

# TWISTED NOVIKOV HOMOLOGY AND CIRCLE-VALUED MORSE THEORY FOR KNOTS AND LINKS

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## Abstract

The Morse-Novikov number  $\mathcal{MN}(L)$  of a smooth link  $L$  in the three-dimensional sphere is by definition the minimal possible number of critical points of a regular circle-valued Morse function on the link complement (the term regular means that the Morse function must have nice behaviour in a tubular neighbourhood of  $L$ ). Novikov homology provides lower bounds for  $\mathcal{MN}(L)$ . In the present paper we introduce the notion of twisted Novikov homology, which allows to obtain better lower bounds for  $\mathcal{MN}(L)$  than the usual Novikov homology. Our twisted Novikov homology is a module over the Novikov ring  $\mathbf{Z}((t))$  but it contains the information coming from the non abelian homological algebra of the group ring of the fundamental group of the link complement. Using this technique we prove that the Morse-Novikov number of the knot  $n\mathfrak{C}$  (the connected sum of  $n$  copies of the Conway knot) is not less than  $2n/5$  for every positive integer  $n$ . We prove also that  $\mathcal{MN}(n\mathfrak{C})$  is not greater than  $2n$ . The same estimates hold for the Morse-Novikov numbers of the connected sum of  $n$  copies of the Kinoshita-Terasaka knot.

## 1. Introduction

Let  $L$  be an oriented link, that is, a  $C^\infty$  embedding of disjoint union of the oriented circles in  $S^3$ . The link  $L$  is called *fibred* if there is a fibration  $\phi: C_L = S^3 \setminus L \rightarrow S^1$  behaving “nicely” in a neighborhood of  $L$  (see Definition 2.5). If  $L$  is not fibred, it is still possible to construct a Morse map  $f: C_L \rightarrow S^1$  behaving nicely in a neighborhood of  $L$ ; such a map has necessarily a finite number of critical points. The minimal number of critical points of such map is an invariant of the link, called *Morse-Novikov number* of  $L$  and denoted by  $\mathcal{MN}(L)$ ; it was first introduced and studied in [21]. This invariant can be studied via the methods of the *Morse-Novikov theory*, in particular the Novikov inequalities [17] provide the lower estimates for the number  $\mathcal{MN}(L)$ . This inequalities can be considered as *fibering obstructions* for the link.

Recall that if a knot  $K$  is fibred, then its Alexander polynomial is monic (see [25], 10.G.9 or [1], Ch. 8, P.8.16). So the Alexander polynomial provides another fibering obstruction for knots, which is sufficiently powerful as to detect all non-fibred knots among the knots with  $\leq 10$  crossings (Kanenobu’s theorem, see [12]). There are

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non-fibred knots with 11 crossings having the trivial Alexander polynomial, for example the Kinoshita-Terasaka knot and the Conway knot (we shall study these knots in details in the present paper, see Section 5).

It is not difficult to prove that the Alexander polynomial of a knot  $K$  is monic if and only if the Novikov homology of  $K$  vanishes, so the fibering obstruction provided by the Novikov homology is equivalent to the one coming from the Alexander polynomial. The advantage of the Novikov homology is that it can give computable lower bounds for  $\mathcal{MN}(K)$  in the case of non-fibred knots (see [21] for examples of knots with arbitrarily large Morse-Novikov number).

Both the Alexander polynomial and the Novikov homology above are *abelian invariants*, that is, they are calculated from the homology of the infinite cyclic covering of  $C_L$ . More information is provided by the *non-abelian* coverings, although the corresponding invariants are more complicated.

Several non-abelian versions of the Alexander polynomial were developed in 90s (see the papers by X.S. Lin [16], M. Wada [27], T. Kitano [15], and Kirk-Livingston [14]). According to M. Wada's definition the twisted Alexander polynomial of a link  $L$  is a rational function in one or several commuting variables. It is associated to a representation of  $\pi_1(S^3 \setminus L)$  to  $GL(n, R)$  (where  $R$  is a commutative ring), and it is through this representation that the non-abelian invariants come into play. In the recent preprint [10] by H. Goda, T. Kitano, and T. Morifuji, it was shown that if a knot is fibred, then the twisted Alexander invariant is monic.

In the present paper we develop a version of the Novikov homology, which we call *twisted Novikov homology*. This part (Section 3) may be of independent interest for the Morse-Novikov theory. The twisted Novikov homology which we define is a module over the ring  $\mathbf{Z}((t))$  associated to a representation of the fundamental group, thus it allows to keep track of the non-abelian homological algebra associated to the group ring of the fundamental group of the considered space. As we shall show in this paper there are efficient tools for computing the twisted Novikov homology of the link. Theorem 4.2 gives a lower bound for the Morse-Novikov number  $\mathcal{MN}(L)$  in terms of the twisted Novikov homology.

We show that the twisted Novikov homology is additive with respect to the connected sum of knots. We apply these techniques to study the Morse-Novikov numbers of the knots  $m\mathcal{KT}$ ,  $m\mathcal{C}$ , where  $\mathcal{KT}$  is the Kinoshita-Terasaka knot [13],  $\mathcal{C}$  is the Conway knot [3], and  $mK$  stands for the connected sum of  $m$  copies of the knot  $K$ . We prove that

$$\mathcal{MN}(m\mathcal{KT}) = \mathcal{MN}(m\mathcal{C}) \geq \frac{2m}{5}.$$

The computation of the twisted Novikov homology for these knots was done with the help of Kodama's KNOT program (available at <http://www.math.kobe-u.ac.jp/~kodama/knot.html>).

