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WELL-POSEDNESS OF THE GENERALIZED BENJAMIN-ONO-BURGERS EQUATIONS IN SOBOLEV SPACES OF NEGATIVE ORDER

MASANORI OTANI

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Abstract

We study the well-posedness issue of the generalized Benjamin-Ono-Burgers (gBO-B) equations. We solve the initial-value problem (IVP) of the gBO-B equations with data below $L^2(\mathbf{R})$. Our proof is based on the method of L. Molinet and F. Ribaud, which is analogous to that of J. Bourgain, and C.E. Kenig, G. Ponce, and L. Vega. It is known that such a method cannot be applied to the Benjamin-Ono equation.

1. Introduction

In this paper, we are devoted to the well-posedness issue of the initial-value problem (IVP) for the generalized Benjamin-Ono-Burgers (gBO-B) equations

$$\begin{cases} \partial_t u + u\partial_x u - \partial_x |D_x|^{1+a} u + |D_x|^{2\alpha} u = 0, & (x, t) \in \mathbf{R} \times \mathbf{R}_+, \\ u(x, 0) = u_0(x) \in H^s(\mathbf{R}), \end{cases} \tag{1.1}$$

where $a \geq 0$, $\alpha > 0$, and $|D_x|^k$ is the Fourier multiplier operator with symbol $|\xi|^k$. Equation (1.1) is called the KdV-Burgers (KdV-B) equation and the ordinary BO-B equation when $(a, \alpha) = (1, 1)$ and $(a, \alpha) = (0, 1)$ respectively.

In [23], L. Molinet and F. Ribaud showed that the KdV-B equation is globally well-posed for $s > -1$ [23]. Their method of the proof was analogous to that of J. Bourgain [4] and C.E. Kenig, G. Ponce, and L. Vega [17]. This result of the KdV-B equation is surprising compared with the known results of the KdV equation and the Burgers equation. Kenig, Ponce, and Vega [17] proved that the KdV equation $\partial_t u + u\partial_x u + \partial_x^3 u = 0$ is locally well-posed in $H^s(\mathbf{R})$ with $s > -3/4$. See also [6, 7]. On the other hand, D.B. Dix [9] and D. Bekiranov [1] made it clear that the Burgers equation $\partial_t u + u\partial_x u - \partial_x^2 u = 0$ is locally well-posed for $s \geq -1/2$. The result is optimal since the uniqueness of the solutions fails when $s < -1/2$ [9].

In a previous paper [25], we were concerned with the special forms of (1.1):

$$\partial_t u + u\partial_x u - \partial_x |D_x|^{1+a} u - \partial_x^2 u = 0. \tag{1.2}$$

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In [25], we have proved that the gBO-B equations (1.2) are globally well-posed for $s > -(1+a)/2$ by applying the argument by L. Molinet and F. Ribaud [21, 22, 23]. Here we note that when $a > 0$, the value $s = -(1+a)/2$ is lower than the threshold $s = -1/2$ for the well-posedness of the Burgers equation. This result is due to the effect of the dispersive term of (1.2).

The purpose of this paper is to generalize the dissipative term $\partial_x^2 u$ of (1.2). This kind of generalization when $a = 1$ was treated by Molinet and Ribaud [21] (Case of the dissipative KdV), and they showed that the dissipative KdV equation is globally well-posed for $s > -3/4$, which is the same as that of the KdV [17]. The smoothing property of the KdV equation is strong so that they did not make use of the dissipative term to solve the dissipative KdV equation in paper [21]. In our study, we shall use not only dispersive property but dissipative one to solve the gBO-B equation (1.1), and define the following function space:

DEFINITION 1.1. For $s, b \in \mathbf{R}$, $X^{s,b}$ denotes the completion of the Schwartz space $\mathcal{S}(\mathbf{R}^2)$ with respect to the norm

$$\|F\|_{X^{s,b}} = \left(\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left\langle i\left(\tau - \xi|\xi|^{1+a}\right) + |\xi|^{2\alpha}\right\rangle^{2b}\langle\xi\rangle^{2s}\left|\widehat{F}(\xi,\tau)\right|^2 d\xi\, d\tau\right)^{1/2}, \tag{1.3}$$

where $\langle\cdot\rangle = (1+|\cdot|^2)^{1/2}$.

For $T > 0$, we define the localized space $X_T^{s,b}$ with the norm

$$\|f\|_{X_T^{s,b}} = \inf_{g\in X^{s,b}}\{\|g\|_{X^{s,b}} : g(t) = f(t) \text{ on } [0,T]\}. \tag{1.4}$$

Hereafter, $\widehat{\cdot}$ or $\mathcal{F}$ denotes the Fourier transform with respect to space-time variables. Note that $i(\tau - \xi|\xi|^{1+a}) + |\xi|^{2\alpha}$ is the symbol of the linear part of the gBO-B equation.

To solve the dissipative KdV equation, Molinet and Ribaud set a space-time function space equipped with the norm $\left\|\langle\tau-\xi^3\rangle^b\langle\xi\rangle^s\widehat{F}(\xi,\tau)\right\|_{L^2(\mathbf{R}^2)}$. This space is essentially suitable for the KdV equation, and was used by Kenig, Ponce, and Vega [17].

In the case of the gBO-B equations (1.1) with $0 \le a < 1$, it is difficult to adopt the approach based on the dispersive property since the dispersive effect is weaker than that of the KdV equation. Indeed, L. Molinet, J.-C. Saut, and N. Tzvetkov [24] showed that the following generalized Benjamin-Ono (gBO) equations

$$\partial_t u + u\partial_x u - \partial_x|D_x|^{1+a}u = 0, \quad 0 \le a < 1 \tag{1.5}$$

cannot be solved by the Picard iteration scheme when initial data is in $H^s(\mathbf{R})$, $s \in \mathbf{R}$. Therefore, when $0 \le a < 1$, we cannot solve the gBO-B equations (1.1) in function spaces with the norm $\left\|\langle\tau-\xi|\xi|^{1+a}\rangle^b\langle\xi\rangle^s\widehat{F}(\xi,\tau)\right\|_{L^2(\mathbf{R}^2)}$ by the iteration scheme. However, once we adopt the function spaces defined above (Definition 1.1), we can avoid the difficulty with the aid of the dissipative term.

DEFINITION 1.2. Let $U(t) = \exp(t\partial_x|D_x|^{1+a})$ be the unitary operator associated with the linear gBO equation. We denote by $\mathcal{W}(t)$ the semigroup associated with the linear gBO-B equation;

$$\mathcal{F}_x(\mathcal{W}(t)\phi)(\xi) = \exp\bigl[-|\xi|^{2\alpha}t + i\xi|\xi|^{1+a}t\bigr]\mathcal{F}_x(\phi)(\xi), \quad t \geq 0, \quad \phi \in \mathcal{S}. \tag{1.6}$$

And we extend $\mathcal{W}(t)$ to a linear operator defined on $\mathbf{R}$ by setting

$$\mathcal{F}_x(W(t)\phi)(\xi) = \exp\bigl[-|\xi|^{2\alpha}|t| + i\xi|\xi|^{1+a}t\bigr]\mathcal{F}_x(\phi)(\xi), \quad t \in \mathbf{R}, \quad \phi \in \mathcal{S}. \tag{1.7}$$

Here $\mathcal{F}_x$ denotes the Fourier transform with respect to x.

The following is the main result in this paper.

Theorem 1.1. *Let $s > -(a+2\alpha-1)/2$ with $a+2\alpha \leq 3$ and $\alpha > (3-a)/4 \geq 1/2$. Then for any $u_0 \in H^s(\mathbf{R})$, there exist $T = T(\|u_0\|_{H^s}) > 0$, $b \in (1/2, 1)$, and a unique solution $u(t)$ of the IVP* (1.1) *satisfying*

$$u(t) \in C([0,T], H^s(\mathbf{R})) \cap C((0,T], H^\infty(\mathbf{R})), \tag{1.8}$$

$$u \in X^{s-\alpha(2b-1),b}, \tag{1.9}$$

$$u\partial_x u \in X^{s-\alpha(2b-1),b-1}, \quad \partial_t u, \partial_x^2 u \in X^{s-\alpha(2b+1),b-1}. \tag{1.10}$$

Moreover, the flow map $u_0 \mapsto u(t)$ is locally Lipschitz from $H^s(\mathbf{R})$ to $C([0,T], H^s(\mathbf{R})) \cap C((0,T], H^\infty(\mathbf{R})) \cap X^{s-\alpha(2b-1),b}$. If the solution u is real-valued, $u \in C((0,+\infty), H^\infty(\mathbf{R}))$.

REMARK 1.1. Note that for any $u_0 \in H^s(\mathbf{R})$, the solution $u(t)$ belongs not to $X^{s,b}$ but to $X^{s-\alpha(2b-1),b}$. This loss of the regularity follows from Proposition 2.1. See Remark 2.1.

To prove the well-posedness, we solve by a contraction mapping principle the corresponding integral equation associated with the IVP (1.1):

$$u(t) = \mathcal{W}(t)u_0(x) - \frac{1}{2}\int_0^t \mathcal{W}(t-t')\partial_x\bigl(u^2(t')\bigr)\,dt', \quad t \geq 0. \tag{1.11}$$

However, we actually prove that the following map F is a contraction on a suitable function space:

$$F(\omega) = \psi(t)\left[W(t)u_0(x) - \frac{\chi_{\mathbf{R}_+(t)}}{2}\int_0^t W(t-t')\partial_x(\psi_T\omega(t'))^2\,dt'\right] \tag{1.12}$$

for $t \in \mathbf{R}$, where ψ is a cut-off function satisfying

$$\psi \in C_0^\infty(\mathbf{R}), \quad \operatorname{supp}\psi \subset [-2,2], \quad \psi \equiv 1 \quad \text{on} \quad [-1,1],$$

and $\psi_\delta(t) = \psi(t/\delta)$, and $\chi_{\mathbf{R}_+}(t)$ is the characteristic function of the interval $[0, \infty)$.

Finally, we shall collect some corollaries derived from Theorem 1.1.

Corollary 1.1 ([23], [25]). *When* $\alpha = 1$, *the gBO-B equations with* $0 \le a \le 1$

$$\partial_t u + u\partial_x u - \partial_x |D_x|^{1+a} u - \partial_x^2 u = 0 \tag{1.13}$$

are locally well-posed for $s > -(a+1)/2$.

Corollary 1.2. *When* $a = 0$, *the gBO-B equations with* $3/4 < \alpha \le 3/2$

$$\partial_t u + u\partial_x u - \partial_x |D_x| u + |D_x|^{2\alpha} u = 0 \tag{1.14}$$

are locally well-posed for $s > (1 - 2\alpha)/2$.

REMARK 1.2. Corollary 1.2 implies that the gBO-B equations (1.14) are locally well-posed in Sobolev spaces of negative order. Hence, in the case of the gBO-B (1.14), we can treat the IVP with more singular data than that of the BO equation so far, see Remark 1.4 below. Moreover, the author expects from the arguments in [1, 9] that the generalized Burgers equations

$$\partial_t u + u\partial_x u + |D_x|^{2\alpha} u = 0 \tag{1.15}$$

are locally well-posed for $s \ge (3 - 4\alpha)/2$ with $\alpha > 3/4$; note that $(3 - 4\alpha)/2 > (1 - 2\alpha)/2$ when $\alpha < 1$. Based on this conjecture, we may see the effect of the dispersive term of the gBO-B (1.14).

REMARK 1.3. From Theorem 1.1, the gBO-B equations can be solved in weaker spaces than that of the gBO equations and of Burgers type equations. The reason of this is due to dispersive-dissipative effects. Bilinear estimates (Proposition 3.1) are crucial ones for the well-posedness, and Lemmas 3.1 to 3.5 are needed in the proof of Proposition 3.1.

In the proofs of these lemmas, we are to use the dispersive-dissipative effects. Roughly speaking, we are to use the dissipative effect in the domain of interaction of low and high frequencies, and dispersive one in the domain of high-high interactions.

However, in the cases of the gBO (resp. Burgers type) equation, we cannot use the dissipative (resp. dispersive) effect. This leads to result that the gBO-B can be solved in weaker spaces. See Remark 3.4 for more details.

REMARK 1.4. According to C.E. Kenig and K.D. Koenig [16], the gBO equation (1.3) is locally well-posed for $s > 9/8 - 3a/8$. In particular, T. Tao [26] has shown that the ordinary BO equation (when $a = 0$) is globally well-posed in $H^1(\mathbf{R})$.

As is mentioned above, the gBO with $0 \le a < 1$ is not solved by iteration scheme as in [4, 17].[1]

However, J. Colliander, C. Kenig, and G. Staffilani [8] showed by the iteration scheme that the gBO equations with $0 < a < 1$ is locally well-posed in some weighted Sobolev space which is smaller than $H^{1/2}(\mathbf{R})$. Moreover, S. Herr [12] has shown the local well-posedness of the equation (1.3) with $0 < a < 1$ in a Sobolev-like space whose high frequency corresponds to that of $H^s(\mathbf{R})$ with $s > (1-a)/2$. Herr's study seems to be motivated by the work of K. Kato [15] on the existence of the solutions of the BO equation.[2]

REMARK 1.5. It is known that the flow map of the ordinary BO is not uniformly continuous when the initial data is in $H^s(\mathbf{R})$ with $s > 0$ [19] and $s < -1/2$ [3]. Recently, N. Kita and J. Segata [18] have proved that the BO equation is locally well-posed in some weighted Sobolev spaces which are smaller than $H^1(\mathbf{R})$ but contain the soliton solution. In this case, the flow map is locally Lipschitz.

Corollary 1.3. *When* $a = 1$, *the generalized KdV-Burgers equations with* $1/2 < \alpha \le 1$

$$\partial_t u + u\partial_x u - \partial_x^3 u + |D_x|^{2\alpha} u = 0 \tag{1.16}$$

are locally well-posed for $s > -\alpha$.

According to Molinet and Ribaud [21], the stronger statement follows when $a = 1$ and $0 < \alpha \le 3/4$:

Theorem 1.2 ([21]). *Let* $s > -3/4$. *Then the same results as in* Theorem 1.1 *are valid for the IVP* (1.1) *with* $a = 1$ *and* $\alpha > 0$.

REMARK 1.6. It follows from Corollary 1.3 that we improve the former results by Molinet and Ribaud [21] of the IVP (1.1) with $a = 1$ and $\alpha > 3/4$. Under the assumption of Theorem 1.1, we can not take $a > 1$. In our proof, when $a = 1$, the assumption $\alpha > 1/2$ truns out to be unnecessary. See Remark 3.3.

NOTATIONS. If there exists a harmless positive constant $c > 0$ such that $A \le cB$ (resp. $A \ge cB$) for any positive A and B, we denote $A \lesssim B$ (resp. $A \gtrsim B$) for abbreviation. The notation $A \sim B$ means that $A \lesssim B \lesssim A$.

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[1] Since the submission of this paper, remarkable studies on the Benjamin-Ono equation have appeared. The reader is referred to [5, 14, 20]. The papers [14] and [20] deal with the global well-posedness in $L^2(\mathbf{R})$ and in $L^2(\mathbf{T})$ respectively.

[2] See also [13], which is an improvement of [12]. Herr has improved the exponents s down to $s > -3a/4$.

The rest of this paper is organized as follows: Section 2 contains some linear estimates. In the proof of Theorem 1.1, bilinear estimates (Proposition 3.1) are key estimates. Section 3 includes the preparatory lemmas for the construction of the bilinear estimates. Section 4 is devoted to the proof of the bilinear estimates. Theorem 1.1 will be proved in Section 5.

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2. Linear estimates

In this section, we shall collect a few linear estimates for the proof of Theorem 1.1. The estimates corresponding to the case of $0 \le a \le 1$ and $\alpha = 1$ are given in [25]. We also treat a linear estimate (Lemma 2.3) to construct Proposition 3.1.

Proposition 2.1. *Let $s \in \mathbf{R}$, $\alpha > 0$, and $b \in [1/2, 1]$. There exists $C > 0$ such that*

$$\|\psi(t)W(t)\phi\|_{X^{s,b}} \le C\|\phi\|_{H^{s+\alpha(2b-1)}} \tag{2.1}$$

for any $\phi \in H^{s+\alpha(2b-1)}(\mathbf{R})$.

