Marcus, A.
Osaka J. Math.
**42** (2005), 453–462

# TILTING COMPLEXES FOR GROUP GRADED ALGEBRAS, II

ANDREI MARCUS

(Received October 9, 2003)

**Abstract**

We construct derived equivalences between group graded symmetric algebras, starting from equivalences between their 1-components obtained via a construction of J. Rickard. This applies to the verification of various cases of Broué's abelian defect group conjecture.

## 1. Introduction

This paper is a sequel of [5], and we are concerned with the problem of constructing derived equivalences between two algebras $R$ and $S$ graded by the finite group $G$. This is especially motivated by Broué's abelian defect group conjecture. If $K$ is a normal subgroup of the finite group $H$, with $G = H/K$, and $b$ is a $G$-invariant block with defect group $D$ of the group algebra $kK$, then the Brauer correspondent $c$ of $b$ in $kN_K(D)$ is a $G$-invariant block of $kN_K(D)$; under the assumption that $D$ is abelian, the conjecture predicts that there is a derived equivalence between the block algebras $kKb$ and $kN_K(D)c$; moreover, such an equivalence should be compatible with $p'$-extensions, that is, if $p$ does not divide the order of $G$, then the equivalence can be lifted to a derived equivalence between the $G$-graded $k$-algebras $S = kHb$ and $R = kN_H(D)c$ induced by a bounded complex of $G$-graded $(R, S)$-bimodules.

The main result of [5] is a graded version of Rickard's characterization of derived equivalences, which is then applied to find conditions implying that the tilting complexes constructed by T. Okuyama are compatible with $p'$-extensions as above.

In this paper we do a similar investigation on another method aimed to lift stable equivalences to Rickard equivalences, due to J. Rickard [6]. This method starts by constructing a tilting complex not by characterizing the objects that correspond to free modules under the derived equivalence, but by characterizing the objects that correspond to simple modules. Rickard's method applies to symmetric algebras over a field, preferably algebraically closed, and it has been successful in verifying Broué's conjecture in several cases by J. Chuang [1] (principal $p$-block of $\mathrm{SL}_2(p^2)$), M. Holloway [2] (5-blocks of $2.J_2$, $U_3(4)$ and $\mathrm{Sp}_4(4)$, all having elementary abelian defect group of order 25), and Y. Usami and N. Yoshida (principal 5-blocks of $G_2(2^n)$, where $5 \mid 2^n + 1$ but $25 \nmid 2^n + 1$, again with defect group $D \simeq C_5 \times C_5$).

We shall freeely use the notations and definitions introduced in [5]. We recall here the main result of [5] combined with [4, Theorem 4.7], characterizing *graded derived* *equivalences*, as we rely on them.

**Theorem 1.1.** *Let $k$ be a commutative ring, $G$ a finite group and $R$, $S$ two $G$-graded $k$-algebras. The following statements are equivalent.*
(i) *There is a $G$-graded tilting complex $T \in \mathcal{D}(R\text{-Gr})$ and an isomorphism $S \to \mathrm{End}_{\mathcal{D}(R)}(T)^{\mathrm{op}}$ of $G$-graded algebras.*
(ii) *There is a complex $X$ of $G$-graded $(R, S)$-bimodules such that the functor*

$$X \overset{\mathbf{L}}{\otimes}_S \cdot : \mathcal{D}(S) \to \mathcal{D}(R)$$

*is an equivalence.*
(iii) *There are equivalences $F: \mathcal{D}(S) \to \mathcal{D}(R)$ and $F^{\mathrm{gr}}: \mathcal{D}(S\text{-Gr}) \to \mathcal{D}(R\text{-Gr})$ of triangulated categories such that $F^{\mathrm{gr}}$ is a $G$-graded functor and the diagram*

$$\begin{array}{ccc} \mathcal{D}(S) & \xrightarrow{F} & \mathcal{D}(R) \\ \mathcal{U}\uparrow & & \uparrow\mathcal{U} \\ \mathcal{D}(S\text{-Gr}) & \xrightarrow[F^{\mathrm{gr}}]{} & \mathcal{D}(R\text{-Gr}) \end{array}$$