Applying the techniques of the papers [7], [8] we prove that

$$\mathcal{MN}(m\mathfrak{RT}) = \mathcal{MN}(m\mathfrak{C}) \leq 2m.$$

We recall the corresponding notions and results from [5], [7] and [8] in Section 2.

In Section 7 we introduce and study the *asymptotic Morse-Novikov number* of a knot.

The final section of the paper contains a discussion about necessary and sufficient condition for a link to be fibred.

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## 2. Heegaard splitting for sutured manifold

The basic Morse theory gives a relationship of a Morse map and a handle decomposition for a manifold. In this section, we review the notion of Heegaard splitting for sutured manifold introduced in [7] and [8], and reveal the relationship with circle-valued Morse maps. Moreover, we present some properties which are used to determine Morse-Novikov numbers.

In this section, we assume that a link is always non-split.

Here, we recall the definition of a sutured manifold which was defined by D. Gabai [5].

**DEFINITION 2.1.** A *sutured manifold*  $(M, \gamma)$  is a compact oriented 3-dimensional manifold  $M$  together with a set  $\gamma \subset \partial M$  of mutually disjoint annuli  $A(\gamma)$  and tori  $T(\gamma)$ . In this paper, we treat the case  $T(\gamma) = \emptyset$ . The core curves of  $A(\gamma)$ , say  $s(\gamma)$ , are called the *sutures*. Every component of  $R(\gamma) = \text{cl}(\partial M - A(\gamma))$  is oriented, and  $R_+(\gamma)$  ( $R_-(\gamma)$  resp.) denotes the union of the components whose normal vectors point out (into resp.)  $M$ . Moreover, the orientation of  $R(\gamma)$  is coherent with respect to the orientations of  $s(\gamma)$ .

We say that a sutured manifold  $(M, \gamma)$  is a *product sutured manifold* if  $(M, \gamma)$  is homeomorphic to  $(F \times [0, 1], \partial F \times [0, 1])$  with  $R_+(\gamma) = F \times \{1\}$ ,  $R_-(\gamma) = F \times \{0\}$ ,  $A(\gamma) = \partial F \times [0, 1]$ , where  $F$  is a compact surface.

Let  $L$  be an oriented link in  $S^3$ , and let  $R$  be a Seifert surface of  $L$ . Set  $R_E = R \cap E(L)$  ( $E(L) = \text{cl}(S^3 - N(L))$ ), then  $(P, \delta) = (N(R_E, E(L)), N(\partial R_E, \partial E(L)))$  has a product sutured manifold structure  $(R_E \times [0, 1], \partial R_E \times [0, 1])$ . We call  $(P, \delta)$  a *product sutured manifold* for  $R$ . Thus  $P$  is homeomorphic to  $R_E \times [0, 1]$ , and then we denote

by  $R_+(\delta)$  ( $R_-(\delta)$  resp.) the surface  $R_E \times \{1\}$  ( $R_E \times \{0\}$  resp.). Let

$$(M, \gamma) = (\text{cl}(E(L) - P), \text{cl}(\partial E(L) - \delta))$$

with  $R_\pm(\gamma) = R_\mp(\delta)$ . We call  $(M, \gamma)$  a *complementary sutured manifold* for  $R$ . In this paper, we call this a *sutured manifold* for short.

Here we denote by  $N(X, Y)$  a regular neighborhood of  $X$  in  $Y$ .

In [2], the notion of compression body was introduced by A. Casson and C. Gordon. It is a generalization of a handlebody, and important to define a Heegaard splitting for 3-manifolds with boundaries.

DEFINITION 2.2. A *compression body*  $W$  is a cobordism rel  $\partial$  between surfaces  $\partial_+ W$  and  $\partial_- W$  such that  $W \cong \partial_+ W \times [0, 1] \cup 2\text{-handles} \cup 3\text{-handles}$  and  $\partial_- W$  has no 2-sphere components. We can see that if  $\partial_- W \neq \emptyset$  and  $W$  is connected,  $W$  is obtained from  $\partial_- W \times [0, 1]$  by attaching a number of 1-handles along the disks on  $\partial_- W \times \{1\}$  where  $\partial_- W$  corresponds to  $\partial_- W \times \{0\}$ .

We denote the number of these 1-handles by  $h(W)$ .

These notions enable us to define a Heegaard splitting for sutured manifold.

DEFINITION 2.3 ([7]).  $(W, W')$  is a Heegaard splitting for  $(M, \gamma)$  if

- (i)  $W, W'$  are connected compression bodies,
- (ii)  $W \cup W' = M$ ,
- (iii)  $W \cap W' = \partial_+ W = \partial_+ W'$ ,  $\partial_- W = R_+(\gamma)$ , and  $\partial_- W' = R_-(\gamma)$ .

DEFINITION 2.4 ([8]). Set  $h(R) = \min\{h(W) (= h(W')) \mid (W, W') \text{ is a Heegaard splitting for the sutured manifold of } R\}$ . We call  $h(R)$  the *handle number* of  $R$ .

Handle numbers of Seifert surfaces are studied in [7] and [8].

In order to state the relationship between the handle number and the Morse-Novikov number, we recall some definitions on circle-valued Morse map according to [21].

