface of R is the upper half-plane $\mathbb{H}$ that admits the hyperbolic metric. We denote the hyperbolic length of an arc c on R by $l(c)$. We say that R satisfies the *lower bound condition* if the injectivity radius at every point of R except cusp neighborhoods is uniformly bounded away from zero, and R satisfies the *upper bound condition* if there exists a subdomain $\check{R}$ of R such that the injectivity radius at every point of $\check{R}$ is uniformly bounded from above and that the simple closed curves in $\check{R}$ carry the fundamental group of R. If the injectivity radius at any point of R is uniformly bounded from above, then clearly R satisfies the upper bound condition. The lower and upper bound conditions are invariant under quasiconformal deformations. For a non-trivial and non-cuspidal simple closed curve c on R, we denote the simple closed geodesic that is freely homotopic to c by c_*.

The *Teichmüller space* $T(R)$ is the set of all equivalence classes $[f]$ of quasiconformal homeomorphisms f on R. Here we say that two quasiconformal homeomorphisms f_1 and f_2 on R are *equivalent* if there exists a conformal homeomorphism $h: f_1(R) \to f_2(R)$ such that $f_2^{-1} \circ h \circ f_1$ is homotopic to the identity by a homotopy that keeps every point of the ideal boundary at infinity fixed throughout. The distance between two points $[f_1]$ and $[f_2]$ in $T(R)$ is defined by $d([f_1],[f_2]) = (1/2) \log K(f)$, where f is an extremal quasiconformal homeomorphism in the sense that its maximal dilatation $K(f)$ is minimal in the homotopy class of $f_2 \circ f_1^{-1}$. Then d is a complete metric on $T(R)$, which is called the Teichmüller distance.

The *quasiconformal mapping class* $[g]$ is a homotopy class of quasiconformal automorphisms g of a Riemann surface R, and the *quasiconformal mapping class group* $\mathrm{MCG}(R)$ is the group of all quasiconformal mapping classes on R. Here we also consider homotopy classes relative to the ideal boundary at infinity. A mapping class $[g]$ is said to be *eventually trivial* if there exists a compact subsurface $V_g \subset R$ such that, for each connected component W of $R - V_g$ that is not a cusp neighborhood, the restriction $g|_W: W \to R$ is homotopic to the inclusion map $\mathrm{id}|_W: W \to R$. The *eventually trivial mapping class group* $E(R)$ of R is the group of all eventually trivial mapping classes on R. Furthermore the *pure mapping class group* $P(R)$ of R is the group of mapping classes $[g]$ such that g fix all non-cuspidal ends of R.

Every element $[g] \in \mathrm{MCG}(R)$ induces a biholomorphic automorphism $[g]_*$ of $T(R)$ by $[f] \mapsto [f \circ g^{-1}]$, which is also an isometry with respect to the Teichmüller distance. Let $\mathrm{Aut}(T(R))$ be the group of all biholomorphic automorphisms of $T(R)$. Then we have a homomorphism $\iota: \mathrm{MCG}(R) \to \mathrm{Aut}(T(R))$ by $[g] \mapsto [g]_*$ and define the Teichmüller modular group by $\mathrm{Mod}(R) = \iota(\mathrm{MCG}(R))$. It is known that the homomorphism ι is injective except for a few low dimensional cases. Thus we may identify $\mathrm{Mod}(R)$ with $\mathrm{MCG}(R)$.

We say that a subgroup $G \subset \mathrm{MCG}(R)$ acts at a point $p \in T(R)$ *discontinuously* if there exists a neighborhood U of p such that the number of elements $[g] \in G$ satisfying $[g]_*(U) \cap U \neq \emptyset$ is finite. This is equivalent to that there exist no distinct elements $[g_n] \in G$ such that $d([g_n]_*(p), p) \to 0$ as $n \to \infty$ (see [5]). We say that G acts on $T(R)$ discontinuously if G acts at every point in $T(R)$ discontinuously. If R has the ideal boundary at infinity, then the action of $\mathrm{MCG}(R)$ is discontinuous at no points of $T(R)$.

3. A Riemann surface with length parameters

In this section, we construct a Riemann surface R from pairs of pants whose quasiconformal mapping class group $\mathrm{MCG}(R)$ has a cyclic non-stationary subgroup G that acts on $T(R)$ discontinuously. Although this property itself is weaker than that of the Riemann surface as in Theorem 1.4, the surface in Proposition 3.1 below has the advantage of flexibility: by changing length parameters, we have quasiconformal mapping classes of various types as is explained in Remark 3.4 below.

Hereafter, $P(l_1, l_2, l_3)$ denotes a pair of pants whose geodesic boundary components have the hyperbolic lengths l_1, l_2 and l_3. We allow the case $l_i = 0$, which means that the boundary component degenerates into a puncture. A pair of pants P has three symmetry axes, which are the shortest geodesic arcs connecting two boundary components and which divide P into two congruent polygons.