*is commutative, where $\mathcal{U}$ is the ungrading (grade-forgetting) functor.*
(iv) (*provided that $R$ and $S$ are strongly graded*) *There are* (*bounded*) *complexes $X_1$ of $\Delta(R\otimes_k S^{\mathrm{op}})$ modules and $Y_1$ of $\Delta(S\otimes_k R^{\mathrm{op}})$ modules, and isomorphisms $X_1 \overset{\mathbf{L}}{\otimes}_{S_1} Y_1 \simeq R_1$ in $\mathcal{D}^b(\Delta(R \otimes_k R^{\mathrm{op}}))$ and $Y_1 \overset{\mathbf{L}}{\otimes}_{R_1} X_1 \simeq S_1$ in $\mathcal{D}^b(\Delta(S \otimes_k S^{\mathrm{op}}))$.*

The first three statements above are from [5, Theorem 2.4]. Concerning the last statement, we refer to [3, Chapter 8, written by B. Keller] for the connection between bounded and unbounded derived equivalences.

## 2. $G$-graded tilting complexes

Throughout the paper, $R = \bigoplus_{g\in G} R_g$ denotes a $G$-graded crossed product over the algebraically closed field $k$, such that $A := R_1$ is a finite-dimensional algebra.

The first statement of the next result is an analogue of the equivalence between (ii) and (iv) in Theorem 1.1.

**Proposition 2.1.** *Let $T$ be a $G$-invariant object of $\mathcal{H}^b(A)$, and denote $\widetilde{T} = R \otimes_A T$ and $S = \mathrm{End}_{\mathcal{H}(R)}(\widetilde{T})^{\mathrm{op}}$.*
a) *$T$ is a tilting complex for $A$ if and only if $\widetilde{T}$ is a $G$-graded tilting complex for $R$.*
b) *If $T$ is a tilting complex for $A$ and $R$ is a symmetric crossed product, then $S$ is a symmetric crossed product of $B := S_1 \simeq \mathrm{End}_{\mathcal{H}(A)}(T)^{\mathrm{op}}$ and $G$.*

Proof. a) Since the functor $R \otimes_A \cdot : A\text{-Mod} \to R\text{-Gr}$ is an equivalence, and a $G$-graded $R$-module is projective in $R$-Gr if and only if it is projective in $R$-Mod, it is clear that $T$ is a bounded complex of finitely generated projective $A$-modules if and only if $\widetilde{T}$ is a bounded complex of finitely generated projective $R$-modules.

Next, for each $m \in \mathbb{Z}$, we have

$$\mathrm{Hom}_{\mathcal{H}(R)}(\widetilde{T}, \widetilde{T}[m]) = \bigoplus_{g \in G} \mathrm{Hom}_{\mathcal{H}(R\text{-Gr})}(\widetilde{T}, \widetilde{T}[m](g)),$$

where, by the equivalence $R \otimes_A \cdot$, for each $g \in G$,

$$\mathrm{Hom}_{\mathcal{H}(R\text{-Gr})}(\widetilde{T}, \widetilde{T}[m](g)) \simeq \mathrm{Hom}_{\mathcal{H}(A)}(T, R_g \otimes_A T[m]).$$

Since $T$ is $G$-invariant, we see that for $m \neq 0$, $\mathrm{Hom}_{\mathcal{H}(A)}(T, R_g \otimes_A T[m]) = 0$ if and only if $\mathrm{Hom}_{\mathcal{H}(R)}(\widetilde{T}, \widetilde{T}[m]) = 0$.

If $A$ belongs to the triangulated subcategory generated by $\mathrm{add}(T)$, then using again the equivalence $R \otimes_A \cdot$, we obtain that $R$ belongs to the triangulated subcategory of $\mathcal{D}(R\text{-Gr})$ generated by $\mathrm{add}(\widetilde{T})$. Hence, by forgetting the gradings, $R$ belongs to the triangulated subcategory of $\mathcal{D}(R)$ generated by $\mathrm{add}(\widetilde{T})$. Conversely, assume that $R$ belongs to the triangulated subcategory of $\mathcal{D}(R)$ generated by $\mathrm{add}(\widetilde{T})$. Then, by restriction of scalars, ${}_AR$ belongs to the triangulated subcategory of $\mathcal{D}(A)$ generated by $\mathrm{add}({}_A\widetilde{T})$. But ${}_AR$ is a finite direct sum of copies of $A$, and ${}_A\widetilde{T}$ is a finite direct sum of copies of $T$, hence $A$ belongs to the subcategory generated by $\mathrm{add}(T)$.