DEFINITION 2.5 ([21]). Let  $L$  be a link. A Morse map  $f: C_L \rightarrow S^1$  is said to be *regular* if there is a diffeomorphism

$$\phi: L \times D^2 \rightarrow U$$

where  $U$  is a neighbourhood of  $L$  in  $S^3$ , such that

$$f \circ \phi(x, y) = \frac{y}{|y|} \quad \text{for } y \neq 0.$$

A link  $L$  is called *fibred* if there is a regular Morse map  $f: S^3 \setminus L \rightarrow S^1$  without critical points.

For a regular Morse map we denote by  $S_i(f)$  the set of all critical points of index  $i$  and by  $m_i(f)$  the cardinality of  $S_i(f)$ . We say that a Morse map  $f: C_L \rightarrow S^1$  is *minimal* if it is regular and for every  $i$  the number  $m_i(f)$  is minimal possible among all regular maps homotopic to  $f$ .

We define  $\mathcal{MN}(L)$  as the number of critical points of the minimal Morse map.

**DEFINITION 2.6.** A regular Morse map  $f: C_L \rightarrow S^1$  is said to be *moderate* if it satisfies the all of the following:

- (i)  $m_0(f) = m_3(f) = 0$ ;
- (ii) all critical values corresponding to critical points of the same index coincide;
- (iii)  $f^{-1}(x)$  is a connected Seifert surface for any regular value  $x \in S^1$ .

**Theorem 2.7** ([21]). *Every link has a minimal Morse map which is moderate.*

**Corollary 2.8.** (1) *Let  $f$  be a moderate map, then  $m_1(f) = m_2(f)$ .*

(2) *Let  $f$  be a regular Morse map realizing  $\mathcal{MN}(L)$ , then  $\mathcal{MN}(L) = m_1(f) + m_2(f)$ .*

(3)  $\mathcal{MN}(L) = 2 \times \min\{h(R) \mid R \text{ is a Seifert surface for } L\}$ .

We denote by  $h(L)$  the minimum handle number among all Seifert surfaces of  $L$ . Note that  $L$  is a fibred if and only if  $h(L) = 0$ .

Thus we know that the handle number and Morse-Novikov number are same essentially, that is,

$$\mathcal{MN}(L) = 2 \times h(L).$$

We shall finish this section with some remarks on the behavior of the invariant introduced above with respect to connected sum and plumbing. Let us denote  $\natural$  the operation of plumbing. For a Seifert surface  $R$  of a link  $L$ , we set  $\mathcal{MN}(R) = 2 \times h(R)$ . Conjecture 6.3 in [21] says that

$$\mathcal{MN}(R_1 \natural R_2) \leq \mathcal{MN}(R_1) + \mathcal{MN}(R_2).$$

This conjecture follows from Theorem A in [8] in the case of non-split links. In the recent paper [11] by M. Hirasawa and L. Rudolph, the authors prove the conjecture in general case.

Let  $K_1$  and  $K_2$  be knots in  $S^3$ . Let  $K_1 \sharp K_2$  denote their connected sum. The following natural question is due to M. Boileau and C. Weber:

$$\text{Is it true that } \mathcal{MN}(K_1 \sharp K_2) = \mathcal{MN}(K_1) + \mathcal{MN}(K_2)?$$

As far as we know this question is still unanswered, and we know only that

$$(1) \quad \mathcal{MN}(K_1 \sharp K_2) \leq \mathcal{MN}(K_1) + \mathcal{MN}(K_2).$$

### 3. Twisted Novikov homology

Let  $R$  be a commutative ring, put

$$Q = R[t, t^{-1}], \quad \widehat{Q} = R((t)) = R[[t]][t^{-1}].$$

The ring  $Q$  is isomorphic to the group ring  $R[\mathbf{Z}]$ , via the isomorphism sending  $t \in Q$  to the element  $-1 \in \mathbf{Z}$ . The ring  $\widehat{Q}$  is then identified with the *Novikov completion* of  $R[\mathbf{Z}]$ . In the case when  $R$  is a field,  $\widehat{Q}$  is also a field. When  $R = \mathbf{Z}$ , the ring  $\widehat{Q}$  is PID. We shall need in this paper only the particular cases when  $R = \mathbf{Z}$ , or  $R$  is a field. Let  $X$  be a CW complex; let  $G = \pi_1 X$ , and let  $\xi: G \rightarrow \mathbf{Z}$  be a homomorphism. Let  $\rho: G \rightarrow GL(n, R)$  be a map such that  $\rho(g_1 g_2) = \rho(g_2) \rho(g_1)$  for every  $g_1, g_2 \in G$ . Such map will be called a *right representation* of  $G$ . The homomorphism  $\xi$  extends to a ring homomorphism  $\mathbf{Z}[G] \rightarrow Q$ , which will be denoted by the same symbol  $\xi$ . The tensor product  $\rho \otimes \xi$  (where  $\xi$  is considered as a representation  $G \rightarrow GL(1, Q)$ ) induces a right representation  $\rho_\xi: G \rightarrow GL(n, Q)$ . The composition of this right representation with the natural inclusion  $Q \hookrightarrow \widehat{Q}$  gives a right representation  $\widehat{\rho}_\xi: G \rightarrow GL(n, \widehat{Q})$ . Let us form a chain complex

$$(2) \quad \widehat{C}_*(\widetilde{X}; \xi, \rho) = \widehat{Q}^n \otimes_{\widehat{\rho}_\xi} C_*(\widetilde{X}).$$