Proof. From the definition of the norm,

$$\begin{aligned}
\|\psi(t)W(t)\phi\|_{X^{s,b}} &= \left\| \langle\xi\rangle^s \mathcal{F}_x(\phi)(\xi) \left\| \left\langle i\tau + |\xi|^{2\alpha}\right\rangle^b \mathcal{F}_t\left(\psi(t)e^{-|t|\,|\xi|^{2\alpha}}\right)(\tau) \right\|_{L^2_\tau} \right\|_{L^2_\xi} \\
&\le C \left\| \langle\xi\rangle^s \mathcal{F}_x(\phi)(\xi) \left\| \langle\tau\rangle^b \mathcal{F}_t\left(\psi(t)e^{-|t|\,|\xi|^{2\alpha}}\right)(\tau) \right\|_{L^2_\tau} \right\|_{L^2_\xi} \\
&\quad + C \left\| \langle\xi\rangle^{s+2\alpha b} \mathcal{F}_x(\phi)(\xi) \left\| \mathcal{F}_t\left(\psi(t)e^{-|t|\,|\xi|^{2\alpha}}\right)(\tau) \right\|_{L^2_\tau} \right\|_{L^2_\xi}.
\end{aligned} \tag{2.2}$$

Put $g_\xi(\tau) = \mathcal{F}_t\left(\psi(t)e^{-|t||\xi|^{2\alpha}}\right)(\tau)$. If $|\xi| \ge 1$, it follows from $\langle\tau\rangle^b \le \langle\tau-\tau'\rangle^b + |\tau'|^b$ and Young's inequality that

$$\begin{aligned}
\|g_\xi\|_{H^b_t} &= \left\| \langle\tau\rangle^b \left(\widehat{\psi} \underset{\tau}{*} \mathcal{F}_t\left(e^{-|t|\,|\xi|^{2\alpha}}\right)\right) \right\|_{L^2_\tau} \\
&\le \left\| \langle\tau\rangle^b \widehat{\psi} \right\|_{L^1_\tau} \left\| e^{-|t|\,|\xi|^{2\alpha}} \right\|_{L^2_t} + \left\| \widehat{\psi} \right\|_{L^1_\tau} \left\| e^{-|t|\,|\xi|^{2\alpha}} \right\|_{\dot{H}^b_t} \\
&\le C\left(|\xi|^{-\alpha} + |\xi|^{\alpha(2b-1)}\right) \le C|\xi|^{\alpha(2b-1)},
\end{aligned} \tag{2.3}$$

where we note that for $\lambda > 0$

$$\| f(\lambda t)\|_{\dot{H}^s_t} \sim \lambda^{s-1/2} \| f(t)\|_{\dot{H}^s_t}. \tag{2.4}$$

If $|\xi| \le 1$, it follows that

$$\begin{aligned} \|g_\xi\|_{H_t^b} &= \left\| \psi e^{-|t|\,|\xi|^{2\alpha}} \right\|_{H_t^b} \\ &\le C \sum_{n\ge 0} \frac{|\xi|^{2\alpha n}}{n!} \| |t|^n \psi(t) \|_{H_t^1} \\ &\le C \left(1 + \sum_{n\ge 1} \frac{1}{(n-1)!} \right) \le C. \end{aligned} \tag{2.5}$$

Hence it follows that

$$\left\| \langle \tau \rangle^b g_\xi(\tau) \right\|_{L_\tau^2} \le C \langle \xi \rangle^{\alpha(2b-1)} \quad \text{for} \quad \frac{1}{2} \le b \le 1. \tag{2.6}$$

Combining (2.2) with (2.6), we obtain the desired estimate. □

REMARK 2.1. It follows from this proposition that for any initial data $u_0 \in H^s(\mathbf{R})$ the solution $u(t)$ is in $X^{s-\alpha(2b-1),b}$, which means the loss of the space regularity (in the L^2-based sense). In this proposition, we have estimated $\psi(t)W(t)\phi$ in the L^2-based space. Whereas we can treat it in the general L^p-based sense. Indeed, it is easy to derive the following estimate from the proof of the proposition: $\|\psi(t)W(t)\phi\|_{X_p^{s,b}} \le C\|\phi\|_{H_p^{s+2\alpha(b-1+1/p)}}$, where $\|F\|_{X_p^{s,b}} = \left\| \langle i(\tau - \xi|\xi|^{1+a}) + |\xi|^{2\alpha} \rangle^b \langle \xi \rangle^s \widehat{F}(\xi, \tau) \right\|_{L_{\xi,\tau}^p}$ and $\|f\|_{H_p^s} = \left\| \langle \xi \rangle^s \widehat{f}(\xi) \right\|_{L_\xi^p}$.

Noting that $b = 1/2 + \epsilon$ in the practical use, we see that the regularity loss can be recovered, provided $p > 2$. This may be a reasonable fact since the L^2 framework is suitable to treat the dispersive term.

Linear estimates appearing below can also be translated into that in a L^p-based sense. However, the author think that such a translation can not be applied to bilinear estimates below as long as our method of proof is used. The main reason for using L^2 space here is to use the duality argument in the proof of the bilinear estimates.

In [11], A. Grünrock deals with the modified KdV equation in the L^p framework, but it is open problem whether or not his argument is applicable to the gBO-B equation (1.1).

Proposition 2.2. *Let* $s \in \mathbf{R}$ *and let* $b > 1/2$. *For* $\delta \in (0, 1]$, *we have*

$$\|\psi_\delta F\|_{X^{s,b}} \le C\delta^{(1-2b)/2} \|F\|_{X^{s,b}}. \tag{2.7}$$

Proof. The proof can be done by modifying that of Lemma 2.5 in [10] slightly. □

Lemmas 2.1 and 2.2 are needed for the proof of Proposition 2.3.

Lemma 2.1. *For $w \in \mathcal{S}(\mathbf{R}^2)$, we define k_ξ on $\mathbf{R}$ as follows*:

$$k_\xi(t) = \psi(t) \int_{\mathbf{R}} \frac{e^{it\tau} - e^{-|t|\,|\xi|^{2\alpha}}}{i\tau + |\xi|^{2\alpha}} \widehat{w}(\xi, \tau)\, d\tau. \tag{2.8}$$

Let $\alpha > 0$ and $1/2 \le b < 1$. Then, it holds for any fixed $\xi \in \mathbf{R}$ that

$$\begin{aligned} &\left\| \left\langle i\tau + |\xi|^{2\alpha} \right\rangle^b \mathcal{F}_t(k_\xi) \right\|^2_{L^2_\tau(\mathbf{R})} \\ &\le C \left[\langle \xi \rangle^{2\alpha(2b-1)} \left(\int_{\mathbf{R}} \frac{|\widehat{w}(\xi, \tau)|}{\langle i\tau + |\xi|^{2\alpha} \rangle}\, d\tau \right)^2 + \left(\int_{\mathbf{R}} \frac{|\widehat{w}(\xi, \tau)|^2}{\langle i\tau + |\xi|^{2\alpha} \rangle^{2(1-b)}}\, d\tau \right) \right]. \end{aligned} \tag{2.9}$$

Proof. We rewrite k_ξ in the following way:

$$\begin{aligned} k_\xi(t) &= \psi(t) \int_{|\tau| \le 1} \frac{e^{it\tau} - 1}{i\tau + |\xi|^{2\alpha}} \widehat{w}(\xi, \tau)\, d\tau + \psi(t) \int_{|\tau| \le 1} \frac{1 - e^{-|t|\,|\xi|^{2\alpha}}}{i\tau + |\xi|^{2\alpha}} \widehat{w}(\xi, \tau)\, d\tau \\ &\quad + \psi(t) \int_{|\tau| \ge 1} \frac{e^{it\tau}}{i\tau + |\xi|^{2\alpha}} \widehat{w}(\xi, \tau)\, d\tau - \psi(t) \int_{|\tau| \ge 1} \frac{e^{-|t|\,|\xi|^{2\alpha}}}{i\tau + |\xi|^{2\alpha}} \widehat{w}(\xi, \tau)\, d\tau \\ &= \mathrm{I} + \mathrm{II} + \mathrm{III} - \mathrm{IV}. \end{aligned} \tag{2.10}$$

We have to estimate the contribution of these four terms to the left-hand side of (2.10).

Contribution of IV. Noting that $\langle i\tau + |\xi|^{2\alpha} \rangle \le C \left| i\tau + |\xi|^{2\alpha} \right|$ holds for $|\tau| \ge 1$,

$$\begin{aligned} &\left\| \left\langle i\tau + |\xi|^{2\alpha} \right\rangle^b \mathcal{F}_t(\mathrm{IV}) \right\|^2_{L^2_\tau} \\ &\le C \int_{\mathbf{R}} \left\langle i\tau + |\xi|^{2\alpha} \right\rangle^{2b} \left| \mathcal{F}_t\left(\psi(t) e^{-|t|\,|\xi|^{2\alpha}}\right)(\tau) \right|^2 d\tau \left(\int_{|\tau| \ge 1} \frac{|\widehat{w}(\xi, \tau)|}{\langle i\tau + |\xi|^{2\alpha} \rangle}\, d\tau \right)^2 \end{aligned} \tag{2.11}$$

Set $g_\xi(\tau) = \mathcal{F}_t\left(\psi(t) e^{-|t|\,|\xi|^{2\alpha}}\right)(\tau)$. By using (2.6), we have

$$\begin{aligned} \int_{\mathbf{R}} \left\langle i\tau + |\xi|^{2\alpha} \right\rangle^{2b} |g_\xi(\tau)|^2\, d\tau &\le C \int_{\mathbf{R}} \langle \tau \rangle^{2b} |g_\xi(\tau)|^2\, d\tau + C |\xi|^{4\alpha b} \int_{\mathbf{R}} |g_\xi(\tau)|^2\, d\tau \\ &\le C \langle \xi \rangle^{2\alpha(2b-1)}. \end{aligned} \tag{2.12}$$

Therefore we obtain

$$\left\| \langle i\tau + |\xi|^{2\alpha} \rangle^b \mathcal{F}_t(\mathrm{IV}) \right\|^2_{L^2_\tau} \le C \langle \xi \rangle^{2\alpha(2b-1)} \left(\int_{\mathbf{R}} \frac{|\widehat{w}(\xi, \tau)|}{\langle i\tau + |\xi|^{2\alpha} \rangle}\, d\tau \right)^2. \tag{2.13}$$

Contribution of III. Noting that $\langle i\tau + |\xi|^{2\alpha}\rangle^b \le C\langle\tau'\rangle^b + C\left|i(\tau-\tau') + |\xi|^{2\alpha}\right|^b$ and using Young's inequality,

$$
\begin{aligned}
&\left\|\left\langle i\tau + |\xi|^{2\alpha}\right\rangle^b \mathcal{F}_t(\mathrm{III})\right\|_{L^2_\tau}^2 \\
&= \int_{\mathbf{R}} \left\langle i\tau + |\xi|^{2\alpha}\right\rangle^{2b} \left|\int_{\mathbf{R}} \widehat{\psi}(\tau') \frac{\widehat{w}(\xi,\tau-\tau')\chi_{|\tau-\tau'|\ge 1}}{i(\tau-\tau') + |\xi|^{2\alpha}}\, d\tau'\right|^2 d\tau \\
&\le \int_{\mathbf{R}} \left(\int_{\mathbf{R}} \left|\langle\tau'\rangle^b \widehat{\psi}(\tau')\right| \frac{|\widehat{w}(\xi,\tau-\tau')|}{\left|i(\tau-\tau') + |\xi|^{2\alpha}\right|} \chi_{|\tau-\tau'|\ge 1}\, d\tau'\right)^2 d\tau \\
&\quad + \int_{\mathbf{R}} \left(\int_{\mathbf{R}} \left|\widehat{\psi}(\tau')\right| \frac{|\widehat{w}(\xi,\tau-\tau')|}{\left|i(\tau-\tau') + |\xi|^{2\alpha}\right|^{1-b}} \chi_{|\tau-\tau'|\ge 1}\, d\tau'\right)^2 d\tau \\
&\le C \left\| \frac{\widehat{w}(\xi,\tau)}{\langle i\tau + |\xi|^{2\alpha}\rangle^{1-b}} \right\|_{L^2_\tau}^2,
\end{aligned} \tag{2.14}
$$

where $\left\|\langle\tau\rangle^b \widehat{\psi}\right\|_{L^1} \le C$ for $0 \le b \le 1$.

Contribution of II. It follows from Schwarz inequality that

$$
\begin{aligned}
\left\|\left\langle i\tau + |\xi|^{2\alpha}\right\rangle^b \mathcal{F}_t(\mathrm{II})\right\|_{L^2_\tau}^2 &\le C \left\|\left\langle i\tau + |\xi|^{2\alpha}\right\rangle^b \mathcal{F}_t\left(\psi(t)\left[1 - e^{-|t|\,|\xi|^{2\alpha}}\right]\right)(\tau)\right\|_{L^2_\tau}^2 \\
&\quad \times \frac{\langle |\xi|^{2\alpha}\rangle}{|\xi|^{4\alpha}} \int_{\mathbf{R}} \frac{|\widehat{w}(\xi,\tau)|^2}{\langle i\tau + |\xi|^{2\alpha}\rangle}\, d\tau.
\end{aligned} \tag{2.15}
$$

(i) Case of $|\xi| \ge 1$. It follows that

$$
\begin{aligned}
&\left\|\left\langle i\tau + |\xi|^{2\alpha}\right\rangle^b \mathcal{F}_t\left(\psi(t)\left[1 - e^{-|t|\,|\xi|^{2\alpha}}\right]\right)(\tau)\right\|_{L^2_\tau}^2 \\
&\le \left\|\left\langle i\tau + |\xi|^{2\alpha}\right\rangle^b \mathcal{F}_t(\psi)(\tau)\right\|_{L^2_\tau}^2 + \left\|\left\langle i\tau + |\xi|^{2\alpha}\right\rangle^b \mathcal{F}_t\left(\psi(t) e^{-|t|\,|\xi|^{2\alpha}}\right)(\tau)\right\|_{L^2_\tau}^2 \\
&\le 2\left(\|\psi\|_{H^b_\tau}^2 + |\xi|^{4\alpha b}\|\psi\|_{L^2_\tau}^2\right) + C\langle\xi\rangle^{2\alpha(2b-1)} \\
&\le C\langle\xi\rangle^{4\alpha b},
\end{aligned} \tag{2.16}
$$

where we use (2.12) for the second term. Therefore we have

$$
\left\|\left\langle i\tau + |\xi|^{2\alpha}\right\rangle^b \mathcal{F}_t(\mathrm{II})\right\|_{L^2_\tau}^2 \le C\langle\xi\rangle^{2\alpha(2b-1)} \int_{\mathbf{R}} \frac{|\widehat{w}(\xi,\tau)|^2}{\langle i\tau + |\xi|^{2\alpha}\rangle}\, d\tau. \tag{2.17}
$$

(ii) Case of $|\xi| \leq 1$. It follows that

$$
\begin{aligned}
&\left\| \left\langle i\tau + |\xi|^{2\alpha} \right\rangle^b \mathcal{F}_t\big(\psi(t)\big[1 - e^{-|t|\,|\xi|^{2\alpha}}\big]\big)(\tau) \right\|_{L^2_\tau} \\
&\leq C \left\| \langle \tau \rangle^b \mathcal{F}_t\big(\psi(t)\big[1 - e^{-|t|\,|\xi|^{2\alpha}}\big]\big)(\tau) \right\|_{L^2_\tau} = C \left\| \sum_{n\geq 1} \frac{t^n \psi(t) |\xi|^{2\alpha n}}{n!} \right\|_{H^b_\tau} \\
&\leq C \sum_{n\geq 1} \frac{|\xi|^{2\alpha n}}{n!} \left\| t^n \psi(t) \right\|_{H^1_\tau} \\
&\leq C \sum_{n\geq 0} \frac{|\xi|^{2\alpha}}{n!} < C|\xi|^{2\alpha}.
\end{aligned} \tag{2.18}
$$