First we make a surface S with indefinite parameters $\{l_i\}_{i\in\mathbb{Z}}$ as follows. For every $i \in \mathbb{Z}$, we take two pairs of pants $P_i^- = P_i^-(l_i, 1, 1)$ and $P_i^+ = P_i^+(l_i, 1, 1)$ with geodesic boundary components (a_i^-, b_i^-, c_i^-) and (a_i^+, b_i^+, c_i^+) respectively. Let $\alpha_i^\pm$ be the symmetry axis of $P_i^\pm$ connecting $b_i^\pm$ and $c_i^\pm$. Similarly, $\beta_i^\pm$ is the symmetry axis connecting $c_i^\pm$ and $a_i^\pm$, and $\gamma_i^\pm$ is the one connecting $a_i^\pm$ and $b_i^\pm$. We give an orientation to each boundary component of $P_i^\pm$ counterclockwise when we view from the inside of $P_i^\pm$. Furthermore we parametrize each boundary component of $P_i^\pm$ by a normalized arc length parameter θ $(0 \le \theta \le 1)$ with respect to the hyperbolic metric (that is, the normalization means the variation of the paramter is one) such that $a_i^\pm(0) = a_i^\pm(1) \in \gamma_i^\pm$, $b_i^\pm(0) = b_i^\pm(1) \in \alpha_i^\pm$ and $c_i^\pm(0) = c_i^\pm(1) \in \beta_i^\pm$.

We glue P_i^- and P_i^+ by identifying $a_i^-(\theta)$ with $a_i^+(1-\theta)$ and $b_i^-(\theta)$ with $b_i^+(1-\theta)$ for all θ. Then we obtain a torus A_i with two geodesic boundary components c_i^- and c_i^+ having $a_i^-(\theta) = a_i^+(1-\theta)$ and $b_i^-(\theta) = b_i^+(1-\theta)$ as simple closed geodesics a_i and b_i in it. Note that $l(b_i) = 1$ for all i, but $l(a_i) = l_i$ are indefinite. Furthermore, for each $i \in \mathbb{Z}$, we glue A_i and A_{i+1} by identifying $c_i^+(\theta)$ with $c_{i+1}^-(1-\theta)$ for all θ. Then the resulting surface of infinite genus is denoted by S, which is our Riemann surface of indefinite parameters $\{l_i\}_{i\in\mathbb{Z}}$.

Assume here that all the parameters l_i are the same. Then this surface admits a conformal automorphism g determined by a translation such that $g(A_i) = A_{i+1}$ for all i. We consider this particular mapping class $[g]$ of S hereafter.

After the preparation of those notations, we can state the example of our Riemann surface as follows.

Proposition 3.1. *Let R be a Riemann surface obtained by taking the lengths $\{l_i\}_{i\in\mathbb{Z}}$ of S so that $l_i \to 0$ as $i \to \pm\infty$ and that $1/e^2 \le l_i/l_{i+1} \le e^2$ for every i. Then the mapping class $[g]$ of R belongs to* $\mathrm{MCG}(R)$ *and the cyclic non-stationary subgroup G generated by $[g]$ acts on $T(R)$ discontinuously.*

The following two lemmas, which give certain estimates of the maximal dilatations of quasiconformal homeomorphisms, will be used in the proofs of our statements here and later.

Lemma 3.2 ([2]). *Let $P = P(l_1, l_2, l_3)$ and $P' = P'(l_1', l_2, l_3)$ be pairs of pants (possibly degenerate) with $\max\{l_1, l_1', l_2, l_3\} \le L$. Suppose that $\varepsilon := |\log(l_1/l_1')| \le 2$. Then there exists a quasiconformal homeomorphism $\psi\colon P \to P'$ preserving the symmetry axes such that $K(\psi) \le 1 + C\varepsilon$ for a constant $C = C(L)$ depending only on L*

and that it is identical on each boundary component with respect to the normalized arc length parameter.

Lemma 3.3 ([12], [13]). *Let c be a simple closed geodesic on a Riemann surface R with the hyperbolic length $l(c)$ and $f : R \to R'$ a quasiconformal homeomorphism of R onto another Riemann surface R'. Then the hyperbolic length $l(f(c)_*)$ of the geodesic $f(c)_*$ satisfies*

$$\frac{1}{K(f)} l(c) \leq l(f(c)_*) \leq K(f) l(c).$$

Proof of Proposition 3.1. By applying Lemma 3.2 to each pair of pants, we see that there exists a quasiconformal automorphism h of R in the mapping class $[g]$ such that $K(h|_{A_i}) \leq 1 + C(L_i)\varepsilon_i$ on A_i. Here $\varepsilon_i = |\log(l_i/l_{i+1})| \leq 2$ and $L_i = \max\{l_i, l_{i+1}, 1\}$ for every i. Hence the mapping class $[g]$ belongs to $\mathrm{MCG}(R)$.