Here  $\widetilde{X}$  is the universal cover of  $X$ ,  $C_*(\widetilde{X})$  is a module over  $\mathbf{Z}[G]$ , and  $\widehat{Q}^n$  is a right  $\mathbf{Z}G$ -module via the right representation  $\widehat{\rho}_\xi$ . Then (2) is a chain complex of free left modules over  $\widehat{Q}$ , and the same is true for its homology. The modules

$$\widehat{H}_*(X; \xi, \rho) = H_*(\widehat{C}_*(\widetilde{X}; \xi, \rho)),$$

will be called  *$\rho$ -twisted Novikov homology* or simply *twisted Novikov homology* if no confusion is possible. When these modules are finitely generated (this is the case for example for any  $X$  homotopy equivalent to a finite CW complex) we set

$$\widehat{b}_i(X; \xi, \rho) = \text{rk}_{\widehat{Q}}(\widehat{H}_i(X; \xi, \rho)), \quad \widehat{q}_i(X; \xi, \rho) = \text{t.n.}_{\widehat{Q}}(\widehat{H}_i(X; \xi, \rho))$$

where t.n. stands for the *torsion number* of the  $\widehat{Q}$ -module, that is the minimal possible number of generators of the torsion part over  $\widehat{Q}$ .

The numbers  $\widehat{b}_i(X; \xi, \rho)$  and  $\widehat{q}_i(X; \xi, \rho)$  can be recovered from the canonical decomposition of  $\widehat{H}_i(X; \xi, \rho)$  into a direct sum of cyclic modules. Namely, let

$$\widehat{H}_i(X; \xi, \rho) = \widehat{Q}^{\alpha_i} \oplus \left( \bigoplus_{j=1}^{\beta_i} \widehat{Q} / \lambda_j^{(i)} \widehat{Q} \right)$$

where  $\lambda_j^{(i)}$  are non-zero non-invertible elements of  $\widehat{Q}$  and  $\lambda_{j+1}^{(i)} | \lambda_j^{(i)} \forall j$ . (Such decomposition exists since  $\widehat{Q}$  is a PID.) Then  $\alpha_i = \widehat{b}_i(X; \xi, \rho)$  and  $\beta_i = \widehat{q}_i(X; \xi, \rho)$ . It is not difficult to show that we can always choose  $\lambda_j^{(i)} \in Q \forall i, \forall j$ .

When  $\rho$  is the trivial 1-dimensional representation, we obtain the usual Novikov homology, which can be also calculated from the infinite cyclic covering  $\bar{X}$  associated to  $\xi$ , namely

$$\widehat{H}_*(X; \xi, \rho) = \widehat{Q} \otimes_Q H_*(\bar{X}) \quad \text{for } \rho = 1: G \rightarrow GL(1, R).$$

If  $R$  is a field the numbers  $\widehat{q}_i(X; \xi, \rho)$  vanish (for every right representation  $\rho$ ), and the module  $\widehat{H}_*(X; \xi, \rho)$  is a vector space over the field  $\widehat{Q}$ .

#### 4. Novikov-type inequalities for knots and links

Now we shall apply the algebraic techniques developed in the previous section to the topology of knots and links. Let  $L \subset S^3$  be an oriented link and put  $C_L = S^3 \setminus L$ . Let  $G$  denote  $\pi_1(C_L)$ . There is a unique element  $\xi \in H^1(C_L, \mathbf{Z})$  such that for every positively oriented meridian  $\mu_i$  of a component of  $L$ , we have  $\xi(\mu_i) = 1$ . We shall identify the cohomology class  $\xi$  with the corresponding homomorphism  $G \rightarrow \mathbf{Z}$ .

Let  $\rho: G \rightarrow GL(n, R)$  be any right representation of  $G$  (where  $R = \mathbf{Z}$  or  $R$  is a field). The next theorem follows from the main theorem in [18]. See also Theorem 8.1 of the present paper.

**Theorem 4.1.** *Let  $f: C_L \rightarrow S^1$  be a regular Morse map. There is a free chain complex  $\mathcal{N}_*$  over  $\widehat{Q}$  such that*

1. *for every  $i$  the number of free generators of  $\mathcal{N}_*$  in degree  $i$  equals  $n \times m_i(f)$ ;*
2.  *$H_*(\mathcal{N}_*) \approx \widehat{H}_*(C_L; \xi, \rho)$ .*

We shall denote  $\widehat{H}_*(C_L, \xi, \rho)$  by  $\widehat{H}_*(L, \rho)$ . The numbers  $\widehat{b}_i(C_L, \rho)$  and  $\widehat{q}_i(C_L, \rho)$  will be denoted by  $\widehat{b}_i$  and  $\widehat{q}_i$  (we omit the cohomology class  $\xi$  in the notation since it is determined by the orientation of the link).

The next theorem follows from Theorem 4.1 by a simple algebraic argument.