Hence

$$
\begin{aligned}
\left\| \left\langle i\tau + |\xi|^{2\alpha} \right\rangle^b \mathcal{F}_t(\mathrm{II}) \right\|^2_{L^2_\tau} &\leq C|\xi|^{4\alpha} \frac{\langle |\xi|^{2\alpha} \rangle}{|\xi|^{4\alpha}} \int_{\mathbf{R}} \frac{|\widehat{w}(\xi,\tau)|^2}{\langle i\tau + |\xi|^{2\alpha} \rangle} \, d\tau \\
&\leq C \int_{\mathbf{R}} \frac{|\widehat{w}(\xi,\tau)|^2}{\langle i\tau + |\xi|^{2\alpha} \rangle} \, d\tau.
\end{aligned} \tag{2.19}
$$

From (2.17) and 2.19, we obtain

$$
\begin{aligned}
\left\| \left\langle i\tau + |\xi|^{2\alpha} \right\rangle^b \mathcal{F}_t(\mathrm{II}) \right\|^2_{L^2_\tau} &\leq C \langle \xi \rangle^{2\alpha(2b-1)} \int_{\mathbf{R}} \frac{|\widehat{w}(\xi,\tau)|^2}{\langle i\tau + |\xi|^{2\alpha} \rangle} \, d\tau \\
&\leq C \int_{\mathbf{R}} \frac{|\widehat{w}(\xi,\tau)|^2}{\langle i\tau + |\xi|^{2\alpha} \rangle^{2(1-b)}} \, d\tau.
\end{aligned} \tag{2.20}
$$

Contribution of I. We can rewrite I as

$$
\mathrm{I} = \psi(t) \int_{|\tau|\leq 1} \sum_{n\geq 1} \frac{(it\tau)^n}{n!} \frac{\widehat{w}(\xi,\tau)}{i\tau + |\xi|^{2\alpha}} \, d\tau. \tag{2.21}
$$

It follows from Schwarz inequality that

$$
\begin{aligned}
&\left\| \left\langle i\tau + |\xi|^{2\alpha} \right\rangle^b \mathcal{F}_t(\mathrm{I}) \right\|_{L^2_\tau} \\
&\leq C \|\mathrm{I}\|_{H^b_\tau} + C|\xi|^{2\alpha b} \|\mathrm{I}\|_{L^2_\tau} \\
&\leq C \sum_{n\geq 1} \left[\left\| \frac{t^n \psi(t)}{n!} \right\|_{H^b_\tau} + |\xi|^{2\alpha b} \left\| \frac{t^n \psi(t)}{n!} \right\|_{L^2_\tau} \right] \\
&\quad \times \int_{|\tau|\leq 1} \frac{|i\tau|^n}{\left| i\tau + |\xi|^{2\alpha} \right|} |\widehat{w}(\xi,\tau)| \, d\tau \\
&\leq C\big(1 + |\xi|^{2\alpha b}\big) \left(\int_{|\tau|\leq 1} \frac{|\widehat{w}(\xi,\tau)|^2}{\langle i\tau + |\xi|^{2\alpha} \rangle} \, d\tau \right)^{1/2} \left(\int_{|\tau|\leq 1} \frac{|\tau|^2 \langle i\tau + |\xi|^{2\alpha} \rangle}{|i\tau + |\xi|^{2\alpha}|^2} \, d\tau \right)^{1/2}.
\end{aligned} \tag{2.22}
$$

If $|\xi| \le 1$, (2.22) is bounded by

$$C\left(\int_{\mathbf{R}} \frac{|\widehat{w}(\xi,\tau)|^2}{\langle i\tau + |\xi|^{2\alpha}\rangle}\,d\tau\right)^{1/2}. \tag{2.23}$$

If $|\xi| \ge 1$, (2.22) is bounded by

$$\begin{aligned} &\frac{\langle\xi\rangle^{2\alpha b}}{\langle\xi\rangle^{\alpha}}\left(\int_{\mathbf{R}} \frac{|\widehat{w}(\xi,\tau)|^2}{\langle i\tau + |\xi|^{2\alpha}\rangle}\,d\tau\right)^{1/2} \\ &\le C\left(\int_{\mathbf{R}} \frac{|\widehat{w}(\xi,\tau)|^2}{\langle i\tau + |\xi|^{2\alpha}\rangle^{2(1-b)}}\,d\tau\right)^{1/2}, \end{aligned} \tag{2.24}$$

where we note that

$$\int_{|\tau|\le 1} \frac{|\tau|^2\langle i\tau + |\xi|^{2\alpha}\rangle}{\left|i\tau + |\xi|^{2\alpha}\right|^2}\,d\tau \le \frac{1}{\langle\xi\rangle^{2\alpha}}. \tag{2.25}$$

From (2.23) and (2.24), we get

$$\left\|\left\langle i\tau + |\xi|^{2\alpha}\right\rangle^{b}\mathcal{F}_t(\mathrm{I})\right\|_{L^2_\tau} \le C\left(\int_{\mathbf{R}} \frac{|\widehat{w}(\xi,\tau)|^2}{\langle i\tau + |\xi|^{2\alpha}\rangle^{2(1-b)}}\,d\tau\right)^{1/2}. \tag{2.26}$$

Summing up, from (2.13), (2.14), (2.20) and (2.26), we obtain the desired estimate (2.9). □

Lemma 2.2. *Let $0 \le \sigma \le 1$, $\sigma \ne 1/2$. For $f \in H^\sigma(\mathbf{R})$ with $f(0) = 0$,*

$$\left\|\chi_{\mathbf{R}_+} f\right\|_{H^\sigma} \le C_\sigma \|f\|_{H^\sigma}, \tag{2.27}$$

where $\chi_{\mathbf{R}_+}$ is the characteristic function of $[0,\infty)$.

Proposition 2.3. *Let $s \in \mathbf{R}$, $\alpha > 0$, and let $b > 1/2$.*
(i) *There exists $C > 0$ such that, for any $v \in \mathcal{S}(\mathbf{R}^2)$,*

$$\begin{aligned} &\left\|\chi_{\mathbf{R}_+}(t)\psi(t)\int_0^t W(t-t')v(t')\,dt'\right\|_{X^{s,b}} \\ &\le C\left[\|v\|_{X^{s,b-1}} + \left(\int_{\mathbf{R}} \langle\xi\rangle^{2s+2\alpha(2b-1)}\left(\int_{\mathbf{R}} \frac{|\widehat{v}(\xi,\tau)|}{\langle i(\tau - \xi|\xi|^{1+a}) + |\xi|^{2\alpha}\rangle}\,d\tau\right)^2 d\xi\right)^{1/2}\right]. \end{aligned} \tag{2.28}$$

(ii) *For $0 < \delta < 1/2$, there exists $C_\delta > 0$ such that, for any $v \in X^{s,b-1+\delta}$,*

$$\left\|\chi_{\mathbf{R}_+}(t)\psi(t)\int_0^t W(t-t')v(t')\,dt'\right\|_{X^{s,b}} \le C_\delta\|v\|_{X^{s,b-1+\delta}}. \tag{2.29}$$

Proof. Assume that $v \in \mathcal{S}(\mathbf{R}^2)$. Recall that $U(t) = \exp(t\partial_x |D_x|^{1+a})$. Setting $w(t') = U(-t')v(t')$, we get

$$\begin{aligned}
&\chi_{\mathbf{R}_+}(t)\psi(t)\int_0^t W(t-t')v(t')\,dt' \\
&= U(t)\left[\chi_{\mathbf{R}_+}(t)\psi(t)\int_{\mathbf{R}^2} e^{ix\xi}\frac{e^{it\tau}-e^{-t|\xi|^{2\alpha}}}{i\tau+|\xi|^{2\alpha}}\widehat{w}(\xi,\tau)\,d\xi\,d\tau\right].
\end{aligned} \tag{2.30}$$

Putting

$$k_\xi(t) = \psi(t)\int_{\mathbf{R}}\frac{e^{it\tau}-e^{-|t|\,|\xi|^{2\alpha}}}{i\tau+|\xi|^{2\alpha}}\widehat{w}(\xi,\tau)\,d\tau, \tag{2.31}$$

we can rewrite

$$\chi_{\mathbf{R}_+}(t)\psi(t)\int_0^t W(t-t')v(t')\,dt' = U(t)\mathcal{F}_\xi^{-1}\left(\chi_{\mathbf{R}_+}(t)k_\xi\right)(x,t). \tag{2.32}$$

Since $w(t) = U(-t)v(t) \in \mathcal{S}(\mathbf{R}^2)$, it is clear that for any fixed $\xi \in \mathbf{R}$, k_ξ is continuous on $\mathbf{R}$ and $k_\xi(0) = 0$. By virtue of Lemma 2.2, $\|\chi_{\mathbf{R}_+}k_\xi\|_{H_t^b} \le C_b\|k_\xi\|_{H_t^b}$ holds for $0 \le b \le 1$, $b \ne 1/2$.

Thus we find that

$$\begin{aligned}
&\left\|\chi_{\mathbf{R}_+}(t)\psi(t)\int_0^t W(t-t')v(t')\,dt'\right\|_{X^{s,b}} \\
&= \left\|U(t)\mathcal{F}_\xi^{-1}\left(\chi_{\mathbf{R}_+}(t)k_\xi(t)\right)\right\|_{X^{s,b}} \\
&= \left\|\langle i\tau+|\xi|^{2\alpha}\rangle^b\langle\xi\rangle^s\mathcal{F}_t\left(\chi_{\mathbf{R}_+}(t)k_\xi(t)\right)(\xi,\tau)\right\|_{L^2_{\xi,\tau}} \\
&\le \left\|\langle\xi\rangle^s\left\|\chi_{\mathbf{R}_+}(t)k_\xi(t)\right\|_{H_t^b}\right\|_{L^2_\xi} + \left\|\langle\xi\rangle^{s+2\alpha b}\left\|\chi_{\mathbf{R}_+}(t)k_\xi(t)\right\|_{L^2_t}\right\|_{L^2_\xi} \\
&\le C\left(\left\|\langle\xi\rangle^s\left\|k_\xi(t)\right\|_{H_t^b}\right\|_{L^2_\xi} + \left\|\langle\xi\rangle^{s+2\alpha b}\left\|k_\xi(t)\right\|_{L^2_t}\right\|_{L^2_\xi}\right) \\
&\le C\left\|\langle\xi\rangle^s\left\|\left\langle i\tau+|\xi|^{2\alpha}\right\rangle^b\mathcal{F}_t(k_\xi)(\tau)\right\|_{L^2_\tau}\right\|_{L^2_\xi}.
\end{aligned} \tag{2.33}$$

With the aid of Lemma 2.1, the statement (i) follows if we note that $\widehat{w}(\xi,\tau) = \widehat{v}(\xi,\tau+\xi|\xi|^{1+a})$. By using Schwarz inequality and the density argument, we directly derive (ii) from (i). $\square$

Proposition 2.4. *Let $s \in \mathbf{R}$, $\alpha > 0$, $b \ge 1/2$ and $\delta > 0$. For all $f \in X^{s,b-1+\delta}$,*

$$t \mapsto \int_0^t W(t-t')f(t')\,dt' \in C\left(\mathbf{R}_+, H^{s+2\alpha\delta}(\mathbf{R})\right). \tag{2.34}$$

Moreover, we have

$$\left\| \chi_{\mathbf{R}_+}(t)\psi(t)\int_0^t W(t-t')f(t')\,dt' \right\|_{L^\infty(\mathbf{R}_+,H^{s+2\alpha\delta})} \le C\|f\|_{X^{s,b-1+\delta}}. \tag{2.35}$$

Proof. We can set $s=0$ without loss of generality. It suffices to prove that

$$t \mapsto U(-t)\int_0^t W(t-t')f(t')\,dt'$$

is continuous from $[0,\infty)$ to $H^{2\alpha\delta}(\mathbf{R})$ since U is strongly continuous unitary group in $L^2(\mathbf{R})$.

Put $g(x,t)=(U(-t)f(t))(x)$. The statement follows if we show the continuity of

$$F: t \mapsto \langle\xi\rangle^{2\alpha\delta}\int_0^t e^{-|\xi|^{2\alpha}|t-t'|}\mathcal{F}x(g(\,\cdot\,,t'))(\xi)\,dt' \tag{2.36}$$

for $\langle i\tau+|\xi|^{2\alpha}\rangle^{b-1+\delta}\widehat{g}\in L^2_{\xi,\tau}(\mathbf{R}^2)$. We rewrite, for $t\ge 0$,

$$\begin{aligned} F(t) &= \langle\xi\rangle^{2\alpha\delta}e^{-|\xi|^{2\alpha}t}\int_{\mathbf{R}}\widehat{g}(\xi,\tau)\int_0^t e^{\left(|\xi|^{2\alpha}+i\tau\right)t'}\,dt'\,d\tau \\ &= \langle\xi\rangle^{2\alpha\delta}\int_{\mathbf{R}}\widehat{g}(\xi,\tau)\frac{e^{it\tau}-e^{-|\xi|^{2\alpha}|t|}}{i\tau+|\xi|^{2\alpha}}\,d\tau. \end{aligned} \tag{2.37}$$

Hence

$$F(t_1)-F(t_2)=\langle\xi\rangle^{2\alpha\delta}\int_{\mathbf{R}}\frac{\widehat{g}(\xi,\tau)}{i\tau+|\xi|^{2\alpha}}\left[\left(e^{i\tau t_1}-e^{i\tau t_2}\right)-\left(e^{-|\xi|^{2\alpha}|t_1|}-e^{-|\xi|^{2\alpha}|t_2|}\right)\right]d\tau. \tag{2.38}$$

When $|\xi|\ge 1$, applying Schwarz inequality, we obtain

$$\begin{aligned} &|F(t_1)-F(t_2)| \\ &\le 4\langle\xi\rangle^{2\alpha\delta}\left(\int_{\mathbf{R}}\frac{|\widehat{g}(\xi,\tau)|^2}{\langle i\tau+|\xi|^{2\alpha}\rangle^{2(1-b)-2\delta}}\,d\tau\right)^{1/2}\left(\int_{\mathbf{R}}\frac{\langle i\tau+|\xi|^{2\alpha}\rangle^{2(1-b)-2\delta}}{|i\tau+|\xi|^{2\alpha}|^2}\,d\tau\right)^{1/2} \\ &\le C\langle\xi\rangle^{2\alpha\delta}\left(\int_{\mathbf{R}}\frac{|\widehat{g}(\xi,\tau)|^2}{\langle i\tau+|\xi|^{2\alpha}\rangle^{2(1-b)-2\delta}}\,d\tau\right)^{1/2}|\xi|^{\alpha(1-2b-2\delta)}\left(\int_{\mathbf{R}}\frac{d\theta}{\langle\theta\rangle^{2b+2\delta}}\right)^{1/2}, \end{aligned} \tag{2.39}$$

where we put $\tau=|\xi|^{2\alpha}\theta$. Hence it follows that for $|\xi|\ge 1$

$$|F(t_1)-F(t_2)|\le C\left(\int_{\mathbf{R}}\frac{|\widehat{g}(\xi,\tau)|^2}{\langle i\tau+|\xi|^{2\alpha}\rangle^{2(1-b)-2\delta}}\,d\tau\right)^{1/2}. \tag{2.40}$$

When $|\xi| \leq 1$, we separate the two terms to estimate the right-hand side of (2.38). We may assume that $|t_1 - t_2| < 1$. It follows from mean value theorem and Schwarz inequality that

$$
\begin{aligned}
&\left| \int \frac{\widehat{g}(\xi,\tau)}{i\tau + |\xi|^{2\alpha}} (e^{i\tau t_1} - e^{i\tau t_2})\, d\tau \right| \\
&\leq |t_1 - t_2| \int_{|\tau|\leq 1} \frac{|\tau|\,|\widehat{g}(\xi,\tau)|}{\left|i\tau + |\xi|^{2\alpha}\right|}\, d\tau + 2\int_{|\tau|\geq 1} \frac{|\widehat{g}(\xi,\tau)|}{\left|i\tau + |\xi|^{2\alpha}\right|}\, d\tau \\
&\leq C \left\| \left\langle i\tau + |\xi|^{2\alpha} \right\rangle^{b-1+\delta} \widehat{g}(\xi,\,\cdot\,) \right\|_{L^2_\tau} \\
&\quad \times \left[\left(\int_{|\tau|\leq 1} \langle \tau \rangle^{2(1-b)-2\delta}\, d\tau \right)^{1/2} + \left(\int_{|\tau|\geq 1} \langle \tau \rangle^{-2b-2\delta}\, d\tau \right)^{1/2} \right] \\
&\leq C \left\| \left\langle i\tau + |\xi|^{2\alpha} \right\rangle^{b-1+\delta} \widehat{g}(\xi,\,\cdot\,) \right\|_{L^2_\tau}.
\end{aligned} \tag{2.41}
$$