We will prove that G acts on $T(R)$ discontinuously. First we show that G acts at the base point $o = [\mathrm{id}] \in T(R)$ discontinuously. Suppose to the contrary that there exists a subsequence $\{[g^{n_k}]\}$ such that $d([g^{n_k}]_*(o), o) \to 0$ as $k \to \infty$. Then there exist representatives h_k in the mapping classes $[g^{n_k}]$ such that $K(h_k) \to 1$ as $k \to \infty$. However, since $h_k(a_0)$ is freely homotopic to a_{n_k}, we have $K(h_k) \geq l_0/l_{n_k} \to \infty$ by Lemma 3.3. This is a contradiction.

Next consider an arbitrary point $p = [f] \in T(R)$, where f is a quasiconformal homeomorphism of R. Then, again by Lemma 3.3, the simple closed geodesics $f(a_n)_*$ on $f(R)$ satisfy $l(f(a_n)_*) \to 0$ as $n \to \pm\infty$. Then by the same consideration, we see that G acts at $p \in T(R)$ discontinuously. □

REMARK 3.4. In Proposition 3.1, we can choose the parameters of R as $l_i = l_{-i} = 1/2^i$ for $i \geq 0$. Then the quasiconformal mapping class $[g]$ is not asymptotically comformal. Indeed, since $|\log(l_i/l_{i+1})| = \log 2$ for every i, Theorem 3.6 in [7] yields the assertion. For the definifion of asymptotically comformal homeomorphisms, see [4]. Also, we can set $l_0 = 1$ and $l_i = l_{-i} = 1/i$ for $i \geq 1$ as well. In this case, the quasiconformal mapping class $[g]$ is asymptotically comformal. Indeed, by applying Lemma 3.2 as in the proof of Proposition 3.1, there exists a quasiconformal automorphism h in the mapping class $[g]$ such that $K(h|_{A_i}) \leq 1 + C(1)\varepsilon_i$ on A_i, where $\varepsilon_i = |\log(l_i/l_{i+1})| = |\log((i+1)/i)| \to 0$ as $i \to \infty$.

REMARK 3.5. Let R_1 be a Riemann surface such that the parameters l_i on S are bounded from above and away from zero. Then $G = \langle [g] \rangle \subset \mathrm{MCG}(R_1)$ does not act on $T(R_1)$ discontinuously. Indeed, let R_0 be a Riemann surface with $l_i = 1$ for all $i \in \mathbb{Z}$. Then R_1 is a quasiconformal deformation of R_0 and hence $T(R_1) = T(R_0)$. On the Riemann surface R_0, the mapping class $[g]$ has a conformal representative. Then G does not act discontinuously at $o = [\mathrm{id}] \in T(R_0)$.

4. Proof of main theorem

In this section, we will prove Theorem 1.4. If a Riemann surface R has a sequence of simple closed geodesics whose hyperbolic lengths tend to 0, namely, if R dose not satisfy the lower bound condition, then the action of $\mathrm{MCG}(R)$ on $T(R)$ is not discontinuous (see [5, Theorem 1]). In particular, the Riemann surface as in Proposition 3.1 is not appropriate for Theorem 1.4. The Riemann surface as in Remark 3.5 is not appropriate either by the reason explained there.

Proof of Theorem 1.4. First we define a sequence $\{l_n\}_{n\in\mathbb{N}}$ of positive numbers as follows. Fix a constant $K > 1$ once and for all. Set $l_1 = 1$ and take a degenerate pair of pants $P_1 = P(l_1, l_1, 0)$. Let $\hat{\eta}$ be the supremum of $\eta \le l_1 + 1$ such that there exist K-quasiconformal homeomorphisms $\varphi\colon P_1 \to P(l_1, \eta, 0)$ and $\varphi'\colon P_1 \to P(\eta, \eta, 0)$ that preserve the symmetry axes and that are identical on each boundary component with respect to the normalized arc length parameter. Then we set $l_2 = \hat{\eta}$.

Here the above supremum is actually attained. Indeed, we take a sequence $\{\eta_j\}$ converging to $\hat{\eta}$ such that there exist K-quasiconformal homeomorphisms $\varphi_j\colon P_1 \to P(l_1, \eta_j, 0)$ and $\varphi'_j\colon P_1 \to P(\eta_j, \eta_j, 0)$. It is enough to consider φ_j and φ'_j on the symmetric half D_1 of P_1 and we may assume that their images $\varphi_j(D_1) = D(l_1, \eta_j, 0)$ and $\varphi'_j(D_1) = D(\eta_j, \eta_j, 0)$ are embedded in the hyperbolic plane $\mathbb{H}$ in such a way that $D(l_1, \eta_j, 0)$ and $D(\eta_j, \eta_j, 0)$ converge to pentagons $D(l_1, \hat{\eta}, 0)$ and $D(\hat{\eta}, \hat{\eta}, 0)$ respectively in the sense of Carathéodory. Then φ_j and φ'_j converge to K-quasiconformal homeomorphisms φ_∞ and φ'_∞ respectively (see [11, Theorem 5.2]). Moreover, by an application of the Carathéodory convergence theorem (see [3, Theorem 3.1]), their images $\varphi_\infty(D_1)$ and $\varphi'_\infty(D_1)$ are coincident with $D\big(l_1, \hat{\eta}, 0\big)$ and $D\big(\hat{\eta}, \hat{\eta}, 0\big)$ respectively and they are affine on the two sides of D_1 with respect to the hyperbolic metric. This implies that φ_∞ and φ'_∞ attain the supremum.