**Theorem 4.2.** *Let  $f: C_L \rightarrow S^1$  be any regular map. Then*

$$(3) \quad m_i(f) \geq \frac{1}{n} \left( \widehat{b}_i(L, \rho) + \widehat{q}_i(L, \rho) + \widehat{q}_{i-1}(L, \rho) \right) \quad \text{for every } i.$$

**Corollary 4.3.** *If  $L$  is fibred, then  $\widehat{H}_*(L, \rho) = 0$ , and  $\widehat{b}_i(L, \rho) = \widehat{q}_i(L, \rho) = 0$  for every representation  $\rho$  and every  $i$ .*

**Proposition 4.4.** *The twisted Novikov numbers satisfy the following relations:*

$$(4) \quad \widehat{b}_i(L, \rho) = \widehat{q}_i(L, \rho) = \widehat{q}_2(L, \rho) = 0 \quad \text{for } i = 0, i \geq 3,$$

$$(5) \quad \widehat{b}_1(L, \rho) = \widehat{b}_2(L, \rho).$$

Proof. According to Theorem 3.3 of [21] there is a regular map  $f: C_L \rightarrow S^1$  such that  $f$  has only critical points of indices 1 and 2 and  $m_1(f) = m_2(f)$ . Using Theorem 4.2 we deduce (4). As for the point (5) it follows from the fact that the Euler characteristics of the chain complex  $\mathcal{N}_*$  is equal to 0.  $\square$

In view of the preceding theorem the non-trivial part of the Novikov inequalities is as follows:

$$(6) \quad m_1(f) \geq \frac{1}{n}(\widehat{b}_1(L, \rho) + \widehat{q}_1(L, \rho));$$

$$(7) \quad m_2(f) \geq \frac{1}{n}(\widehat{b}_1(L, \rho) + \widehat{q}_1(L, \rho)).$$

Let us consider some examples and particular cases.

1.  $R = \mathbf{Z}$  and  $\rho$  is the trivial 1-dimensional representation. The chain complex  $\widehat{C}_*(\widetilde{C}_L; \xi, \rho)$  is equal to the chain complex  $\widehat{Q} \otimes_{\widehat{Q}} C_*(\widetilde{C}_L)$ , where  $\widetilde{C}_L$  is the infinite cyclic covering of  $C_L$  associated to the cohomology class  $\xi$ . Thus the twisted Novikov homology in this case coincides with the Novikov homology for links studied in [21]. Theorem 4.2 and Proposition 4.4 in this case are reduced to Proposition 2.1 and the formulas (2)–(6) of [21].

2.  $R$  is a field. In this case the Novikov ring  $R((t))$  is also a field, and the torsion numbers  $\widehat{q}_i(L, \rho)$  vanish for every  $i$  and every representation  $\rho$ . The Novikov inequalities have the simplest possible form:

$$m_1(f) \geq \frac{1}{n} \widehat{b}_1(L, \rho) \leq m_2(f).$$

3. Now we shall investigate the twisted Novikov homology for the connected sum of knots. Let  $K_1, K_2$  be oriented knots in  $S^3$ , and put  $K = K_1 \sharp K_2$ . We have:

$$\pi_1(S^3 \setminus K) = \pi_1(S^3 \setminus K_1) *_Z \pi_1(S^3 \setminus K_2),$$

where  $Z$  is the infinite cyclic group generated by a meridian  $\mu$  of  $K$  (see [1], Ch. 7, Prop. 7.10). In particular the groups  $\pi_1(S^3 \setminus K_1), \pi_1(S^3 \setminus K_2)$  are naturally embedded into  $\pi_1(S^3 \setminus K)$ , and some meridian element  $\mu \in \pi_1(S^3 \setminus K)$  is the image of some meridian elements  $\mu_1 \in \pi_1(S^3 \setminus K_1), \mu_2 \in \pi_1(S^3 \setminus K_2)$ . Now let

$$\rho_1: \pi_1(S^3 \setminus K_1) \rightarrow GL(n, R), \quad \rho_2: \pi_1(S^3 \setminus K_2) \rightarrow GL(n, R)$$



be two right representations. Assume that  $\rho_1(\mu_1) = \rho_2(\mu_2)$ . Form the product representation  $\rho_1 * \rho_2: \pi_1(S^3 \setminus K) \rightarrow GL(n, R)$ .

**Theorem 4.5.**  $\widehat{H}_*(K, \rho_1 * \rho_2) \approx \widehat{H}_*(K_1, \rho_1) \oplus \widehat{H}_*(K_2, \rho_2)$ .

*Proof.* The complement  $C_K$  is the union of two subspaces  $C_1, C_2$  with  $C_i$  having the homotopy type of  $C_{K_i}$  (for  $i = 1, 2$ ). The intersection  $C_1 \cap C_2$  is homeomorphic to the twice punctured sphere  $\Delta' = S^2 \setminus \{*, *\}$ . The universal covering of  $C_K$  is therefore the union of two subspaces, which have the Novikov homology respectively equal to  $\widehat{H}_*(K_1, \rho_1)$  and  $\widehat{H}_*(K_2, \rho_2)$ . The intersection of these two subspaces has the same Novikov homology as  $\Delta'$ , and this module vanishes. Then a standard application of the Mayer-Vietoris sequence proves the result sought.  $\square$

**Corollary 4.6.** Denote by  $mK$  the connected sum of  $m$  copies of the knot  $K$ . Let  $\rho: \pi_1(S^3 \setminus K) \rightarrow GL(n, \mathbf{Z})$  be a representation. Let  $\rho^m: \pi_1(S^3 \setminus mK) \rightarrow GL(n, \mathbf{Z})$  be the product of  $m$  copies of representations  $\rho$ . Then

$$\widehat{q}_1(mK, \rho^m) = m \cdot \widehat{q}_1(K, \rho).$$

*Proof.* This follows from the purely algebraic equality:

$$\text{t.n.}(mN) = m \cdot (\text{t.n.}(N))$$

where  $N$  is any finitely generated module over a principal ideal domain, and  $mN$  stands for the direct sum of  $m$  copies of  $N$ .  $\square$