Similarly it follows that

$$
\begin{aligned}
&\left| \int \frac{\widehat{g}(\xi,\tau)}{i\tau + |\xi|^{2\alpha}} \left(e^{-|\xi|^{2\alpha}|t_1|} - e^{-|\xi|^{2\alpha}|t_2|}\right) d\tau \right| \\
&\leq |t_1 - t_2|\,|\xi|^{2\alpha} \int_{|\tau|\leq 1} \frac{|\widehat{g}(\xi,\tau)|}{\left|i\tau + |\xi|^{2\alpha}\right|}\, d\tau + 2\int_{|\tau|\geq 1} \frac{|\widehat{g}(\xi,\tau)|}{\left|i\tau + |\xi|^{2\alpha}\right|}\, d\tau \\
&\leq C \left\| \left\langle i\tau + |\xi|^{2\alpha} \right\rangle^{b-1+\delta} \widehat{g}(\xi,\,\cdot\,) \right\|_{L^2_\tau}.
\end{aligned} \tag{2.42}
$$

Summing up, we obtain

$$
|F(t_1) - F(t_2)| \leq C \left\| \left\langle i\tau + |\xi|^{2\alpha} \right\rangle^{b-1+\delta} \widehat{g}(\xi,\,\cdot\,) \right\|_{L^2_\tau}. \tag{2.43}
$$

Furthermore, we find that

$$
\|F(t_1) - F(t_2)\|_{L^2(\mathbf{R})} \leq C \left\| \left\langle i\tau + |\xi|^{2\alpha} \right\rangle^{b-1+\delta} \widehat{g}(\xi,\tau) \right\|_{L^2_{\xi,\tau}}. \tag{2.44}
$$

It is clear that the integrant in (2.38) tends to 0 as $|t_1 - t_2| \to 0$, and is bounded uniformly in $|t_1 - t_2|$ by the integrant of the right-hand side of (2.43). Hence, $|F(t_1) - F(t_2)| \to 0$ as $|t_1 - t_2| \to 0$ for almost every $\xi \in \mathbf{R}$. Moreover, from (2.45) and the Lebesgue dominated convergence theorem, we infer that

$$
\|F(t_1) - F(t_2)\|_{L^2(\mathbf{R})} \to 0 \quad \text{as} \quad |t_1 - t_2| \to 0. \tag{2.45}
$$

To show (2.35), we refer to the previous paper [25]. $\square$

Finally, we introduce the following estimate to finish this section. Lemma 2.3 will be used in the proof of Proposition 3.1.

Lemma 2.3. *Let v be with compact support in time in $[-T, T]$. For any $\theta > 0$, there exists $\mu = \mu(\theta) > 0$ such that*

$$\left\| \mathcal{F}^{-1}_{x,t} \left(\frac{\widehat{v}(\xi, \tau)}{\langle \tau - \xi|\xi|^{1+a} \rangle^{\theta}} \right) \right\|_{L^2_{x,t}(\mathbf{R}^2)} \le CT^{\mu} \|v\|_{L^2_{x,t}(\mathbf{R}^2)}. \tag{2.46}$$

Proof. A similar estimate was verified by J. Ginibre, Y. Tsutsumi, and G. Velo [10, Lemma 3.1]. It suffices to modify the proof slightly. Therefore we omit the proof of Lemma 2.3. □

3. Bilinear estimates

Proposition 3.1. *For $s > -(a+2\alpha-1)/2$ with $a+2\alpha \le 3$ and $\alpha > (3-a)/4 \ge 1/2$, there exist $b > 1/2$, C, μ, and $\delta > 0$ such that for any $u, v \in X^{s,b}$ with compact support in $[-T, T]$, we have*

$$\|\partial_x(uv)\|_{X^{s,b-1+\delta}} \le CT^{\mu} \|u\|_{X^{s,b}} \|v\|_{X^{s,b}}. \tag{3.1}$$

By duality argument, it is equivalent to show that for any $w \in X^{-s,1-b-\delta}$ with $\|w\|_{X^{-s,1-b-\delta}} \le 1$,

$$|\mathrm{I}| = |\langle \partial_x(uv), w\rangle| \le CT^{\mu} \|u\|_{X^{s,b}} \|v\|_{X^{s,b}} \|w\|_{X^{-s,1-b-\delta}}. \tag{3.2}$$

Putting

$$\widehat{f}(\xi, \tau) = \left\langle i\left(\tau - \xi|\xi|^{1+a}\right) + |\xi|^{2\alpha} \right\rangle^{b} \langle \xi\rangle^{s} \widehat{u}(\xi, \tau),$$
$$\widehat{g}(\xi, \tau) = \left\langle i\left(\tau - \xi|\xi|^{1+a}\right) + |\xi|^{2\alpha} \right\rangle^{b} \langle \xi\rangle^{s} \widehat{v}(\xi, \tau),$$

and

$$\widehat{h}(\xi, \tau) = \left\langle i\left(\tau - \xi|\xi|^{1+a}\right) + |\xi|^{2\alpha} \right\rangle^{1-b-\delta} \langle \xi\rangle^{-s} \widehat{w}(\xi, \tau),$$

we see that (3.2) is equivalent to

$$|\mathrm{I}| \le CT^{\mu} \|f\|_{L^2_x L^2_t} \|g\|_{L^2_x L^2_t} \|h\|_{L^2_x L^2_t}. \tag{3.3}$$

And we can rewrite

(3.4)

$$\begin{aligned} \mathrm{I} &= \int_{\mathbf{R}^4} \frac{\xi \overline{\widehat{h}(\xi, \tau)} \langle \xi\rangle^{s}}{\langle i(\tau - \xi|\xi|^{1+a}) + |\xi|^{2\alpha}\rangle^{1-b-\delta}} \frac{\widehat{g}(\xi_1, \tau_1)\langle \xi_1\rangle^{-s}}{\langle i(\tau_1 - \xi_1|\xi_1|^{1+a}) + |\xi_1|^{2\alpha}\rangle^{b}} \\ &\quad \times \frac{\widehat{f}(\xi - \xi_1, \tau - \tau_1)\langle \xi - \xi_1\rangle^{-s}}{\langle i(\tau - \tau_1 - (\xi - \xi_1)|\xi - \xi_1|^{1+a}) + |\xi - \xi_1|^{2\alpha}\rangle^{b}} \, d\xi \, d\tau \, d\xi_1 \, d\tau_1 \\ &= \int_{\mathbf{R}^4} \frac{\xi \overline{\widehat{h}(\xi, \tau)} \langle \xi\rangle^{s}}{\langle i\sigma + |\xi|^{2\alpha}\rangle^{1-b-\delta}} \frac{\widehat{g}(\xi_1, \tau_1)\langle \xi_1\rangle^{-s}}{\langle i\sigma_1 + |\xi_1|^{2\alpha}\rangle^{b}} \frac{\widehat{f}(\xi - \xi_1, \tau - \tau_1)\langle \xi - \xi_1\rangle^{-s}}{\langle i\sigma_2 + |\xi - \xi_1|^{2\alpha}\rangle^{b}} \, d\xi \, d\tau \, d\xi_1 \, d\tau_1, \end{aligned}$$

where

$$\sigma = \tau - \xi|\xi|^{1+a}, \quad \sigma_1 = \tau_1 - \xi_1|\xi_1|^{1+a}, \quad \sigma_2 = \tau - \tau_1 - (\xi - \xi_1)|\xi - \xi_1|^{1+a}. \tag{3.5}$$

3.1. Algebraic smoothing relation. The following algebraic relation will be effectively used for the proof of Proposition 3.1:

Proposition 3.2 ([25]). *Let* $|\xi_1| \geq 1$ *and* $|\xi - \xi_1| \geq 1$, *and let* $0 \leq a < 1$. *Then the following relation holds among* σ, σ_1 *and* σ_2 *defined above*:
(i) *If* $\xi_1(\xi - \xi_1) > 0$ *and* $|\xi_1| \geq |\xi - \xi_1|$,

$$\max\{|\sigma|, |\sigma_1|, |\sigma_2|\} \geq \frac{1+a}{3}|\xi_1|^{1+a}|\xi - \xi_1|. \tag{3.6}$$

(ii) *If* $\xi_1(\xi - \xi_1) < 0$ *and* $|\xi_1| \geq |\xi - \xi_1|$,

$$\max\{|\sigma|, |\sigma_1|, |\sigma_2|\} \geq \frac{1+a}{3}|\xi|\,|\xi_1|^{a}|\xi - \xi_1|. \tag{3.7}$$

(iii) *If* $\xi_1(\xi - \xi_1) > 0$ *and* $|\xi_1| \leq |\xi - \xi_1|$,

$$\max\{|\sigma|, |\sigma_1|, |\sigma_2|\} \geq \frac{1+a}{3}|\xi_1|\,|\xi - \xi_1|^{1+a}. \tag{3.8}$$

(iv) *If* $\xi_1(\xi - \xi_1) < 0$ *and* $|\xi_1| \leq |\xi - \xi_1|$,

$$\max\{|\sigma|, |\sigma_1|, |\sigma_2|\} \geq \frac{1+a}{3}|\xi|\,|\xi_1|\,|\xi - \xi_1|^{a}. \tag{3.9}$$

Proof. See [25, Proposition 3.2]. □

REMARK 3.1. When $a = 1$, it follows from $\sigma_1 + \sigma_2 - \sigma = 3\xi\xi_1(\xi - \xi_1)$ that $\max\{|\sigma|, |\sigma_1|, |\sigma_2|\} \geq |\xi\xi_1(\xi - \xi_1)|$. See [4], [17].

REMARK 3.2. Let $\rho = (a+2\alpha-1)/2-(11+5a+10\alpha)\epsilon/2$. The following exponents often appear throughout the proofs of Lemmas 3.1 to 3.4.

$$m_1 = 2\rho - a - 2\alpha + 5(2+a)\epsilon - 4\alpha\epsilon = -1 - (1+14\alpha)\epsilon, \tag{3.10}$$

$$m_2 = 2\rho - a - 2\alpha - 2(2+a)\epsilon + 10\alpha\epsilon = -1 - (15+7a)\epsilon, \tag{3.11}$$

$$\begin{aligned} n_1 &= 4\rho - (1+a)(1-5\epsilon) - 2\alpha(1+2\epsilon) \\ &= a + 2\alpha - 3 - (17+5a+24\alpha)\epsilon. \end{aligned} \tag{3.12}$$

3.2. Preliminaries I. For any fixed (ξ_1, τ_1) with $|\xi_1| \geq 1$, we introduce the following integral region: $A(\xi_1, \tau_1) = \{(\xi, \tau) \in \mathbf{R}^2 \colon |\xi| \leq 2|\xi_1|, \ |\xi - \xi_1| \geq 1\}$.

Lemma 3.1. *Let $\rho = (a + 2\alpha - 1)/2 - (11 + 5a + 10\alpha)\epsilon/2$ and let $1 < a + 2\alpha \leq 3$. If $\max\{|\sigma|, |\sigma_1|, |\sigma_2|\} \gtrsim |\xi_1|^{1+a}|\xi - \xi_1|$ with $|\xi_1| \geq 1$ holds, then for any $\epsilon > 0$ there exists $C > 0$, depending only on ϵ, such that*

$$\mathrm{I} = \frac{\langle \xi_1\rangle^{2\rho}}{\langle i\sigma_1 + |\xi_1|^{2\alpha}\rangle^{1+2\epsilon}} \iint_{A(\xi_1,\tau_1)} \frac{|\xi|^2\langle\xi\rangle^{-2\rho}\langle\xi - \xi_1\rangle^{2\rho}\, d\xi\, d\tau}{\langle i\sigma + |\xi|^{2\alpha}\rangle^{1-5\epsilon}\langle i\sigma_2 + |\xi - \xi_1|^{2\alpha}\rangle^{1+2\epsilon}} \leq C. \tag{3.13}$$

By symmetry between ξ_1 and $\xi - \xi_1$, we can easily derive the following corollary:

Corollary 3.1. *Let $\rho = (a+2\alpha-1)/2-(11+5a+10\alpha)\epsilon/2$ and let $1 < a+2\alpha \leq 3$. If $\max\{|\sigma|, |\sigma_1|, |\sigma_2|\} \gtrsim |\xi_1|\,|\xi - \xi_1|^{1+a}$ with $|\xi_1| \geq 1$, then (3.13) holds.*

Proof of Lemma 3.1. It follows that $|\xi-\xi_1| \leq 3|\xi_1|$ in $A(\xi_1, \tau_1)$. We split $A(\xi_1, \tau_1)$ into three regions;

$$\begin{aligned}
A_1(\xi_1, \tau_1) &= \{(\xi, \tau) \in A(\xi_1, \tau_1) \colon |\sigma| = \max\{|\sigma|, |\sigma_1|, |\sigma_2|\}\},\\
A_2(\xi_1, \tau_1) &= \{(\xi, \tau) \in A(\xi_1, \tau_1) \colon |\sigma_1| = \max\{|\sigma|, |\sigma_1|, |\sigma_2|\}\},\\
A_3(\xi_1, \tau_1) &= \{(\xi, \tau) \in A(\xi_1, \tau_1) \colon |\sigma_2| = \max\{|\sigma|, |\sigma_1|, |\sigma_2|\}\}.
\end{aligned}$$

Estimate in A_1. It follows from the assumption of the lemma that $\langle\sigma\rangle \gtrsim \langle\xi_1\rangle^{1+a}\langle\xi - \xi_1\rangle$ in A_1. With the aid of $\langle\sigma\rangle \gtrsim \langle\xi_1\rangle^{1+a}\langle\xi - \xi_1\rangle$, we have

$$\begin{aligned}
\mathrm{I} &\lesssim \iint_{A_1(\xi_1,\tau_1)} \frac{\langle\xi\rangle^{2-2\rho}\langle\xi_1\rangle^{2\rho-(1+a)(1-5\epsilon)}\langle\xi - \xi_1\rangle^{2\rho-(1-5\epsilon)}}{\langle|\xi_1|^{2\alpha}\rangle^{1+2\epsilon}\langle\sigma_2\rangle^{1+2\epsilon}}\, d\xi\, d\tau\\
&\lesssim \iint_{A_1(\xi_1,\tau_1)} \frac{\langle\xi\rangle^{2-2\rho}\langle\xi_1\rangle^{2\rho-(1+a)(1-5\epsilon)-2\alpha(1+2\epsilon)}\langle\xi - \xi_1\rangle^{2\rho-(1-5\epsilon)}}{\langle\sigma_2\rangle^{1+2\epsilon}}\, d\xi\, d\tau.
\end{aligned} \tag{3.14}$$

Since $\rho \leq 1$ and $a + 2\alpha > 1$, it follows from $|\xi| \leq 2|\xi_1|$ and $|\xi - \xi_1| \leq 3|\xi_1|$ that

$$\begin{aligned}
&\langle\xi\rangle^{2-2\rho}\langle\xi_1\rangle^{2\rho-(1+a)(1-5\epsilon)-2\alpha(1+2\epsilon)}\langle\xi - \xi_1\rangle^{2\rho-(1-5\epsilon)}\\
&\lesssim \langle\xi_1\rangle^{1-a-2\alpha+5(1+a)\epsilon-4\alpha\epsilon}\langle\xi - \xi_1\rangle^{2\rho-(1-5\epsilon)} \lesssim \langle\xi - \xi_1\rangle^{2\rho-a-2\alpha+5(2+a)\epsilon-4\alpha\epsilon}.
\end{aligned} \tag{3.15}$$

Hence it follows from Remark 3.2 that

$$\mathrm{I} \leq C \iint_{A_1(\xi_1,\tau_1)} \frac{d\xi\, d\tau}{\langle\xi - \xi_1\rangle^{1+(1+14\alpha)\epsilon}\langle\sigma_2\rangle^{1+2\epsilon}} \leq C. \tag{3.16}$$

Estimate in A_2. It follows from the assumption of the lemma that $\langle\sigma_1\rangle \gtrsim \langle\xi_1\rangle^{1+a}\langle\xi - \xi_1\rangle$ in A_2.