Assuming that l_n has been determined, we define l_{n+1} as follows. For a degenerate pair of pants $P_n = P(l_n, l_n, 0)$, let l_{n+1} be the supremum of $\eta \le l_n + 1$ (which is actually the maximum by the same reason as above) such that there exist K-quasiconformal homeomorphisms $\varphi\colon P_n \to P(l_n, \eta, 0)$ and $\varphi'\colon P_n \to P(\eta, \eta, 0)$ that preserve the symmetry axes and that are identical on each boundary component with respect to the normalized arc length parameter. In this way, we have l_n for $n \ge 1$ inductively.

Next we prove that $l_n \to \infty$. Suppose to the contrary that $\sup l_n =: \hat{l} < \infty$. Let $C(\,\cdot\,)$ be the constant as in Lemma 3.2. Since $\sup l_n = \hat{l}$, we can take an integer n such that

$$l_n \ge \max\left[\hat{l}\cdot\exp\left(-\frac{K^{1/4}-1}{C\big(\hat{l}\big)}\right), \hat{l} - \frac{1}{2}\right].$$

Then by Lemma 3.2, there exist $K^{1/4}$-quasiconformal homeomorphisms between pairs of pants $P(l_n, l_n, 0)$ and $P\big(l_n, \hat{l}, 0\big)$, and between pairs of pants $P\big(l_n, \hat{l}, 0\big)$ and $P\big(\hat{l}, \hat{l}, 0\big)$.

Furthermore, take a constant $\mu > \hat{l}$ such that

$$\mu \leq \min\left[\hat{l} \cdot \exp\left(\frac{K^{1/4}-1}{C\left(\hat{l}+1/2\right)}\right), \hat{l}+\frac{1}{2}\right].$$

Then by Lemma 3.2, there exist $K^{1/4}$-quasiconformal homeomorphisms between $P\left(\hat{l},\hat{l},0\right)$ and $P\left(\hat{l},\mu,0\right)$, and between $P\left(\hat{l},\mu,0\right)$ and $P(\mu,\mu,0)$. By composing the four $K^{1/4}$-quasiconformal homeomorphisms, we obtain a K-quasiconformal homeomorphism between $P(l_n,l_n,0)$ and $P(\mu,\mu,0)$. Also, since there is a $K^{1/4}$-quasiconformal homeomorphism between $P\left(l_n,\hat{l},0\right)$ and $P(l_n,\mu,0)$, we have a $K^{1/2}$-quasiconformal homeomorphism between $P(l_n,l_n,0)$ and $P(l_n,\mu,0)$. Remark that $\mu - l_n \leq 1$. Since $\mu > \hat{l}$, they contradict the definition of l_{n+1}.

Now we construct the desired Riemann surface R. For each $i \in \mathbb{Z} - \{0\}$, we take degenerate pairs of pants $A_i = P(l_{|i|}, l_{|i|}, 0)$ with geodesic boundary components (a_i^-, a_i^+, x_i) and $B_i = P(l_{|i|}, l_{|i|+1}, 0)$ with geodesic boundary components (b_i^-, b_i^+, y_i). Here x_i and y_i are punctures. For $i = 0$, we set $B_0 = P(1,1,0)$ (namely $l_0 = 1$) with geodesic boundary components (b_0^+, b_0^+, y_0) (the two components have the same name).

Let $s_i^\pm \subset A_i$ be the symmetry axis connecting $a_i^\pm$ with x_i, and let $t_i^\pm \subset B_i$ be the symmetry axis connecting $b_i^\pm$ with y_i. We parametrize the boundary components of A_i and B_i counterclockwise by a normalized arc length parameter θ $(0 \leq \theta \leq 1)$ with respect to the hyperbolic metric such that $a_i^\pm(0) = a_i^\pm(1) \in s_i^\pm$ and $b_i^\pm(0) = b_i^\pm(1) \in t_i^\pm$.

We glue B_0 and A_1 by identifying one $b_0^+(\theta)$ with $a_1^-(1-\theta)$, and glue B_0 and A_{-1} by identifying the other $b_0^+(\theta)$ with $a_{-1}^-(1-\theta)$. For each $i \geq 1$, we glue A_i and B_i by identifying $a_i^+(\theta)$ with $b_i^-(1-\theta)$, and glue B_i and A_{i+1} by identifying $b_i^+(\theta)$ with $a_{i+1}^-(1-\theta)$. Also for each $i \leq -1$, we glue A_i and B_i by identifying $a_i^+(\theta)$ with $b_i^-(1-\theta)$, and glue B_i and A_{i-1} by identifying $b_i^+(\theta)$ with $a_{i-1}^-(1-\theta)$. In this manner, we obtain a planar Riemann surface R.