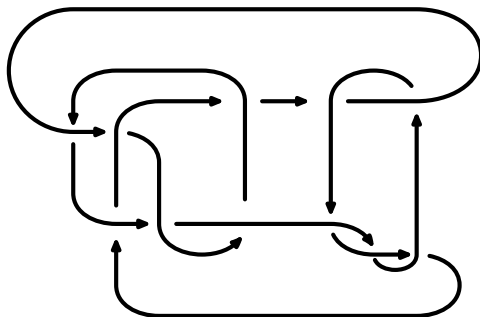
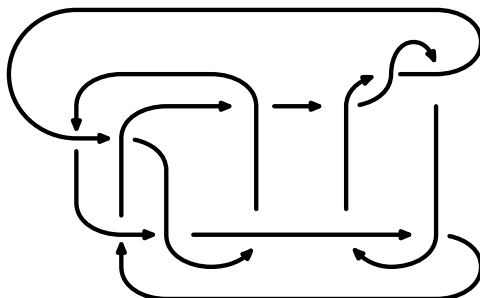
## 5. The Kinoshita-Terasaka and Conway knots: lower bounds for the Morse-Novikov numbers

The Kinoshita-Terasaka knot  $\mathcal{KT}$  was introduced in the paper [13], and the Conway knot  $\mathcal{C}$  was discovered by J. Conway much later [3]. These two knots are very much alike (see the figures below), and many classical invariants have the same value for these knots. Still these knots are different, as was proved by R. Riley in [24], and they can be distinguished by the twisted Alexander polynomials (see [27]).

These knots are not fibred. Indeed, for a fibred knot the degree of its Alexander polynomial equals to twice the genus of the knot, and the Alexander polynomial is trivial for both knots. In this section, we prove the following:

**Theorem 5.1.** *There is a right representation  $\rho: \pi_1(S^3 \setminus \mathcal{C}) \rightarrow SL(5, \mathbf{Z})$  such that  $\widehat{q}_1(\mathcal{C}, \rho) \neq 0$ .*

By Corollary 4.6, this theorem implies

Fig. 1. The Kinoshita-Terasaka knot  $\mathcal{KT}$ Fig. 2. The Conway knot  $\mathcal{C}$ 

$$(8) \quad \mathcal{MN}(m\mathcal{C}) \geq \frac{2m}{5} \quad \text{for every } m.$$

Proof. The Wirtinger presentation for the group  $\pi_1(S^3 \setminus \mathcal{C})$  has 11 generators  $s_i$  and 11 relations:

$$(9) \quad s_1 = s_{10}s_2s_{10}^{-1}, \quad s_2 = s_9^{-1}s_3s_9, \quad s_3 = s_6^{-1}s_4s_6, \quad s_4 = s_7^{-1}s_5s_7,$$

$$(10) \quad s_5 = s_{11}s_6s_{11}^{-1}, \quad s_6 = s_4^{-1}s_7s_4, \quad s_7 = s_2^{-1}s_8s_2, \quad s_8 = s_{11}s_9s_{11}^{-1},$$

$$(11) \quad s_9 = s_7^{-1}s_{10}s_7, \quad s_{10} = s_8s_{11}s_8^{-1}, \quad s_{11} = s_5s_1s_5^{-1}.$$

DEFINITION 5.2. A map  $\phi: G_1 \rightarrow G_2$  between two groups will be called *antihomomorphism* if  $\phi(ab) = \phi(b)\phi(a)$  for every  $a, b \in G_1$ .

There is an antihomomorphism  $h: \pi_1(S^3 \setminus \mathcal{C}) \rightarrow S(5)$  given by the following formulas:

$$(12) \quad h(s_1) = h(s_5) = h(s_6) = h(s_{11}) = (253), \quad h(s_2) = (234),$$

$$(13) \quad h(s_3) = h(s_7) = (123), \quad h(s_4) = (135), \quad h(s_8) = (142),$$

$$(14) \quad h(s_9) = (145), \quad h(s_{10}) = (345).$$

The image of  $h$  is contained in the subgroup  $A(5)$ . The group  $S(5)$  acts by permutation of coordinates on the free  $\mathbf{Z}$ -module  $\mathbf{Z}^5$  and we obtain therefore a right representation  $\rho: \pi_1(S^3 \setminus \mathcal{C}) \rightarrow SL(5, \mathbf{Z})$ . The twisted Novikov homology  $H_1(C_{\mathcal{C}}; \rho)$ , can be computed from the free  $\mathbf{Z}((t))$ -chain complex

$$C_0 \xleftarrow{\partial_1} C_1 \xleftarrow{\partial_2} C_2$$

where  $\text{rk } C_0 = 5$ ,  $\text{rk } C_1 = 55 = \text{rk } C_2$ . The generators of  $C_1$  correspond to  $s_i$ ,  $1 \leq i \leq 11$ , the generators of  $C_2$  correspond to the eleven relations (9)–(11), and the matrix of  $\partial_2$  is obtained by the Fox calculus using these relations. The Novikov homology in degree zero always vanishes, therefore the homomorphism  $\partial_1$  is epimorphic. We deduce that the rank of  $\partial_2$  is not more than 50. The determinant of the  $50 \times 50$ -minor of the matrix of  $\partial_2$  obtained from the matrix by omitting the last five columns and the last five rows, is equal to<sup>1</sup>

$$\begin{aligned} & -5t^{-29} + 14t^{-28} - 15t^{-27} + 16t^{-26} - 19t^{-25} + 10t^{-24} + 5t^{-23} - 24t^{-22} \\ & \quad + 34t^{-21} - 32t^{-20} + 34t^{-19} \\ & -24t^{-18} + 5t^{-17} + 10t^{-16} - 19t^{-15} + 16t^{-14} - 15t^{-13} + 14t^{-12} - 5t^{-11}. \end{aligned}$$