(i) Case of $|\xi-\xi_1| \geq |\xi|$. In this case, it follows that $|\xi-\xi_1| \sim |\xi_1|$. We first note that $\langle\sigma\rangle^{1-5\epsilon}\langle\sigma_1\rangle^{1+2\epsilon} = \langle\sigma\rangle^{1-5\epsilon}\langle\sigma_1\rangle^{1-5\epsilon}\langle\sigma_1\rangle^{7\epsilon} \geq \langle\sigma\rangle^{1+2\epsilon}\langle\sigma_1\rangle^{1-5\epsilon}$ holds since $|\sigma_1| \geq |\sigma|$. With this inequality and $\langle\sigma_1\rangle \gtrsim \langle\xi_1\rangle^{1+a}\langle\xi-\xi_1\rangle$, we have

$$\begin{aligned}
\mathrm{I} &\lesssim \iint_{A_2(\xi_1,\tau_1)} \frac{\langle\xi\rangle^{2-2\rho}\langle\xi_1\rangle^{2\rho}\langle\xi-\xi_1\rangle^{2\rho}}{\langle\sigma\rangle^{1+2\epsilon}\langle\sigma_1\rangle^{1-5\epsilon}\langle i\sigma_2+|\xi-\xi_1|^{2\alpha}\rangle^{1+2\epsilon}}\,d\xi\,d\tau \\
&\lesssim \iint_{A_2(\xi_1,\tau_1)} \frac{\langle\xi\rangle^{2-2\rho}\langle\xi_1\rangle^{2\rho-(1+a)(1-5\epsilon)}\langle\xi-\xi_1\rangle^{2\rho-(1-5\epsilon)}}{\langle\sigma\rangle^{1+2\epsilon}\langle|\xi-\xi_1|^{2\alpha}\rangle^{1+2\epsilon}}\,d\xi\,d\tau \\
&\lesssim \iint_{A_2(\xi_1,\tau_1)} \frac{\langle\xi\rangle^{2-2\rho}\langle\xi_1\rangle^{2\rho-(1+a)(1-5\epsilon)}\langle\xi-\xi_1\rangle^{2\rho-(1-5\epsilon)-2\alpha(1+2\epsilon)}}{\langle\sigma\rangle^{1+2\epsilon}}\,d\xi\,d\tau.
\end{aligned} \tag{3.17}$$

Since $\rho \leq 1$, it follows from $|\xi| \leq 2|\xi_1|$ and $|\xi-\xi_1| \sim |\xi_1|$ that

$$\begin{aligned}
&\langle\xi\rangle^{2-2\rho}\langle\xi_1\rangle^{2\rho-(1+a)(1-5\epsilon)}\langle\xi-\xi_1\rangle^{2\rho-(1-5\epsilon)-2\alpha(1+2\epsilon)} \\
&\lesssim \langle\xi_1\rangle^{2-(1+a)(1-5\epsilon)}\langle\xi-\xi_1\rangle^{2\rho-(1-5\epsilon)-2\alpha(1+2\epsilon)} \sim \langle\xi-\xi_1\rangle^{2\rho-a-2\alpha+5(2+a)\epsilon-4\alpha\epsilon}.
\end{aligned} \tag{3.18}$$

Hence it follows from Remark 3.2 that

$$\mathrm{I} \leq C\iint_{A_2(\xi_1,\tau_1)} \frac{d\xi\,d\tau}{\langle\xi-\xi_1\rangle^{1+(1+14\alpha)\epsilon}\langle\sigma\rangle^{1+2\epsilon}} \leq C. \tag{3.19}$$

(ii) Case of $|\xi-\xi_1| \leq |\xi|$. In this case, it follows that $|\xi| \sim |\xi_1|$ holds. With the aid of $\langle\sigma_1\rangle \gtrsim \langle\xi_1\rangle^{1+a}\langle\xi-\xi_1\rangle$, we have

$$\begin{aligned}
\mathrm{I} &\lesssim \iint_{A_2(\xi_1,\tau_1)} \frac{\langle\xi\rangle^{2-2\rho}\langle\xi_1\rangle^{2\rho-(1+a)(1+2\epsilon)}\langle\xi-\xi_1\rangle^{2\rho-(1+2\epsilon)}}{\langle|\xi|^{2\alpha}\rangle^{1-5\epsilon}\langle\sigma_2\rangle^{1+2\epsilon}}\,d\xi\,d\tau \\
&\lesssim \iint_{A_2(\xi_1,\tau_1)} \frac{\langle\xi\rangle^{2-2\rho-2\alpha(1-5\epsilon)}\langle\xi_1\rangle^{2\rho-(1+a)(1+2\epsilon)}\langle\xi-\xi_1\rangle^{2\rho-(1+2\epsilon)}}{\langle\sigma_2\rangle^{1+2\epsilon}}\,d\xi\,d\tau.
\end{aligned} \tag{3.20}$$

Since $a+2\alpha > 1$, it follows from $|\xi| \sim |\xi_1|$ and $|\xi-\xi_1| \leq |\xi|$ that

$$\begin{aligned}
&\langle\xi\rangle^{2-2\rho-2\alpha(1-5\epsilon)}\langle\xi_1\rangle^{2\rho-(1+a)(1+2\epsilon)}\langle\xi-\xi_1\rangle^{2\rho-(1+2\epsilon)} \\
&\sim \langle\xi\rangle^{1-a-2\alpha-2(1+a)\epsilon+10\alpha\epsilon}\langle\xi-\xi_1\rangle^{2\rho-(1+2\epsilon)} \lesssim \langle\xi-\xi_1\rangle^{2\rho-a-2\alpha-2(2+a)\epsilon+10\alpha\epsilon}.
\end{aligned} \tag{3.21}$$

Hence it follows from Remark 3.2 that

$$\mathrm{I} \leq C\iint_{A_2(\xi_1,\tau_1)} \frac{d\xi\,d\tau}{\langle\xi-\xi_1\rangle^{1+(15+7a)\epsilon}\langle\sigma_2\rangle^{1+2\epsilon}} \leq C. \tag{3.22}$$

Estimate in A_3. It follows from the assumption of the lemma that $\langle\sigma_2\rangle \gtrsim \langle\xi_1\rangle^{1+a}\langle\xi-\xi_1\rangle$ in A_3. We first note that $\langle\sigma\rangle^{1-5\epsilon}\langle\sigma_2\rangle^{1+2\epsilon} = \langle\sigma\rangle^{1-5\epsilon}\langle\sigma_2\rangle^{1-5\epsilon}\langle\sigma_2\rangle^{7\epsilon} \geq$

$\langle\sigma\rangle^{1+2\epsilon}\langle\sigma_2\rangle^{1-5\epsilon}$ holds since $|\sigma_2| \ge |\sigma|$. With this inequality and $\langle\sigma_2\rangle \gtrsim \langle\xi_1\rangle^{1+a}\langle\xi-\xi_1\rangle$, we have

$$\begin{aligned} \mathrm{I} &\lesssim \iint_{A_3(\xi_1,\tau_1)} \frac{\langle\xi\rangle^{2-2\rho}\langle\xi_1\rangle^{2\rho-(1+a)(1-5\epsilon)}\langle\xi-\xi_1\rangle^{2\rho-(1-5\epsilon)}}{\langle\sigma\rangle^{1+2\epsilon}\langle|\xi_1|^{2\alpha}\rangle^{1+2\epsilon}}\,d\xi\,d\tau \\ &\lesssim \iint_{A_3(\xi_1,\tau_1)} \frac{\langle\xi\rangle^{2-2\rho}\langle\xi_1\rangle^{2\rho-(1+a)(1-5\epsilon)-2\alpha(1+2\epsilon)}\langle\xi-\xi_1\rangle^{2\rho-(1-5\epsilon)}}{\langle\sigma\rangle^{1+2\epsilon}}\,d\xi\,d\tau. \end{aligned} \tag{3.23}$$

Since $\rho \le 1$ and $a+2\alpha > 1$, it follows from $|\xi| \le 2|\xi_1|$ and $|\xi-\xi_1| \le 3|\xi_1|$ that

$$\begin{aligned} &\langle\xi\rangle^{2-2\rho}\langle\xi_1\rangle^{2\rho-(1+a)(1-5\epsilon)-2\alpha(1+2\epsilon)}\langle\xi-\xi_1\rangle^{2\rho-(1-5\epsilon)} \\ &\lesssim \langle\xi_1\rangle^{1-a-2\alpha+5(1+a)\epsilon-4\alpha\epsilon}\langle\xi-\xi_1\rangle^{2\rho-(1-5\epsilon)} \lesssim \langle\xi-\xi_1\rangle^{2\rho-a-2\alpha+5(2+a)\epsilon-4\alpha\epsilon}. \end{aligned} \tag{3.24}$$

Hence it follows from Remark 3.2 that

$$\mathrm{I} \le C \iint_{A_3(\xi_1,\tau_1)} \frac{d\xi\,d\tau}{\langle\xi-\xi_1\rangle^{1+(1+14\alpha)\epsilon}\langle\sigma\rangle^{1+2\epsilon}} \le C. \tag{3.25}$$

Summing up, we have the desired result. □

Lemma 3.2. *Let* $\rho = (a+2\alpha-1)/2-(11+5a+10\alpha)\epsilon/2$ *and let* $1 < a+2\alpha \le 3$. *If* $\max\{|\sigma|,|\sigma_1|,|\sigma_2|\} \gtrsim |\xi|\,|\xi_1|^a|\xi-\xi_1|$ *with* $|\xi_1| \ge 1$ *holds, then for any* $\epsilon > 0$ *there exists* $C > 0$, *depending only on* ϵ, *such that*

$$\mathrm{I} = \frac{\langle\xi_1\rangle^{2\rho}}{\langle i\sigma_1+|\xi_1|^{2\alpha}\rangle^{1+2\epsilon}} \iint_{A(\xi_1,\tau_1)} \frac{|\xi|^2\langle\xi\rangle^{-2\rho}\langle\xi-\xi_1\rangle^{2\rho}\,d\xi\,d\tau}{\langle i\sigma+|\xi|^{2\alpha}\rangle^{1-5\epsilon}\langle i\sigma_2+|\xi-\xi_1|^{2\alpha}\rangle^{1+2\epsilon}} \le C. \tag{3.26}$$

By symmetry between ξ_1 and $\xi-\xi_1$, we can easily derive the following corollary:

Corollary 3.2. *Let* $\rho = (a+2\alpha-1)/2-(11+5a+10\alpha)\epsilon/2$ *and let* $1 < a+2\alpha \le 3$. *If* $\max\{|\sigma|,|\sigma_1|,|\sigma_2|\} \gtrsim |\xi|\,|\xi_1|\,|\xi-\xi_1|^a$ *with* $|\xi_1| \ge 1$, *then* (3.26) *holds*.

Proof of Lemma 3.2. We find that $|\xi-\xi_1| \le 3|\xi_1|$ holds in $A(\xi_1,\tau_1)$. As in the proof of Lemma 3.1, we split $A(\xi_1,\tau_1)$ into three regions $A_1(\xi_1,\tau_1)$, $A_2(\xi_1,\tau_1)$ and $A_3(\xi_1,\tau_1)$.

Estimate in A_1. It follows from the assumption of the lemma that $\langle\sigma\rangle \gtrsim |\xi|\langle\xi_1\rangle^a\langle\xi-\xi_1\rangle$ in A_1. With the aid of $\langle\sigma\rangle \gtrsim |\xi|\langle\xi_1\rangle^a\langle\xi-\xi_1\rangle$, we have

$$\begin{aligned} \mathrm{I} &\lesssim \iint_{A_1(\xi_1,\tau_1)} \frac{|\xi|^{1+5\epsilon}\langle\xi\rangle^{-2\rho}\langle\xi_1\rangle^{2\rho-a(1-5\epsilon)}\langle\xi-\xi_1\rangle^{2\rho-(1-5\epsilon)}}{\langle|\xi_1|^{2\alpha}\rangle^{1+2\epsilon}\langle\sigma_2\rangle^{1+2\epsilon}}\,d\xi\,d\tau \\ &\lesssim \iint_{A_1(\xi_1,\tau_1)} \frac{\langle\xi\rangle^{1-2\rho+5\epsilon}\langle\xi_1\rangle^{2\rho-a(1-5\epsilon)-2\alpha(1+2\epsilon)}\langle\xi-\xi_1\rangle^{2\rho-(1-5\epsilon)}}{\langle\sigma_2\rangle^{1+2\epsilon}}\,d\xi\,d\tau. \end{aligned} \tag{3.27}$$

When $\rho \le 1/2$, it follows from $|\xi| \le 2|\xi_1|$ and $|\xi - \xi_1| \le 3|\xi_1|$ that

$$\begin{aligned}(3.28)\qquad &\langle\xi\rangle^{1-2\rho+5\epsilon}\langle\xi_1\rangle^{2\rho-a(1-5\epsilon)-2\alpha(1+2\epsilon)}\langle\xi-\xi_1\rangle^{2\rho-(1-5\epsilon)}\\ &\lesssim \langle\xi_1\rangle^{1-a-2\alpha+5(1+a)\epsilon-4\alpha\epsilon}\langle\xi-\xi_1\rangle^{2\rho-(1-5\epsilon)} \lesssim \langle\xi-\xi_1\rangle^{2\rho-a-2\alpha+5(2+a)\epsilon-4\alpha\epsilon},\end{aligned}$$

where we note that $a + 2\alpha > 1$. When $\rho \ge 1/2$, it follows from $|\xi - \xi_1| \le 3|\xi_1|$ and $|\xi| \le 2|\xi_1|$ that

$$\begin{aligned}(3.29)\qquad &\langle\xi\rangle^{1-2\rho+5\epsilon}\langle\xi_1\rangle^{2\rho-a(1-5\epsilon)-2\alpha(1+2\epsilon)}\langle\xi-\xi_1\rangle^{2\rho-(1-5\epsilon)}\\ &\lesssim \langle\xi\rangle^{1-2\rho+5\epsilon}\langle\xi_1\rangle^{4\rho-(1+a)(1-5\epsilon)-2\alpha(1+2\epsilon)} \lesssim \langle\xi\rangle^{2\rho-a-2\alpha+5(2+a)\epsilon-4\alpha\epsilon},\end{aligned}$$

where we note that Remark 3.2 and $a + 2\alpha \le 3$ in the last term.

Hence it follows from Remark 3.2 that

$$(3.30)\qquad \mathrm{I} \le C \iint_{A(\xi_1,\tau_1)} \frac{d\xi\, d\tau}{\langle\min\{|\xi|, |\xi-\xi_1|\}\rangle^{1+(1+14\alpha)\epsilon}\langle\sigma_2\rangle^{1+2\epsilon}} \le C.$$

Estimate in A_2. It follows from the assumption of the lemma that $\langle\sigma_1\rangle \gtrsim |\xi|\langle\xi_1\rangle^a\langle\xi-\xi_1\rangle$ in A_2.