Let v be the geodesic line consisting of all the symmetry axes of A_i and B_i other than $s_i^\pm$ and $t_i^\pm$. If the hyperbolic length of v is infinite, then R has no ideal boundary at infinity. Otherwise, we reconstruct R as follows. For each i, we prepare more than $1/l(v \cap A_i)$ copies of A_i and glue them in the same way as above to obtain $\tilde{A}_i$ whose boundary components other than punctures are more than one apart in the hyperbolic distance. Then, replacing A_i with $\tilde{A}_i$, we make R. In this sense, we may assume that the Riemann surface R constructed above has no ideal boundary at infinity.

The union of the symmetry axes $s_i^+ \cup t_i^-$ $(i \neq 0)$ makes a geodesic line connecting the punctures x_i with y_i. Similarly, $t_{i-1}^+ \cup s_i^-$ $(i \geq 1)$ or $t_i^+ \cup s_{i-1}^-$ $(i \leq 0)$ makes a geodesic line connecting y_{i-1} with x_i $(i \geq 1)$ or y_i with x_{i-1} $(i \leq 0)$. All these geodesic lines together with v divide the Riemann surface R into the symmetric halves R° and $R^\bullet$, which are simply connected. Also they divide the pair of pants A_i into the symmetric halves $A_i^\circ = A_i \cap R^\circ$ and $A_i^\bullet = A_i \cap R^\bullet$, and divide the pair of pants B_i as well.

The quasiconformal mapping class group $\mathrm{MCG}(R)$ is non-stationary. Indeed, by the definition of the sequence $\{l_n\}$, there exist K^2-quasiconformal homeomorphisms between A_i and B_i ($i \neq 0$), between B_{i-1} and A_i ($i \geq 1$) and between B_i and A_{i-1} ($i \leq 0$). Hence there exists a K^2-quasiconformal automorphism g of R that maps $\{A_i\}$ to $\{B_i\}$. Clearly this mapping class $[g] \in \mathrm{MCG}(R)$ is non-stationary.

Next we will prove that $\mathrm{MCG}(R)$ acts on $T(R)$ discontinuously. To see this, we use the following.

Proposition 4.1. *The Riemann surface R satisfies the lower and upper bound conditions.*

Proof. The hyperbolic distances between geodesic arcs in A_i and B_i ($i \neq 0$) satisfy

$$\cosh d(s_i^+, a_i^-) = \frac{2\cosh(l_{|i|}/2)}{\sinh(l_{|i|}/2)};$$

$$\cosh d(s_i^-, a_i^+) = \frac{2\cosh(l_{|i|}/2)}{\sinh(l_{|i|}/2)};$$

$$\cosh d(t_i^+, b_i^-) = \frac{\cosh(l_{|i|}/2) + \cosh(l_{|i|+1}/2)}{\sinh(l_{|i|}/2)};$$

$$\cosh d(t_i^-, b_i^+) = \frac{\cosh(l_{|i|}/2) + \cosh(l_{|i|+1}/2)}{\sinh(l_{|i|+1}/2)}.$$

These are obtained by the combination of formulae for hyperbolic pentagons (see [1, Theorem 7.18.1]). Since $l_{|i|+1} \leq l_{|i|} + 1$, the above four distances are uniformly bounded from above and away from zero. In fact, we have

$$\limsup_{i \to \pm\infty} \cosh d(t_i^+, b_i^-) \leq 1 + e^{1/2};$$

$$\liminf_{i \to \pm\infty} \cosh d(t_i^-, b_i^+) \geq 1 + e^{-1/2}.$$

First we prove that R satisfies the lower bound condition. We will show that the hyperbolic lengths $l(c)$ of all simple closed geodesics c on R are uniformly bounded away from zero. Take c arbitrarily other than $a_i^{\pm}$ or $b_i^{\pm}$. (Remark that $l(a_i^{\pm}) \geq 1$ and $l(b_i^{\pm}) \geq 1$.) Let i ($\neq 0$) be an integer of the largest absolute value satisfying either $c \cap A_i \neq \emptyset$ or $c \cap B_i \neq \emptyset$. In the case where $c \cap A_i \neq \emptyset$ and $c \cap B_i = \emptyset$, we consider the connected components of $c \cap A_i^{\circ}$ and $c \cap A_i^{\bullet}$, which are simple geodesic arcs. Then at least one of them, say c', connects either s_i^+ with a_i^- or s_i^+ with $v \cap A_i$. Indeed, otherwise both $c \cap A_i^{\circ}$ and $c \cap A_i^{\bullet}$ connect s_i^+ and s_i^-, which means that c surrounds only one puncture x_i. If c' connects s_i^+ with a_i^-, then $l(c') \geq d(s_i^+, a_i^-) \geq \operatorname{arccosh} 2$ by the above formula. If c' connects s_i^+ with $v \cap A_i$, then $l(c') \geq l(a_i^+)/2 \geq 1/2$. In both cases, we have $l(c) \geq 1/2$. Also in the case where $c \cap B_i \neq \emptyset$, we can apply the same argument since $d(t_i^+, b_i^-) \geq \operatorname{arccosh} 2$ and $l(b_i^+)/2 \geq 1/2$. Hence in all cases, we have $l(c) \geq 1/2$ and conclude that R satisfies the lower bound condition.