This polynomial is a non-invertible element of  $\mathbf{Z}((t))$ , since the leading coefficient is  $-5 \neq \pm 1$ . Therefore the torsion part of the twisted Novikov homology in dimension 1 is not zero, and  $\widehat{q}_1(\mathcal{C}; \rho) > 0$ . By Corollary 4.6 we deduce the inequality (8).  $\square$

By using the Kodama's KNOT program, we can show that the Kinoshita-Terasaka knot  $\mathcal{K}\mathfrak{T}$  has also a right representation  $\rho: \pi_1(S^3 \setminus \mathcal{K}\mathfrak{T}) \rightarrow SL(5, \mathbf{Z})$  such that  $\widehat{q}_1(\mathcal{K}\mathfrak{T}, \rho) \neq 0$ .

## 6. The Kinoshita-Terasaka and Conway knots: upper bounds for the Morse-Novikov numbers

In this section, we show that both  $\mathcal{MN}(\mathcal{K}\mathfrak{T})$  and  $\mathcal{MN}(\mathcal{C})$  are less than or equal to 2. Therefore  $\mathcal{MN}(m\mathcal{K}\mathfrak{T}) = \mathcal{MN}(m\mathcal{C}) \leq 2m$  by the inequality (1).

Here we use the minimal genus Seifert surfaces for  $\mathcal{K}\mathfrak{T}$  and  $\mathcal{C}$ , which were found in [6]. See Figs. 3 and 4. Since the proofs are same, we consider only the Conway knot  $\mathcal{C}$ .

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<sup>1</sup>This computation is provided by the Kodama's KNOT program, and also verified independently with the help of MAPLE.

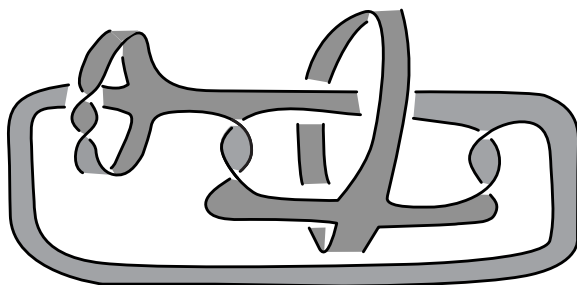
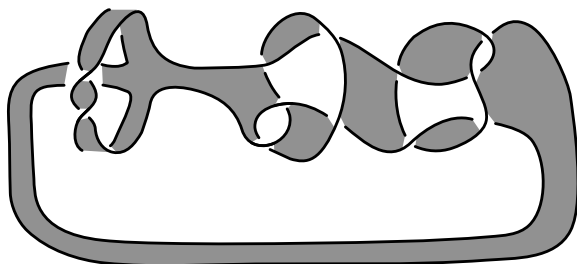
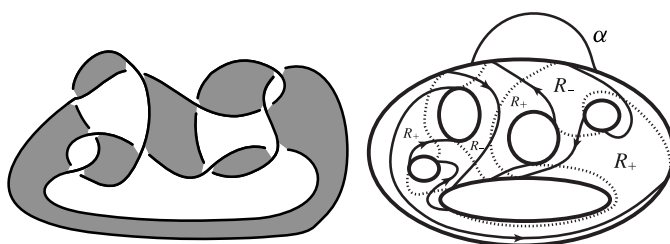
Fig. 3. The Kinoshita-Terasaka knot  $\mathcal{KT}$ Fig. 4. The Conway knot  $\mathcal{C}$ 

Fig. 5.

Since the Hopf band is a fiber surface, we may calculate the handle number of the Seifert surface illustrated in left-hand Fig. 5 by Theorem B in [8]. We call  $R$  this Seifert surface, and denote by  $(M, \gamma)$  the sutured manifold for  $R$ . Further, we name  $R_+(\gamma)$  and  $R_-(\gamma)$ , and let  $\alpha$  be an arc properly embedded in  $M$  as in Fig. 5. (We

abbreviate  $R_{\pm}(\gamma)$  to  $R_{\pm}$ , and  $\gamma$  to its core circles.) Then,  $N(R_+(\gamma) \cup \alpha, M)$  becomes a compression body  $W$  with  $h(W) = 1$ . By using Lemma 2.4 in [7], we can observe that  $\text{cl}(M - W)$  is a compression body  $W'$  such that  $\partial_- W' = R_-(\gamma)$  and  $h(W') = 1$ . Hence we have  $h(R) \leq 1$ , namely,  $\mathcal{MN}(\mathfrak{C}) \leq 2$ . This completes the proof.

## 7. Asymptotic Morse-Novikov number of a knot

Let  $K \subset S^3$  be an oriented knot. Let  $mK$  denote the connected sum of  $m$  copies of  $K$ . Observe that

$$\mathcal{MN}(m_1 K) + \mathcal{MN}(m_2 K) \geq \mathcal{MN}((m_1 + m_2)K),$$

therefore the sequence

$$\mu_m(K) = \frac{\mathcal{MN}(mK)}{m}$$

converges to some number (see [22], B.1, Ex. 98). This number will be called *asymptotic Morse-Novikov number of  $K$*  and denoted by  $\mu(K)$ .