(i) Case of $|\xi - \xi_1| \ge |\xi|$. In this case, it follows that $|\xi_1| \sim |\xi - \xi_1|$. We first note that $\langle\sigma\rangle^{1-5\epsilon}\langle\sigma_1\rangle^{1+2\epsilon} \ge \langle\sigma\rangle^{1+2\epsilon}\langle\sigma_1\rangle^{1-5\epsilon}$ holds since $|\sigma_1| \ge |\sigma|$. Hence we get

$$(3.31)\qquad \mathrm{I} \lesssim \iint_{A_2(\xi_1,\tau_1)} \frac{|\xi|^2\langle\xi\rangle^{-2\rho}\langle\xi_1\rangle^{2\rho}\langle\xi-\xi_1\rangle^{2\rho}}{\langle\sigma\rangle^{1+2\epsilon}\langle\sigma_1\rangle^{1-5\epsilon}\langle i\sigma_2 + |\xi-\xi_1|^{2\alpha}\rangle^{1+2\epsilon}}\, d\xi\, d\tau.$$

With the aid of $\langle\sigma_1\rangle \gtrsim |\xi|\langle\xi_1\rangle^a\langle\xi-\xi_1\rangle$, (3.31) is bounded by

$$\begin{aligned}(3.32)\qquad &\iint_{A_2(\xi_1,\tau_1)} \frac{|\xi|^{1+5\epsilon}\langle\xi\rangle^{-2\rho}\langle\xi_1\rangle^{2\rho-a(1-5\epsilon)}\langle\xi-\xi_1\rangle^{2\rho-(1-5\epsilon)}}{\langle\sigma\rangle^{1+2\epsilon}\langle|\xi-\xi_1|^{2\alpha}\rangle^{1+2\epsilon}}\, d\xi\, d\tau\\ &\lesssim \iint_{A_2(\xi_1,\tau_1)} \frac{\langle\xi\rangle^{1-2\rho+5\epsilon}\langle\xi_1\rangle^{2\rho-a(1-5\epsilon)}\langle\xi-\xi_1\rangle^{2\rho-(1-5\epsilon)-2\alpha(1+2\epsilon)}}{\langle\sigma\rangle^{1+2\epsilon}}\, d\xi\, d\tau.\end{aligned}$$

It follows from $|\xi_1| \sim |\xi - \xi_1|$ and $|\xi| \le 2|\xi_1|$ that

$$\begin{aligned}(3.33)\qquad &\langle\xi\rangle^{1-2\rho+5\epsilon}\langle\xi_1\rangle^{2\rho-a(1-5\epsilon)}\langle\xi-\xi_1\rangle^{2\rho-(1-5\epsilon)-2\alpha(1+2\epsilon)}\\ &\sim \langle\xi\rangle^{1-2\rho+5\epsilon}\langle\xi_1\rangle^{4\rho-(1+a)(1-5\epsilon)-2\alpha(1+2\epsilon)} \lesssim \langle\xi\rangle^{2\rho-a-2\alpha+5(2+a)\epsilon-4\alpha\epsilon},\end{aligned}$$

where we note that Remark 3.2 and $a + 2\alpha \le 3$ in the last term.

Therefore it follows from Remark 3.2 that

$$(3.34)\qquad \mathrm{I} \le C \iint_{A_2(\xi_1,\tau_1)} \frac{d\xi\, d\tau}{\langle\xi\rangle^{1+(1+14\alpha)\epsilon}\langle\sigma\rangle^{1+2\epsilon}} \le C.$$

(ii) Case of $|\xi-\xi_1|\le|\xi|$. In this case, it follows that $|\xi|\sim|\xi_1|$. By virtue of $\langle\sigma_1\rangle\gtrsim|\xi|\langle\xi_1\rangle^a\langle\xi-\xi_1\rangle$, we have

$$\begin{aligned}\mathrm{I}&\lesssim\iint_{A_2(\xi_1,\tau_1)}\frac{|\xi|^{1-2\epsilon}\langle\xi\rangle^{-2\rho}\langle\xi_1\rangle^{2\rho-a(1+2\epsilon)}\langle\xi-\xi_1\rangle^{2\rho-(1+2\epsilon)}}{\langle|\xi|^{2\alpha}\rangle^{1-5\epsilon}\langle\sigma_2\rangle^{1+2\epsilon}}\,d\xi\,d\tau\\&\lesssim\iint_{A_2(\xi_1,\tau_1)}\frac{\langle\xi\rangle^{1-2\rho-2\epsilon-2\alpha(1-5\epsilon)}\langle\xi_1\rangle^{2\rho-a(1+2\epsilon)}\langle\xi-\xi_1\rangle^{2\rho-(1+2\epsilon)}}{\langle\sigma_2\rangle^{1+2\epsilon}}\,d\xi\,d\tau.\end{aligned}\tag{3.35}$$

Since $a+2\alpha>1$, it follows from $|\xi|\sim|\xi_1|$ and $|\xi-\xi_1|\le|\xi|$ that

$$\begin{aligned}&\langle\xi\rangle^{1-2\rho-2\epsilon-2\alpha(1-5\epsilon)}\langle\xi_1\rangle^{2\rho-a(1+2\epsilon)}\langle\xi-\xi_1\rangle^{2\rho-(1+2\epsilon)}\\&\sim\langle\xi\rangle^{1-a-2\alpha-2(1+a)\epsilon+10\alpha\epsilon}\langle\xi-\xi_1\rangle^{2\rho-(1+2\epsilon)}\lesssim\langle\xi-\xi_1\rangle^{2\rho-a-2\alpha-2(2+a)\epsilon+10\alpha\epsilon}.\end{aligned}\tag{3.36}$$

Hence it follows from Remark 3.2 that

$$\mathrm{I}\le C\iint_{A_2(\xi_1,\tau_1)}\frac{d\xi\,d\tau}{\langle\xi-\xi_1\rangle^{1+(15+7a)\epsilon}\langle\sigma_2\rangle^{1+2\epsilon}}\le C.\tag{3.37}$$

Estimate in A_3. It follows from the assumption of the lemma that $\langle\sigma_2\rangle\gtrsim|\xi|\langle\xi_1\rangle^a\langle\xi-\xi_1\rangle$ in A_3.

(i) Case of $|\xi-\xi_1|\ge|\xi|$. In this case, it follows that $|\xi_1|\sim|\xi-\xi_1|$. Since $|\xi_1|\sim|\xi-\xi_1|$, this case is proved as in the region A_2 (i) above by using the symmetry between σ_1 and σ_2.

(ii) Case of $|\xi-\xi_1|\le|\xi|$. In this case, it follows that $|\xi|\sim|\xi_1|$. We first note that $\langle\sigma\rangle^{1-5\epsilon}\langle\sigma_2\rangle^{1+2\epsilon}\ge\langle\sigma\rangle^{1+2\epsilon}\langle\sigma_2\rangle^{1-5\epsilon}$ holds since $|\sigma_2|\ge|\sigma|$. Hence we get

$$\mathrm{I}\lesssim\iint_{A_3(\xi_1,\tau_1)}\frac{|\xi|^2\langle\xi\rangle^{-2\rho}\langle\xi_1\rangle^{2\rho}\langle\xi-\xi_1\rangle^{2\rho}}{\langle\sigma\rangle^{1+2\epsilon}\langle i\sigma_1+|\xi_1|^{2\alpha}\rangle^{1+2\epsilon}\langle\sigma_2\rangle^{1-5\epsilon}}\,d\xi\,d\tau.\tag{3.38}$$

By virtue of $\langle\sigma_2\rangle\gtrsim|\xi|\langle\xi_1\rangle^a\langle\xi-\xi_1\rangle$, we have

$$\begin{aligned}\mathrm{I}&\lesssim\iint_{A_3(\xi_1,\tau_1)}\frac{|\xi|^{1+5\epsilon}\langle\xi\rangle^{-2\rho}\langle\xi_1\rangle^{2\rho-a(1-5\epsilon)}\langle\xi-\xi_1\rangle^{2\rho-(1-5\epsilon)}}{\langle\sigma\rangle^{1+2\epsilon}\langle|\xi_1|^{2\alpha}\rangle^{1+2\epsilon}}\,d\xi\,d\tau\\&\lesssim\iint_{A_3(\xi_1,\tau_1)}\frac{\langle\xi\rangle^{1-2\rho+5\epsilon}\langle\xi_1\rangle^{2\rho-a(1-5\epsilon)-2\alpha(1+2\epsilon)}\langle\xi-\xi_1\rangle^{2\rho-(1-5\epsilon)}}{\langle\sigma\rangle^{1+2\epsilon}}\,d\xi\,d\tau.\end{aligned}\tag{3.39}$$

Since $a+2\alpha>1$, it follows from $|\xi|\sim|\xi_1|$ and $|\xi-\xi_1|\le|\xi|$ that

$$\begin{aligned}&\langle\xi\rangle^{1-2\rho+5\epsilon}\langle\xi_1\rangle^{2\rho-a(1-5\epsilon)-2\alpha(1+2\epsilon)}\langle\xi-\xi_1\rangle^{2\rho-(1-5\epsilon)}\\&\sim\langle\xi\rangle^{1-a-2\alpha+5(1+a)\epsilon-4\alpha\epsilon}\langle\xi-\xi_1\rangle^{2\rho-(1-5\epsilon)}\lesssim\langle\xi-\xi_1\rangle^{2\rho-a-2\alpha+5(2+a)\epsilon-4\alpha\epsilon}.\end{aligned}\tag{3.40}$$

Hence it follows from Remark 3.2 that

$$\mathrm{I}\le C\iint_{A_3(\xi_1,\tau_1)}\frac{d\xi\,d\tau}{\langle\xi-\xi_1\rangle^{1+(1+14\alpha)\epsilon}\langle\sigma\rangle^{1+2\epsilon}}\le C.\tag{3.41}$$

Thus we finish the proof. □

3.3. Preliminaries II. For any fixed (ξ, τ), we introduce the following integral region: $B(\xi, \tau) = \{(\xi_1, \tau_1) \in \mathbf{R}^2 \colon 2|\xi_1| \leq |\xi|,\ \ |\xi_1| \geq 1,\ \ |\xi - \xi_1| \geq 1\}$.

Lemma 3.3. *Let $\rho = (a + 2\alpha - 1)/2 - (11 + 5a + 10\alpha)\epsilon/2$ and let $\alpha > 1/2$. If $\max\{|\sigma|, |\sigma_1|, |\sigma_2|\} \gtrsim |\xi_1|^{1+a}|\xi - \xi_1|$ holds, then for any $\epsilon > 0$ there exists $C > 0$, depending only on ϵ, such that*

$$\mathrm{I} = \frac{|\xi|^2\langle\xi\rangle^{-2\rho}}{\langle i\sigma + |\xi|^{2\alpha}\rangle^{1-5\epsilon}} \iint_{B(\xi,\tau)} \frac{\langle\xi_1\rangle^{2\rho}\langle\xi - \xi_1\rangle^{2\rho}\, d\xi_1\, d\tau_1}{\langle i\sigma_1 + |\xi_1|^{2\alpha}\rangle^{1+2\epsilon}\langle i\sigma_2 + |\xi - \xi_1|^{2\alpha}\rangle^{1+2\epsilon}} \leq C. \tag{3.42}$$

By symmetry between ξ_1 and $\xi - \xi_1$, we can easily derive the following corollary:

Corollary 3.3. *Let $\rho = (a + 2\alpha - 1)/2 - (11 + 5a + 10\alpha)\epsilon/2$ and let $\alpha > 1/2$. If $\max\{|\sigma|, |\sigma_1|, |\sigma_2|\} \gtrsim |\xi_1|\,|\xi - \xi_1|^{1+a}$, then* (3.42) *holds.*

Proof of Lemma 3.3. It follows that $|\xi| \sim |\xi - \xi_1|$ in $B(\xi, \tau)$. We split $B(\xi, \tau)$ into three regions;

$$B_1(\xi, \tau) = \{(\xi_1, \tau_1) \in B(\xi, \tau) \colon |\sigma| = \max\{|\sigma|, |\sigma_1|, |\sigma_2|\}\},$$
$$B_2(\xi, \tau) = \{(\xi_1, \tau_1) \in B(\xi, \tau) \colon |\sigma_1| = \max\{|\sigma|, |\sigma_1|, |\sigma_2|\}\},$$
$$B_3(\xi, \tau) = \{(\xi_1, \tau_1) \in B(\xi, \tau) \colon |\sigma_2| = \max\{|\sigma|, |\sigma_1|, |\sigma_2|\}\}.$$

Estimate in B_1. It follows from the assumption of the lemma that $\langle\sigma\rangle \gtrsim \langle\xi_1\rangle^{1+a}\langle\xi - \xi_1\rangle$ in B_1. With the aid of $\langle\sigma\rangle \gtrsim \langle\xi_1\rangle^{1+a}\langle\xi - \xi_1\rangle$, we have

$$\begin{aligned}\mathrm{I} &\lesssim \iint_{B_1(\xi,\tau)} \frac{\langle\xi\rangle^{2-2\rho}\langle\xi_1\rangle^{2\rho-(1+a)(1-5\epsilon)}\langle\xi - \xi_1\rangle^{2\rho-(1-5\epsilon)}}{\langle\sigma_1\rangle^{1+2\epsilon}\langle|\xi - \xi_1|^{2\alpha}\rangle^{1+2\epsilon}}\, d\xi_1\, d\tau_1 \\ &\lesssim \iint_{B_1(\xi,\tau)} \frac{\langle\xi\rangle^{2-2\rho}\langle\xi_1\rangle^{2\rho-(1+a)(1-5\epsilon)}\langle\xi - \xi_1\rangle^{2\rho-(1-5\epsilon)-2\alpha(1+2\epsilon)}}{\langle\sigma_1\rangle^{1+2\epsilon}}\, d\xi_1\, d\tau_1.\end{aligned} \tag{3.43}$$

Since $2\alpha > 1$, it follows from $|\xi| \sim |\xi - \xi_1|$ and $2|\xi_1| \leq |\xi|$ that

$$\begin{aligned}&\langle\xi\rangle^{2-2\rho}\langle\xi_1\rangle^{2\rho-(1+a)(1-5\epsilon)}\langle\xi - \xi_1\rangle^{2\rho-(1-5\epsilon)-2\alpha(1+2\epsilon)} \\ &\sim \langle\xi\rangle^{1-2\alpha+5\epsilon-4\alpha\epsilon}\langle\xi_1\rangle^{2\rho-(1+a)(1-5\epsilon)} \lesssim \langle\xi_1\rangle^{2\rho-a-2\alpha+5(2+a)\epsilon-4\alpha\epsilon}.\end{aligned} \tag{3.44}$$

Hence it follows from Remark 3.2 that

$$\mathrm{I} \leq C \iint_{B_1(\xi,\tau)} \frac{d\xi_1\, d\tau_1}{\langle\xi_1\rangle^{1+(1+14\alpha)\epsilon}\langle\sigma_1\rangle^{1+2\epsilon}} \leq C. \tag{3.45}$$

Estimate in B_2. It follows from the assumption of the lemma that $\langle\sigma_1\rangle \gtrsim \langle\xi_1\rangle^{1+a}\langle\xi-\xi_1\rangle$ in B_2. With the aid of $\langle\sigma_1\rangle \gtrsim \langle\xi_1\rangle^{1+a}\langle\xi-\xi_1\rangle$, we have

$$\begin{aligned}\mathrm{I} &\lesssim \iint_{B_2(\xi,\tau)} \frac{\langle\xi\rangle^{2-2\rho}\langle\xi_1\rangle^{2\rho-(1+a)(1+2\epsilon)}\langle\xi-\xi_1\rangle^{2\rho-(1+2\epsilon)}}{\langle|\xi|^{2\alpha}\rangle^{1-5\epsilon}\langle\sigma_2\rangle^{1+2\epsilon}}\,d\xi_1\,d\tau_1 \\ &\lesssim \iint_{B_2(\xi,\tau)} \frac{\langle\xi\rangle^{2-2\rho-2\alpha(1-5\epsilon)}\langle\xi_1\rangle^{2\rho-(1+a)(1+2\epsilon)}\langle\xi-\xi_1\rangle^{2\rho-(1+2\epsilon)}}{\langle\sigma_2\rangle^{1+2\epsilon}}\,d\xi_1\,d\tau_1.\end{aligned} \tag{3.46}$$

Since $2\alpha > 1$, it follows from $|\xi| \sim |\xi-\xi_1|$ and $2|\xi_1| \le |\xi|$ that

$$\begin{aligned}&\langle\xi\rangle^{2-2\rho-2\alpha(1-5\epsilon)}\langle\xi_1\rangle^{2\rho-(1+a)(1+2\epsilon)}\langle\xi-\xi_1\rangle^{2\rho-(1+2\epsilon)} \\ &\sim \langle\xi\rangle^{1-2\alpha+10\alpha\epsilon-2\epsilon}\langle\xi_1\rangle^{2\rho-(1+a)(1+2\epsilon)} \lesssim \langle\xi_1\rangle^{2\rho-a-2\alpha-2(2+a)\epsilon+10\alpha\epsilon}.\end{aligned} \tag{3.47}$$

Hence it follows from Remark 3.2 that

$$\mathrm{I} \le C \iint_{B_2(\xi,\tau)} \frac{d\xi_1\,d\tau_1}{\langle\xi_1\rangle^{1+(15+7a)\epsilon}\langle\sigma_2\rangle^{1+2\epsilon}} \le C. \tag{3.48}$$

Estimate in B_3. It follows from the assumption of the lemma that $\langle\sigma_2\rangle \gtrsim \langle\xi_1\rangle^{1+a}\langle\xi-\xi_1\rangle$ in B_3. By symmetry between $i\sigma_1 + |\xi_1|^{2\alpha}$ and $i\sigma_2 + |\xi-\xi_1|^{2\alpha}$, we can prove this case by following the analogous argument in B_2.