Next we prove that R satisfies the upper bound condition. We consider a dividing simple closed geodesic ζ_{2i} $(i \neq 0)$ that bounds a doubly-connected domain together with $s_i^+ \cup t_i^-$, which surrounds x_i and y_i. Also we take a simple closed geodesic ζ_{2i-1} surrounding either y_{i-1} and x_i $(i \geq 1)$ or y_i and x_{i-1} $(i \leq 0)$ in the same manner as above. For each integer $m \neq 0$, let Z_m be one of the connected components of $R - \zeta_m$ that contains the two punctures, and set

$$\check{R} = B_0 \cup \bigcup_{m \neq 0} Z_m.$$

The homomorphism $\pi_1(\check{R}) \to \pi_1(R)$ induced by the inclusion map $\check{R} \hookrightarrow R$ is surjective because the connected components of the complement $R - \check{R}$ are simply connected. Hence we have only to show that the injectivity radii of all points in $\check{R}$ are uniformly bounded from above.

We will show that the hyperbolic lengths of ζ_m are uniformly bounded from above. For disjoint geodesic arcs s and a in the simply-connected domain R°, we denote by $e\langle s \to a \rangle \in a$ the end point of the shortest geodesic arc connecting s with a. Then we see that

$$l(\zeta_{2i}) \leq 2\{d(s_i^-, a_i^+) + d(e\langle s_i^- \to a_i^+ \rangle, e\langle t_i^+ \to b_i^- \rangle) + d(t_i^+, b_i^-)\}$$

for example. Hence we have only to estimate the distances between these end points.

By a formula for the Lambert quadrilaterals (see [1, Theorem 7.17.1 (i)]), we have

$$\begin{aligned}
&d(e\langle s_i^- \to a_i^+ \rangle, e\langle t_i^+ \to b_i^- \rangle) \\
&\quad = \operatorname{arcsinh}\left\{\frac{1}{\sinh d(s_i^-, a_i^+)}\right\} - \operatorname{arcsinh}\left\{\frac{1}{\sinh d(t_i^+, b_i^-)}\right\} \quad (i \neq 0); \\
&d(e\langle t_i^- \to b_i^+ \rangle, e\langle s_{i+1}^+ \to a_{i+1}^- \rangle) \\
&\quad = \operatorname{arcsinh}\left\{\frac{1}{\sinh d(t_i^-, b_i^+)}\right\} - \operatorname{arcsinh}\left\{\frac{1}{\sinh d(s_{i+1}^+, a_{i+1}^-)}\right\} \quad (i \geq 1); \\
&d(e\langle t_i^- \to b_i^+ \rangle, e\langle s_{i-1}^+ \to a_{i-1}^- \rangle) \\
&\quad = \operatorname{arcsinh}\left\{\frac{1}{\sinh d(t_i^-, b_i^+)}\right\} - \operatorname{arcsinh}\left\{\frac{1}{\sinh d(s_{i-1}^+, a_{i-1}^-)}\right\} \quad (i \leq 0),
\end{aligned}$$

which are uniformly bounded from above. Hence we conclude that $l(\zeta_m) \leq \delta$ for some constant $\delta > 0$.

Since the hyperbolic area of Z_m is 2π, there exists a constant $r > 0$ independent of m such that the radius of any embedded disk in any Z_m is not greater than

r. This means that, for every $z \in Z_m$, there exists either a non-trivial closed curve passing through z whose length is not greater than $2r$, or an arc connecting z with $\zeta_m = \partial Z_m$ whose length is not greater than r. Hence, for every $z \in Z_m$, there is a non-trivial closed curve passing through z whose length is not greater than $2r + \delta$. Thus we conclude that the injectivity radii of all points of $\check{R}$ are uniformly bounded from above. □

Proof of Theorem 1.4 continued. We prove that $\mathrm{MCG}(R)$ acts on $T(R)$ discontinuously. First we show that $\mathrm{MCG}(R)$ acts at the base point $o = [\mathrm{id}] \in T(R)$ discontinuously. Suppose to the contrary that there is a sequence of distinct elements $[g_n] \in \mathrm{MCG}(R)$ such that $d([g_n]_*(o), o) \to 0$ as $n \to \infty$. If the sequence $\{[g_n]\}$ is stationary, namely, if there exists a compact subsurface W of R such that $g_n(W) \cap W \neq \emptyset$ for every representative g_n for every n, then we have a contradiction by Proposition 1.3 (applied to the sequence instead of a subgroup) and Proposition 4.1. Thus we may assume that the sequence $\{[g_n]\}$ is non-stationary.