**Corollary 7.1.** *The asymptotic Morse-Novikov numbers of Kinoshita-Terasaka knot  $\mathfrak{KT}$  and of the Conway knot  $\mathfrak{C}$  satisfy*

$$\frac{2}{5} \leq \mu(\mathfrak{KT}), \quad \mu(\mathfrak{C}) \leq 2.$$

## 8. Detecting fibred knots

In the previous sections we gave the estimates for the Morse-Novikov numbers of regular maps arising from linear representations of the fundamental group. In the present section we explore an approach which starts from the most general form of the Novikov homology and which should give the best possible lower bounds, although the corresponding invariants can be more difficult to compute.

Let us recall first the construction of the Novikov completion of a group  $G$  with respect to a homomorphism  $\xi: G \rightarrow \mathbf{Z}$ . Set  $\Lambda = \mathbf{Z}G$  and denote by  $\widehat{\Lambda}$  the abelian group of all functions  $G \rightarrow \mathbf{Z}$ . Equivalently,  $\widehat{\Lambda}$  is the set of all formal linear combinations  $\lambda = \sum_{g \in G} n_g g$  (not necessarily finite) of the elements of  $G$  with integral coefficients. For  $\lambda \in \widehat{\Lambda}$  and  $C \in \mathbf{R}$  set

$$\text{supp}(\lambda, C) = \{g \in G \mid n_g \neq 0, \quad \xi(g) \geq C\}.$$

Set

$$(15) \quad \widehat{\Lambda}_{\xi} = \{\lambda \in \widehat{\Lambda} \mid \text{supp}(\lambda, C) \text{ is finite for every } C \in \mathbf{R}\}.$$

Then  $\widehat{\Lambda}_\xi$  has a natural structure of a ring, containing  $\Lambda$  as a subring.

**Theorem 8.1.** *Let  $L$  be an oriented link in  $S^3$ . Let  $f: C_L \rightarrow S^1$  be a regular Morse map. Then there is a chain complex  $\mathcal{N}_*$  of free finitely generated  $\widehat{\Lambda}_\xi$ -modules (the Novikov complex), such that*

1. *the number of free generators in each degree  $i$  equals  $m_i(f)$ ,*
2. *there is a chain homotopy equivalence*

$$\phi: \mathcal{N}_* \longrightarrow \widehat{\Lambda}_\xi \otimes_{\Lambda} S_*(\widetilde{C}_L),$$

(where  $S_*(\widetilde{C}_L)$  stands for the singular chain complex of the universal covering  $\widetilde{C}_L$  of  $C_L$ ).

The analog of this theorem for the case of Morse maps  $f: M \rightarrow S^1$  of closed manifolds to a circle was first proved in [18]. The manifold  $C_L$  has a non-empty boundary, so the results of [18] can not be applied directly. Nevertheless one can show that the proof of the main theorem of [18] works also in the present case.

**REMARK 8.2.** In the paper [18], we worked with the convention that the fundamental group acts on the universal covering on the right. Theorem 8.1 above is the translation of the results of [18] to the language of the left modules.

**Corollary 8.3.** *Let  $L$  be an oriented link in  $S^3$ . If  $L$  is fibred, then*

$$H_* \left( \widehat{\Lambda}_\xi \otimes_{\Lambda} S_*(\widetilde{C}_L) \right) = 0.$$

**Conjecture 8.4.** An oriented link  $L$  is fibred if and only if

$$(16) \quad H_* \left( \widehat{\Lambda}_\xi \otimes_{\Lambda} S_*(\widetilde{C}_L) \right) = 0.$$

**REMARK 8.5.** It is known that for the general problem of fibering of an arbitrary closed manifold over a circle the vanishing of the Novikov homology is a condition which is only necessary but not in general sufficient. When this condition is fulfilled there is a secondary obstruction to fibering, which lies in the Whitehead group of the Novikov ring (see the paper [20] of A. Ranicki and A. Pajitnov). For the case of closed manifolds of dimension  $\geq 6$  the vanishing of this secondary obstruction is sufficient for the existence of the fibration (see [4], [19], [23], [20]).

However, combining the main theorem of [20] with the classical theorem of Waldhausen [28] (the Whitehead group of the link group vanishes) one can show that this secondary obstruction vanishes in the case of knots and links in  $S^3$ . Thus in the case of links the total obstruction to fibering provided by the Novikov complex is reduced

to the Novikov homology, and this gives the motivation for Conjecture 8.4.

The Novikov ring  $\widehat{\Lambda}_\xi$  is a complicated algebraic object, and the verification of the condition (16) is certainly a difficult algebraic task. The twisted Novikov homology as introduced and studied in the previous sections provides an effectively computable tools for evaluating the Novikov homology, and as we have seen in many examples, the twisted Novikov homology is often sufficient to compute the Morse-Novikov number. Thus we are led to the following problem.

**Problem 8.6.** Is it true that vanishing of the  $\rho$ -twisted Novikov homology for every right representation  $\rho$  implies the condition (16)?

A natural and a very interesting question would be to investigate the relations between the Problem 8.6 and the Problem 1.1 of [9], which asks whether the information contained in the twisted Alexander polynomials for all  $SL(2k, \mathbf{F})$ -representations (where  $\mathbf{F}$  is a field) is sufficient to decide whether a link is fibred.

ADDED IN PROOF. After this paper was accepted for publication, J.-Cl. Sikorav informed that Conjecture 8.4 had been solved affirmatively in [26].

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