Summing up, our statement is established. □

Lemma 3.4. *Let* $\rho = (a+2\alpha-1)/2 - (11+5a+10\alpha)\epsilon/2$ *and let* $\alpha > 0$. *If* $\max\{|\sigma|, |\sigma_1|, |\sigma_2|\} \gtrsim |\xi|\,|\xi_1|^a|\xi-\xi_1|$ *holds, then for any* $\epsilon > 0$ *there exists* $C > 0$, *depending only on* ϵ, *such that*

$$\mathrm{I} = \frac{|\xi|^2\langle\xi\rangle^{-2\rho}}{\langle i\sigma + |\xi|^{2\alpha}\rangle^{1-5\epsilon}} \iint_{B(\xi,\tau)} \frac{\langle\xi_1\rangle^{2\rho}\langle\xi-\xi_1\rangle^{2\rho}\,d\xi_1\,d\tau_1}{\langle i\sigma_1 + |\xi_1|^{2\alpha}\rangle^{1+2\epsilon}\langle i\sigma_2 + |\xi-\xi_1|^{2\alpha}\rangle^{1+2\epsilon}} \le C. \tag{3.49}$$

By symmetry between ξ_1 and $\xi-\xi_1$, we can easily derive the following corollary:

Corollary 3.4. *Let* $\rho = (a+2\alpha-1)/2 - (11+5a+10\alpha)\epsilon/2$ *and let* $\alpha > 0$. *If* $\max\{|\sigma|, |\sigma_1|, |\sigma_2|\} \gtrsim |\xi|\,|\xi_1|\,|\xi-\xi_1|^a$, *then* (3.49) *holds.*

Proof of Lemma 3.4. It follows that $|\xi| \sim |\xi-\xi_1|$ in $B(\xi,\tau)$. As in the proof of Lemma 3.3, we split $B(\xi,\tau)$ into three regions $B_1(\xi,\tau)$, $B_2(\xi,\tau)$ and $B_3(\xi,\tau)$.

Estimate in B_1. It follows from the assumption of the lemma that $\langle\sigma\rangle \gtrsim \langle\xi\rangle\langle\xi_1\rangle^a\langle\xi-\xi_1\rangle$ in B_1. With the aid of $\langle\sigma\rangle \gtrsim \langle\xi\rangle\langle\xi_1\rangle^a\langle\xi-\xi_1\rangle$, we have

$$\begin{aligned}\mathrm{I} &\lesssim \iint_{B_1(\xi,\tau)} \frac{\langle\xi\rangle^{1+5\epsilon-2\rho}\langle\xi_1\rangle^{2\rho-a(1-5\epsilon)}\langle\xi-\xi_1\rangle^{2\rho-(1-5\epsilon)}}{\langle\sigma_1\rangle^{1+2\epsilon}\langle|\xi-\xi_1|^{2\alpha}\rangle^{1+2\epsilon}}\,d\xi_1\,d\tau_1 \\ &\lesssim \iint_{B_1(\xi,\tau)} \frac{\langle\xi\rangle^{1+5\epsilon-2\rho}\langle\xi_1\rangle^{2\rho-a(1-5\epsilon)}\langle\xi-\xi_1\rangle^{2\rho-(1-5\epsilon)-2\alpha(1+2\epsilon)}}{\langle\sigma_1\rangle^{1+2\epsilon}}\,d\xi_1\,d\tau_1.\end{aligned} \tag{3.50}$$

Since $|\xi| \sim |\xi - \xi_1|$ and $2|\xi_1| \le |\xi|$ hold, we have

$$\begin{aligned} &\langle \xi \rangle^{1+5\epsilon-2\rho} \langle \xi_1 \rangle^{2\rho-a(1-5\epsilon)} \langle \xi - \xi_1 \rangle^{2\rho-(1-5\epsilon)-2\alpha(1+2\epsilon)} \\ &\sim \langle \xi \rangle^{10\epsilon-2\alpha(1+2\epsilon)} \langle \xi_1 \rangle^{2\rho-a(1-5\epsilon)} \lesssim \langle \xi_1 \rangle^{2\rho-a-2\alpha+5(2+a)\epsilon-4\alpha\epsilon}. \end{aligned} \tag{3.51}$$

Hence it follows from Remark 3.2 that

$$\mathrm{I} \le C \iint_{B_1(\xi,\tau)} \frac{d\xi_1\, d\tau_1}{\langle \xi_1 \rangle^{1+(1+14\alpha)\epsilon} \langle \sigma_1 \rangle^{1+2\epsilon}} \le C. \tag{3.52}$$

Estimate in B_2. It follows from the assumption of the lemma that $\langle \sigma_1 \rangle \gtrsim \langle \xi \rangle \langle \xi_1 \rangle^a \langle \xi - \xi_1 \rangle$ in B_2. By virtue of $\langle \sigma_1 \rangle \gtrsim \langle \xi \rangle \langle \xi_1 \rangle^a \langle \xi - \xi_1 \rangle$, we have

$$\begin{aligned} \mathrm{I} &\lesssim \iint_{B_2(\xi,\tau)} \frac{\langle \xi \rangle^{1-2\rho-2\epsilon} \langle \xi_1 \rangle^{2\rho-a(1+2\epsilon)} \langle \xi - \xi_1 \rangle^{2\rho-(1+2\epsilon)}}{\langle |\xi|^{2\alpha} \rangle^{1-5\epsilon} \langle \sigma_2 \rangle^{1+2\epsilon}} d\xi_1\, d\tau_1 \\ &\lesssim \iint_{B_2(\xi,\tau)} \frac{\langle \xi \rangle^{1-2\rho-2\epsilon-2\alpha(1-5\epsilon)} \langle \xi_1 \rangle^{2\rho-a(1+2\epsilon)} \langle \xi - \xi_1 \rangle^{2\rho-(1+2\epsilon)}}{\langle \sigma_2 \rangle^{1+2\epsilon}} d\xi_1\, d\tau_1. \end{aligned} \tag{3.53}$$

Since $|\xi| \sim |\xi - \xi_1|$ and $2|\xi_1| \le |\xi|$ hold, we obtain

$$\begin{aligned} &\langle \xi \rangle^{1-2\rho-2\epsilon-2\alpha(1-5\epsilon)} \langle \xi_1 \rangle^{2\rho-a(1+2\epsilon)} \langle \xi - \xi_1 \rangle^{2\rho-(1+2\epsilon)} \\ &\sim \langle \xi \rangle^{-4\epsilon-2\alpha(1-5\epsilon)} \langle \xi_1 \rangle^{2\rho-a(1+2\epsilon)} \lesssim \langle \xi_1 \rangle^{2\rho-a-2\alpha-2(2+a)\epsilon+10\alpha\epsilon}. \end{aligned} \tag{3.54}$$

Hence it follows from Remark 3.2 that

$$\mathrm{I} \le C \iint_{B_2(\xi,\tau)} \frac{d\xi_1\, d\tau_1}{\langle \xi_1 \rangle^{1+(15+7a)\epsilon} \langle \sigma_2 \rangle^{1+2\epsilon}} \le C. \tag{3.55}$$

Estimate in B_3. It follows from the assumption of the lemma that $\langle \sigma_2 \rangle \gtrsim \langle \xi \rangle \langle \xi_1 \rangle^a \langle \xi - \xi_1 \rangle$ in B_3. By symmetry between $i\sigma_1 + |\xi_1|^{2\alpha}$ and $i\sigma_2 + |\xi - \xi_1|^{2\alpha}$, we can prove this case by following the analogous argument in B_2.

Summing up, we finish the proof. □

3.4. Preliminaries III.

Lemma 3.5. *Let* $\rho = (a + 2\alpha - 1)/2 - (11 + 5a + 10\alpha)\epsilon/2$ *with* $a + 2\alpha \le 3$ *and* $\alpha > (3 - a)/4 \ge 1/2$. *For any fixed* (ξ, τ), *we introduce the following integral region*:

$$D(\xi, \tau) = \{(\xi_1, \tau_1) \in \mathbf{R}^2,\ \ |\xi_1| \le 1\}.$$

Then for any $\epsilon > 0$ *there exists* $C > 0$, *depending only on* ϵ, *such that*

$$\mathrm{I} = \frac{|\xi|^2 \langle \xi \rangle^{-2\rho}}{\langle i\sigma + |\xi|^{2\alpha} \rangle^{1-5\epsilon}} \iint_{D(\xi,\tau)} \frac{\langle \xi_1 \rangle^{2\rho} \langle \xi - \xi_1 \rangle^{2\rho}\, d\xi_1\, d\tau_1}{\langle i\sigma_1 + |\xi_1|^{2\alpha} \rangle^{1+2\epsilon} \langle i\sigma_2 + |\xi - \xi_1|^{2\alpha} \rangle^{1+2\epsilon}} \le C. \tag{3.56}$$

Proof. By direct calculations, we have

$$
\begin{aligned}
\mathrm{I} &\le C \frac{\langle |\xi|^{2\alpha}\rangle^{(1-\rho)/\alpha}}{\langle i\sigma + |\xi|^{2\alpha}\rangle^{1-5\epsilon}} \int_{\tau_1}\int_{|\xi_1|\le 1} \frac{\langle |\xi-\xi_1|^{2\alpha}\rangle^{\rho/\alpha}\, d\xi_1\, d\tau_1}{\langle i\sigma_1 + |\xi_1|^{2\alpha}\rangle^{1+2\epsilon}\langle i\sigma_2 + |\xi-\xi_1|^{2\alpha}\rangle^{1+2\epsilon}} \\
&\le C \frac{1}{\langle i\sigma + |\xi|^{2\alpha}\rangle^{1-(1-\rho)/\alpha-5\epsilon}} \\
&\quad\times \int_{\tau_1}\int_{|\xi_1|\le 1} \frac{d\xi_1\, d\tau_1}{\langle i\sigma_1 + |\xi_1|^{2\alpha}\rangle^{1+2\epsilon}\langle i\sigma_2 + |\xi-\xi_1|^{2\alpha}\rangle^{1-\rho/\alpha+2\epsilon}}.
\end{aligned}
\tag{3.57}
$$

Note that $1-(1-\rho)/\alpha - 5\epsilon = 2-(3-a)/2\alpha - (11+5a+20\alpha)\epsilon/2\alpha > 0$ from the assumption. Hence it follows that

$$
\begin{aligned}
\mathrm{I} &\le C \int_{\tau_1}\int_{|\xi_1|\le 1} \frac{d\xi_1\, d\tau_1}{\langle\sigma_1\rangle^{1+2\epsilon}\langle\sigma_2\rangle^{1-\rho/\alpha+2\epsilon}} \\
&\le C \int_{|\xi_1|\le 1}\int_{\tau_1} \frac{d\tau_1}{\langle \min\{|\sigma_1|, |\sigma_2|\}\rangle^{2-\rho/\alpha+4\epsilon}}\, d\xi_1 \\
&= C \int_{|\xi_1|\le 1}\int_{\tau_1} \frac{d\tau_1}{\langle\tau_1\rangle^{2-\rho/\alpha+4\epsilon}}\, d\xi_1 \le C,
\end{aligned}
\tag{3.58}
$$

where $2-\rho/\alpha+4\epsilon = 1+(1-a)/2\alpha + (11+5a+18\alpha)\epsilon/2\alpha > 1$ from the assumption $a \le 1$.

Hence we establish our statement. □

REMARK 3.3. When $a=1$, we do not need to assume that $\alpha > 1/2$ in Lemma 3.5. Indeed, by following the proof of [17, Lemma 2.4], we can prove (3.56) without the assumption $\alpha > 1/2$. On the other hand, for the construction of the bilinear estimates (Proposition 3.1), Lemmas 3.1 and 3.3 are not needed when $a=1$. Hence, we need not impose the assumption $\alpha > 1/2$ on the bilinear estimates when $a=1$.

REMARK 3.4. Lemmas 3.1 to 3.4 are estimates over the domain of interactions of high and high frequencies ($|\xi_1| \ge 1$ and $|\xi - \xi_1| \ge 1$), and Lemma 3.5 over the domain of low and high interactions ($|\xi_1| < 1$ or $|\xi-\xi_1| < 1$). The point of the proofs of these lemmas is to utilize the dissipative property in the low-high interactions and the dispersive-dissipative one in the high-high interactions.

The dissipative effect plays an important role in the proof of Lemma 3.5. In fact, the proof of Lemma 3.5 is independent of the dispersive property, that is

$$
\frac{|\xi|^2\langle\xi\rangle^{-2\rho}}{\langle i\tau + |\xi|^{2\alpha}\rangle^{1-5\epsilon}} \iint_{D(\xi,\tau)} \frac{\langle\xi_1\rangle^{2\rho}\langle\xi-\xi_1\rangle^{2\rho}\, d\xi_1\, d\tau_1}{\langle i\tau_1 + |\xi_1|^{2\alpha}\rangle^{1+2\epsilon}\langle i(\tau-\tau_1) + |\xi-\xi_1|^{2\alpha}\rangle^{1+2\epsilon}} \le C
\tag{3.59}
$$

holds for the same exponents ρ, α.

If the dispersive effect is missing, the proofs of Lemmas 3.1 to 3.4 (high-high interactions) break down for $\rho > 0$. In the proofs of Lemmas 3.1 to 3.4, we use the dispersive-dissipative effect rather than the dispersive one. We shall take Lemma 3.1 for example. In Lemma 3.1, we have proved that

$$\frac{\langle\xi_1\rangle^{2\rho}}{\langle i\sigma_1+|\xi_1|^{2\alpha}\rangle^{1+2\epsilon}}\iint_{A(\xi_1,\tau_1)}\frac{|\xi|^2\langle\xi\rangle^{-2\rho}\langle\xi-\xi_1\rangle^{2\rho}\,d\xi\,d\tau}{\langle i\sigma+|\xi|^{2\alpha}\rangle^{1-5\epsilon}\langle i\sigma_2+|\xi-\xi_1|^{2\alpha}\rangle^{1+2\epsilon}}\le C. \tag{3.60}$$

In the KdV-B case ($\sigma=\tau-\xi^3$, $\alpha=1$), the weight $\langle i(\tau-\xi^3)+\xi^2\rangle$ has dispersive and dissipative characteristics. Hence, as we have seen in the proof, we can take $\rho<1$ with the relation $\langle\tau-\xi^3\rangle\ge|\xi\xi_1(\xi-\xi_1)|$ and cancellation by $|\xi|^2$.

On the other hand, to show the well-posedness of the KdV equation in our method, we need to show for example that

$$\frac{\langle\xi_1\rangle^{2\rho}}{\left\langle\tau_1-\xi_1^3\right\rangle^{1+2\epsilon}}\iint_{A(\xi_1,\tau_1)}\frac{|\xi|^2\langle\xi\rangle^{-2\rho}\langle\xi-\xi_1\rangle^{2\rho}\,d\xi\,d\tau}{\langle\tau-\xi^3\rangle^{1-5\epsilon}\langle\tau-\tau_1-(\xi-\xi_1)^3\rangle^{1+2\epsilon}}\le C. \tag{3.61}$$

Since we cannot use the dissipative effect in the KdV case, we should set $\rho<3/4$ as was proved by Kenig, Ponce, and Vega [17]. The condition for the convergence of integral (3.61) is more restrictive than that of the KdV-B case for the lack of the dissipative term. Thus, the KdV-B equation can be solved in weaker spaces than that of the KdV equation.