Let X_i and Y_i be horocyclic cusp neighborhoods of x_i and y_i respectively whose hyperbolic areas are 1. For $k \geq 1$, set

$$W_k = (B_0 - Y_0) \cup \bigcup_{1 \leq |i| \leq k} \{(A_i - X_i) \cup (B_i - Y_i)\},$$

which is a compact subsurface of R. Then there exist $n_k \in \mathbb{N}$ and a representative $g_{n_k} \in [g_{n_k}]$ such that $g_{n_k}(W_k) \cap W_k = \emptyset$. In particular, $g_{n_k}(c_0)_* \cap W_k = \emptyset$, where $c_0 := b_0^+$ is a geodesic boundary component of B_0 and $g_{n_k}(c_0)_*$ is the simple closed geodesic that is freely homotopic to $g_{n_k}(c_0)$. Without loss of generality, we may assume that $g_{n_k}(c_0)_*$ belongs to $\bigcup_{i=i_k}^{\infty} \{(A_i - X_i) \cup (B_i - Y_i)\}$, where $i_k \geq k + 1$ is the minimum integer satisfying this property. We may also assume that $g_{n_k}(c_0)_*$ is neither $a_i^{\pm}$ nor $b_i^{\pm}$, for if $g_{n_k}(c_0)_*$ is either $a_i^{\pm}$ or $b_i^{\pm}$ then the estimate below is obvious.

First we consider the case where $g_{n_k}(c_0)_* \cap A_{i_k} \neq \emptyset$. The geodesic $g_{n_k}(c_0)_*$ has intersection with $s_{i_k}^-$. Indeed, otherwise, the homotopy class of $g_{n_k}(c_0)$ has a closed curve that is shorter than $g_{n_k}(c_0)_*$. We consider the connected components of $g_{n_k}(c_0)_* \cap R^{\circ}$ and $g_{n_k}(c_0)_* \cap R^{\bullet}$, which are simple geodesic arcs. Then one of these arcs, which is denoted by c_k', connects $s_{i_k}^-$ with v. Indeed, suppose that $g_{n_k}(c_0)_*$ has no intersection with v. Then one connected component of $R - g_{n_k}(c_0)_*$ has only finitely many punctures. However, since c_0 divides R into two connected components both of which have infinitely many punctures and since g_{n_k} is homeomorphic, $g_{n_k}(c_0)_*$ has the same property as c_0. This is a contradiction. Also in the case where $g_{n_k}(c_0)_* \cap B_{i_k} \neq \emptyset$ but $g_{n_k}(c_0)_* \cap A_{i_k} = \emptyset$, by applying the same argument as above, we conclude that one of the simple geodesic arcs $g_{n_k}(c_0)_* \cap R^{\circ}$ and $g_{n_k}(c_0)_* \cap R^{\bullet}$ connects $t_{i_k}^-$ with v.

Here we see that $l(c_k') \geq (1/2)l(a_{i_k}^{\pm})$ since $a_{i_k}^{\pm}$ restricted to R° or $R^{\bullet}$ are shortest geodesic arcs connecting $s_{i_k}^-$ with v and $t_{i_k}^-$ with v. Then we have $l(g_{n_k}(c_0)_*) \geq (1/2)l_{i_k} \to \infty$ as $k \to \infty$. On the other hand, we can choose representatives $g_{n_k}' \in [g_{n_k}]$

such that $K(g'_{n_k}) \to 1$ as $k \to \infty$. However by Lemma 3.3, we have

$$K(g'_{n_k}) \geq \frac{l(g'_{n_k}(c_0)_*)}{l(c_0)} = l(g_{n_k}(c_0)_*),$$

which is a contradiction. Hence we conclude that MCG(R) acts at the base point $o = [\mathrm{id}] \in T(R)$ discontinuously.

For an arbitrary point $p = [f] \in T(R)$, the Riemann surface $f(R)$ satisfies the lower and upper bound conditions and $l(f(a_i^{\pm})_*) = \infty$ as $i \to \pm\infty$ because these properties are quasiconformally invariant. Thus we can apply the same argument as above and conclude that MCG(R) acts at p discontinuously. □

5. A stationary countable mapping class group

In this section, we will prove that MCG(R) is stationary for the Riemann surface R that was constructed in [10]. This surface R has a property that MCG(R) consists only of a countable number of elements, and as a consequence, MCG(R) acts on $T(R)$ discontinuously (see [10, Theorem 1]).