Similarly, we also consider the BO and gBO-B equations. In the BO case, the integral

$$\frac{\langle\xi_1\rangle^{2\rho}}{\langle\tau_1-\xi_1|\xi_1|\rangle^{1+2\epsilon}}\iint_{A(\xi_1,\tau_1)}\frac{|\xi|^2\langle\xi\rangle^{-2\rho}\langle\xi-\xi_1\rangle^{2\rho}\,d\xi\,d\tau}{\langle\tau-\xi|\xi|\rangle^{1-5\epsilon}\langle\tau-\tau_1-(\xi-\xi_1)|\xi-\xi_1|\rangle^{1+2\epsilon}} \tag{3.62}$$

is not convergent. On the other hand, in the gBO-B case with $a=0$ ($\sigma=\tau-\xi|\xi|$), the integral (3.60) is certainly convergent with the aid of the dissipative part $|\xi|^{2\alpha}$. The restriction of $s=-\rho>(1-2\alpha)/2$, $\alpha>3/4$ is a necessary condition for the convergence of the integral (3.60).

4. Proof of Proposition 3.1

Let $s>-(a+2\alpha-1)/2$. In this section, we shall prove

$$|\mathrm{I}|\le CT^{\mu}\|f\|_{L_x^2L_t^2}\|g\|_{L_x^2L_t^2}\|h\|_{L_x^2L_t^2}, \tag{4.1}$$

where

(4.2)

$$\mathrm{I}=\int_{\mathbf{R}^4}\frac{\xi\overline{\widehat{h}(\xi,\tau)}\langle\xi\rangle^s}{\langle i\sigma+|\xi|^{2\alpha}\rangle^{1-b-\delta}}\frac{\widehat{g}(\xi_1,\tau_1)\langle\xi_1\rangle^{-s}}{\langle i\sigma_1+|\xi_1|^{2\alpha}\rangle^{b}}\frac{\widehat{f}(\xi-\xi_1,\tau-\tau_1)\langle\xi-\xi_1\rangle^{-s}}{\langle i\sigma_2+|\xi-\xi_1|^{2\alpha}\rangle^{b}}\,d\xi\,d\tau\,d\xi_1\,d\tau_1.$$

It suffices to show (4.1) only in the case $s=-\rho=-(a+2\alpha-1)/2+(11+5a+10\alpha)\epsilon/2$. By Fubini's theorem, we can assume that $\widehat{f},\widehat{g},\widehat{h}\ge0$.

We divide $\mathbf{R}^4$ into five regions D_1, D_2, D_3, D_4, and D_5;

$$\begin{aligned}
D_1 &= \{(\xi,\xi_1,\tau,\tau_1)\in\mathbf{R}^4\colon |\xi_1|\ge 1,\ |\xi-\xi_1|\ge 1,\ \text{(3.6) holds}\},\\
D_2 &= \{(\xi,\xi_1,\tau,\tau_1)\in\mathbf{R}^4\colon |\xi_1|\ge 1,\ |\xi-\xi_1|\ge 1,\ \text{(3.7) holds}\},\\
D_3 &= \{(\xi,\xi_1,\tau,\tau_1)\in\mathbf{R}^4\colon |\xi_1|\ge 1,\ |\xi-\xi_1|\ge 1,\ \text{(3.8) holds}\},\\
D_4 &= \{(\xi,\xi_1,\tau,\tau_1)\in\mathbf{R}^4\colon |\xi_1|\ge 1,\ |\xi-\xi_1|\ge 1,\ \text{(3.9) holds}\},\\
D_5 &= \{(\xi,\xi_1,\tau,\tau_1)\in\mathbf{R}^4\colon |\xi_1|\le 1 \text{ or } |\xi-\xi_1|\le 1\}.
\end{aligned}$$

REMARK 4.1. When $a = 1$, we have only to set $D_1 = D_3 = \phi$. See Remark 3.1.

Furthermore we split these regions into two parts respectively;

$$D_j = D_{j,A}\cup D_{j,B} \quad (j = 1, 2, 3, 4),$$

where

$$D_{j,A} = \{(\xi,\xi_1,\tau,\tau_1)\in D_j : |\xi|\le 2|\xi_1|\},\quad D_{j,B} = \{(\xi,\xi_1,\tau,\tau_1)\in D_j : |\xi|\ge 2|\xi_1|\}.$$

And we need not divide D_5. According to these integral regions, we divide the integral $\mathrm{I} = \sum_{j=1}^4 \mathrm{I}_{D_{j,A}} + \sum_{j=1}^4 \mathrm{I}_{D_{j,B}} + \mathrm{I}_{D_5}$, where

$$\mathrm{I}_{\widetilde{D}} = \int_{\widetilde{D}} \frac{\xi\overline{\widehat{h}(\xi,\tau)}\langle\xi\rangle^{-\rho}}{\langle i\sigma+|\xi|^{2\alpha}\rangle^{1/2-2\epsilon}}\frac{\widehat{g}(\xi_1,\tau_1)\langle\xi_1\rangle^{\rho}}{\langle i\sigma_1+|\xi_1|^{2\alpha}\rangle^{1/2+\epsilon}}\frac{\widehat{f}(\xi-\xi_1,\tau-\tau_1)\langle\xi-\xi_1\rangle^{\rho}}{\langle i\sigma_2+|\xi-\xi_1|^{2\alpha}\rangle^{1/2+\epsilon}}\,d\xi\,d\tau\,d\xi_1\,d\tau_1$$

and we set $s = -\rho = -(a+2\alpha-1)/2 + (11+5a+10\alpha)\epsilon/2$, $\delta = \epsilon$ and $b = 1/2+\epsilon$. Each integral $\mathrm{I}_{\widetilde{D}}$ is estimated according to the following two cases:

(I) Case of $\widetilde{D} = D_{1,A}\cup D_{2,A}\cup D_{3,A}\cup D_{4,A}$. Using Schwarz inequality and applying four lemmas in Section 3.2, we have

$$\begin{aligned}
\mathrm{I}_{\widetilde{D}} \le \sup_{\xi_1,\tau_1}\Bigg[&\frac{\langle\xi_1\rangle^{\rho}}{\langle i\sigma_1+|\xi_1|^{2\alpha}\rangle^{1/2+\epsilon}}\\
&\times\left(\int_{\widetilde{D}(\xi_1,\tau_1)}\frac{|\xi|^2\langle\xi-\xi_1\rangle^{2\rho}\langle\xi\rangle^{-2\rho}}{\langle i\sigma+|\xi|^{2\alpha}\rangle^{1-5\epsilon}\langle i\sigma_2+|\xi-\xi_1|^{2\alpha}\rangle^{1+2\epsilon}}\,d\xi\,d\tau\right)^{1/2}\Bigg]\\
&\times\int_{\mathbf{R}^2}\widehat{g}(\xi_1,\tau_1)\left(\int_{\mathbf{R}^2}\frac{\big|\widehat{h}(\xi,\tau)\big|^2}{\langle\sigma\rangle^{\epsilon}}\big|\widehat{f}(\xi-\xi_1,\tau-\tau_1)\big|^2\,d\xi\,d\tau\right)^{1/2}d\xi_1\,d\tau_1,
\end{aligned}\tag{4.3}$$

where $\widetilde{D}(\xi_1,\tau_1) = \big\{(\xi,\tau)\in\mathbf{R}^2\colon (\xi,\xi_1,\tau,\tau_1)\in\widetilde{D}\big\}$. Moreover from Schwarz inequality, Fubini's theorem and Lemma 2.3, we obtain

$$\mathrm{I}_{\widetilde{D}} \le CT^{\mu}\|f\|_{L^2(\mathbf{R}^2)}\|g\|_{L^2(\mathbf{R}^2)}\|h\|_{L^2(\mathbf{R}^2)}.\tag{4.4}$$

(II) Case of $\widetilde{D} = D_{1,B} \cup D_{2,B} \cup D_{3,B} \cup D_{4,B} \cup D_5$. In the same way, we can show that

$$
\begin{aligned}
\mathrm{I}_{\widetilde{D}} \le \sup_{\xi,\tau} \Bigg[& \frac{|\xi|\langle\xi\rangle^{-\rho}}{\langle i\sigma + |\xi|^{2\alpha}\rangle^{1/2-5\epsilon/2}} \\
& \times \left(\int_{\widetilde{D}(\xi,\tau)} \frac{\langle\xi_1\rangle^{2\rho}\langle\xi-\xi_1\rangle^{2\rho}}{\langle i\sigma_1 + |\xi_1|^{2\alpha}\rangle^{1+2\epsilon}\langle i\sigma_2 + |\xi-\xi_1|^{2\alpha}\rangle^{1+2\epsilon}} \, d\xi_1 \, d\tau_1 \right)^{1/2} \Bigg] \\
& \times \int_{\mathbf{R}^2} \frac{\overline{\widehat{h}(\xi,\tau)}}{\langle\sigma\rangle^{\epsilon/2}} \left(\int_{\mathbf{R}^2} |\widehat{g}(\xi_1,\tau_1)|^2 \left| \widehat{f}(\xi-\xi_1,\tau-\tau_1) \right|^2 d\xi_1 \, d\tau_1 \right)^{1/2} d\xi \, d\tau,
\end{aligned} \tag{4.5}
$$

where $\widetilde{D}(\xi,\tau) = \left\{(\xi_1,\tau_1) \in \mathbf{R}^2 : (\xi,\xi_1,\tau,\tau_1) \in \widetilde{D}\right\}$. Moreover, by virtue of five lemmas in Sections 3.3, 3.4 and Lemma 2.3, we obtain

$$
\mathrm{I}_{\widetilde{D}} \le CT^{\mu} \|f\|_{L^2(\mathbf{R}^2)} \|g\|_{L^2(\mathbf{R}^2)} \|h\|_{L^2(\mathbf{R}^2)}. \tag{4.6}
$$

Therefore Proposition 3.1 follows from (4.4) and (4.6). □

5. Proof of Theorem 1.1

5.1. Existence. Let $u_0(x) \in H^s(\mathbf{R})$ with $s > -(a+2\alpha-1)/2$, $a+2\alpha \le 3$ and $\alpha > (3-a)/4 \ge 1/2$. We may assume $T < 1$. Let us choose $0 < 8\alpha\epsilon < s+(a+2\alpha-1)/2$ and take b such that $2b-1 = 2\epsilon$.

We define the map

$$
F(\omega) = \psi(t)W(t)u_0 - \frac{1}{2}\chi_{\mathbf{R}_+}(t)\psi(t)\int_0^t W(t-t')\partial_x(\psi_T\omega(t'))^2 \, dt' \tag{5.1}
$$

and suppose ω is in the ball

$$
\mathcal{B}_M = \left\{ u \in X^{s-\alpha(2b-1),b} : \|u\|_{X^{s-\alpha(2b-1),b}} \le M \right\}, \tag{5.2}
$$

where $M = 2C_0\|u_0\|_{H^s}$. In what follows, we shall show that $F(\omega)$ is a contraction on the ball $\mathcal{B}_M$ for $[0,T]$.

By virtue of Propositions 2.1, 2.3, 3.1 and 2.2, we have

$$
\begin{aligned}
\|F(u)\|_{X^{s-\alpha(2b-1),b}} &\le \|\psi(t)W(t)u_0\|_{X^{s-\alpha(2b-1),b}} \\
&\quad + \frac{1}{2}\left\| \chi_{\mathbf{R}_+}(t)\psi(t)\int_0^t W(t-t')\partial_x(\psi_T u(t'))^2 \, dt' \right\|_{X^{s-\alpha(2b-1),b}} \\
&\le C_0\|u_0\|_{H^s} + C_\delta \left\| \partial_x(\psi_T u)^2 \right\|_{X^{s-\alpha(2b-1),b-1+\delta}} \\
&\le C_0\|u_0\|_{H^s} + C_\delta T^{\mu} \|\psi_T u\|^2_{X^{s-\alpha(2b-1),b}} \\
&\le C_0\|u_0\|_{H^s} + C_1 T^{\mu-2\epsilon} \|u\|^2_{X^{s-\alpha(2b-1),b}},
\end{aligned} \tag{5.3}
$$

where $s-\alpha(2b-1) > -(a+2\alpha-1)/2+6\alpha\epsilon$ and $1-2b=-2\epsilon$. Therefore, for $u \in \mathcal{B}_M$

$$\|F(u)\|_{X^{s-\alpha(2b-1),b}} \le \frac{M}{2} + C_1 T^{\mu-2\epsilon} M^2. \tag{5.4}$$

Hence it follows that for $T = (4MC_1)^{-1/(\mu-2\epsilon)}$, $F(u) \in \mathcal{B}_M$.

Similarly, it follows that for $u,\ v \in \mathcal{B}_M$

$$\begin{aligned} \|F(u)-F(v)\|_{X^{s-\alpha(2b-1),b}} &\le 2MC_1T^{\mu-2\epsilon}\|u-v\|_{X^{s-\alpha(2b-1),b}} \\ &= \frac{1}{2}\|u-v\|_{X^{s-\alpha(2b-1),b}}, \end{aligned} \tag{5.5}$$

from which F is a contraction on $\mathcal{B}_M$. By virtue of the contraction mapping principle, $F(u)$ has the fixed point in the ball $\mathcal{B}_M$. Therefore there exists a unique solution $u(t)$ in $\mathcal{B}_M$ for $T < (4MC_1)^{-1/(\mu-2\epsilon)}$ satisfying

$$u(t) = \psi(t)\left[W(t)u_0 - \frac{1}{2}\chi_{\mathbf{R}_+}(t)\int_0^t W(t-t')\partial_x(\psi_T u(t'))^2\,dt'\right]. \tag{5.6}$$

Hence $u(t)$ solves the integral equation associated with the IVP (1.1) in the time interval $[0,T]$.

5.2. Continuous dependence. In this section, we shall show the continuous dependence upon the initial data. We choose $0 < 8\alpha\epsilon < s+(a+2\alpha-1)/2$ and take b such that $2b-1=2\epsilon$. Let u and v be the solutions obtained in Section 5.1 with data u_0 and v_0 respectively.

As in Section 5.1, with the aid of Propositions 2.1, 2.3, 3.1 and 2.2, we obtain

$$\begin{aligned} \|u-v\|_{X^{s-\alpha(2b-1),b}} &\le C_0\|u_0-v_0\|_{H^s} + 2MC_1T^{\mu-2\epsilon}\|u-v\|_{X^{s-\alpha(2b-1),b}} \\ &\le C_0\|u_0-v_0\|_{H^s} + \frac{1}{2}\|u-v\|_{X^{s-\alpha(2b-1),b}} \end{aligned} \tag{5.7}$$

for $u, v \in \mathcal{B}_M$ and for $T < (4MC_1)^{-1/(\mu-2\epsilon)}$. Hence

$$\|u-v\|_{X^{s-\alpha(2b-1),b}} \le 2C_0\|u_0-v_0\|_{H^s}. \tag{5.8}$$

Moreover, by virtue of Propositions 2.4, 3.1, 2.2 and (5.8), we have

$$\begin{aligned} \|u(t)-v(t)\|_{H^s} &\le \|\psi(t)W(t)(u_0-v_0)\|_{H^s} \\ &\quad + \frac{1}{2}\left\|\psi(t)\chi_{\mathbf{R}_+}(t)\int_0^t W(t-t')\partial_x\left(\psi_T^2(u-v)(u+v)(t')\right)dt'\right\|_{H^s} \\ &\le C\|W(t)(u_0-v_0)\|_{H^s} + C\left\|\partial_x\left(\psi_T^2(u-v)(u+v)\right)\right\|_{X^{s-2\alpha\epsilon,b-1+\epsilon}} \\ &\le C_0\|u_0-v_0\|_{H^s} + C_1T^{\mu-2\epsilon}\|u-v\|_{X^{s-2\alpha\epsilon,b}}\|u+v\|_{X^{s-2\alpha\epsilon,b}} \\ &\le C_0\|u_0-v_0\|_{H^s} + 2C_1T^{\mu-2\epsilon}M\|u-v\|_{X^{s-2\alpha\epsilon,b}} \\ &\le C_0\|u_0-v_0\|_{H^s} + \frac{1}{2}\|u-v\|_{X^{s-2\alpha\epsilon,b}} \\ &\le 2C_0\|u_0-v_0\|_{H^s}, \end{aligned} \tag{5.9}$$

which implies the continuous dependence on the initial data.

5.3. Uniqueness and global existence. The proof of these parts are the same as in the previous paper. We refer to [25]. See also [2] for uniqueness.

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Department of Mathematics
Tokyo University of Science
26 Wakamiya-cho, Shinjuku-ku, Tokyo, 162-0827
Japan
e-mail: otan@aug.rikadai.jp