The Riemann surface R was constructed as follows. Set $P_0 = P(1, 1, 1)$ and $P_n = P(n!, (n+1)!, (n+1)!)$ for every integer $n \geq 1$. We denote the geodesic boundary component of length $n!$ in each pair of pants by c_n. We prepare 2^{n+1} copies of P_n for each $n \geq 0$ and glue the geodesic boundary components as follows: We glue the geodesic boundary components c_0 of the 2 copies of P_0 together. The resulting hyperbolic surface with 4 geodesic boundary components c_1 is denoted by R_1. Next we glue the geodesic boundary component c_1 of each copy of P_1 to the 4 boundary components of R_1. The resulting hyperbolic surface with 8 geodesic boundary components c_2 is denoted by R_2. Continuing this process, for every integer $n \geq 1$, we obtain a hyperbolic surface R_n with 2^{n+1} geodesic boundary components c_n which is made of R_{n-1} and 2^n copies of P_{n-1}. Then take the exhaustion of these compact subsurfaces R_n, which is $R = \bigcup_{n=0}^{\infty} R_n$. Each connected component of $R - R_n$ is denoted by E_n. At each step of gluing, we give an appropriate amount of twist along the geodesic boundaries so that R is a complete hyperbolic surface without ideal boundary at infinity. Then R has the following property.

Lemma 5.1 ([10, Theorem 3]). *Let $g\colon R \to R$ be a K-quasiconformal automorphism of the Riemann surface R. Then, on each component E_n of $R - R_n$ for $n \geq \max\{K, 5\}$, the g restricted to E_n is homotopic to a conformal homeomorphism of E_n onto another component of $R - R_n$.*

We will prove the following.

Proposition 5.2. *Let R be the Riemann surface constructed above. Then* MCG(R) *is stationary.*

Proof. Let R_1 be the compact subsurface defined as above. We will prove that $g(R_1) \cap R_1 \neq \emptyset$ for every representatives g of every element $[g] \in \mathrm{MCG}(R)$. Suppose to the contrary that there exists some $[g]$ such that $g(R_1) \cap R_1 = \emptyset$. Let K be the maximal dilatation of g and take an integer n with $n \geq \max\{K, 5\}$. The number of the components E_n of $R - R_n$ is 2^{n+1} and precisely $2^{n+1}/4$ of them belong to each of the four components E_1 of $R - R_1$.

By Lemma 5.1, $[g]$ gives a permutation of the 2^{n+1} components E_n. Since g is homeomorphic, there are $2^{n+1}/4$ components E_n in each of the four components of $R - g(R_1)$. By the assumption that $g(R_1) \cap R_1 = \emptyset$, the image $g(R_1)$ belongs to some E_1. Then we see that there should be at least $3 \cdot 2^{n+1}/4$ components E_n belonging to this E_1. This is a contradiction. Hence we conclude that $g(R_1) \cap R_1 \neq \emptyset$ for every representatives g of every $[g] \in \mathrm{MCG}(R)$, which means that $\mathrm{MCG}(R)$ is stationary. □

References

[1] A.F. Beardon: The Geometry of Discrete Groups, Graduate Texts in Mathematics **91**, Springer, 1983.

[2] C.J. Bishop: *Quasiconformal mapping of Y-pieces*, Rev. Mat. Iberoamericana **18** (2002), 627–652.

[3] P. Duren: Univalent Functions, Grundlehren der Mathematischen Wissenschaften **259**, Springer, 1983.

[4] F.P. Gardiner and N. Lakic: Quasiconformal Teichmüller Theory, Mathematical Surveys and Monographs **76**, Amer. Math. Soc., 2000.

[5] E. Fujikawa: *Limit sets and regions of discontinuity of Teichmüller modular groups*, Proc. Amer. Math. Soc. **132** (2004), 117–126.

[6] E. Fujikawa: *Modular groups acting on infinite dimensional Teichmüller spaces*; in In the tradition of Ahlfors and Bers, III, Contemp. Math. **355**, Amer. Math. Soc., Providence, RI, 2004, 239–253.

[7] E. Fujikawa: *The action of geometric automorphisms of asymptotic Teichmüller spaces*, Michigan Math. J. **54** (2006), 269–282.

[8] E. Fujikawa: *Pure mapping class group acting on Teichmüller space*, preprint.

[9] E. Fujikawa, H. Shiga and M. Taniguchi: *On the action of the mapping class group for Riemann surfaces of infinite type*, J. Math. Soc. Japan **56** (2004), 1069–1086.

[10] K. Matsuzaki: *A countable Teichmüller modular group*, Trans. Amer. Math. Soc. **357** (2005), 3119–3131.

[11] O. Lehto and K.I. Virtanen: Quasiconformal Mappings in the Plane, Grundlehren der Mathematischen Wissenschaften **126**, Springer, 1973.

[12] T. Sorvali: *The boundary mapping induced by an isomorphism of covering groups*, Ann. Acad. Sci. Fennicæ, Series A I **526** (1972).

[13] S.A. Wolpert: *The length spectra as moduli for compact Riemann surfaces*, Ann. of Math. (2) **109** (1979), 323–351.

Ege Fujikawa
Department of Mathematics
Sophia University
7–1 Kioi-cho, Chiyoda-ku
Tokyo 102–8554
Japan
e-mail: fujikawa@mm.sophia.ac.jp

Katsuhiko Matsuzaki
Department of Mathematics
Okayama University
3–1–1 Tsushima-Naka
Okayama 700–8530
Japan
e-mail: matsuzak@math.okayama-u.ac.jp