b) Since $\widetilde{T}$ is $G$-invariant, by [5, Lemma 1.7] $S$ is a $G$-graded crossed product, with

$$S_1 = \mathrm{End}_{\mathcal{H}(R\text{-Gr})}(\widetilde{T}, \widetilde{T})^{\mathrm{op}} \simeq \mathrm{End}_{\mathcal{H}(A)}(T)^{\mathrm{op}},$$

again since $R \otimes_A \cdot$ is an equivalence. The symmetry of $R$ means that $R^\vee := \mathrm{Hom}_k(R, k) \simeq R$ as $G$-graded $(R, R)$-bimodules. There is a derived equivalence between $R \otimes_k R^{\mathrm{op}}$ and $S \otimes_k S^{\mathrm{op}}$, sending $R$ to $S$ and $R^\vee$ to $S^\vee$. (This is due to Rickard, but we refer to [9] for a proof in a more general situation.) This derived equivalence is actually $G \times G$-graded (see [4, Corollary 4.9 c)]), so we conclude that $S \simeq S^\vee$ as $G$-graded $(S, S)$-bimodules. □

REMARK 2.2. The first implication in Proposition 2.1 a) is in fact true under more general assumptions. Assume that $R$ is strongly graded, that is, for each $g \in G$, $R_g$ is a direct summand of a finite direct sum of copies of $A$. Then it is not difficult to show that if $T$ is a tilting complex for $A$, then $\widetilde{T}$ is also weakly $G$-invariant, or equivalently, $S$ is strongly graded too, and $\widetilde{T}$ is a graded tilting complex for $R$.

On the other hand, if $R$ is an arbitrary $G$-graded algebra, and $\widetilde{T}$ is a $G$-graded tilting complex for $R$ with endomorphism ring $S$, without assuming that $R$ and $S$ are strongly graded, we cannot conclude in general that $R_1$ and $S_1$ are derived equivalent.

**2.3.** Next we come to Rickard's construction [6]. Under a derived equivalence between the $k$-algebras $A$ and $B$, the objects $X_i \in \mathcal{D}^b(A\text{-mod})$, $i \in I$, corresponding to simple $B$ modules must satisfy the following conditions.
(2.3.a) $\mathrm{Hom}(X_i, X_j[m]) = 0$ for $m < 0$.
(2.3.b) $\mathrm{Hom}(X_i, X_j) = k$ if $i = j$ and 0 otherwise.
(2.3.c) $X_i$, $i \in I$ generate $\mathcal{D}^b(A\text{-mod})$ as a triangulated category.

In order to obtain a graded derived equivalence, we also need to consider the action of $G$.

**Theorem 2.4.** *Let $R$ be a symmetric crossed product of $A$ and $G$, let $I$ be a finite $G$-set, and let $X_i \in \mathcal{D}^b(A\text{-mod})$, $i \in I$, be objects satisfying* (2.3.a), (2.3.b) *and* (2.3.c). *Assume that the objects $X_i$ satisfy the additional condition*
(2.4.a) $R_g \otimes_A X_i \simeq X_{{}^g i}$ *in* $\mathcal{D}^b(A\text{-mod})$, *for all $i \in I$ and $g \in G$.*

*Then there is another symmetric crossed product $R'$ of $A'$ and $G$, and a $G$-graded derived equivalence between $R$ and $R'$, whose restriction to $A$ sends $X_i$, $i \in I$, to the simple $A'$-modules.*

Proof. By the proof of [6, Theorem 5.1], there is a tilting complex $T = \bigoplus_{i \in I} T_i$ for $A$ satisfying
(2.4.b) $\mathrm{Hom}(T_i, X_j[m]) = k$ if $i = j$ and $m = 0$, and 0 otherwise.

By Proposition 2.1, it is enough to show that $T$ also satisfies
(2.4.c) $R_g \otimes_A T_i \simeq T_{{}^g i}$ in $\mathcal{D}^b(A\text{-mod})$, for all $i \in I$ and $g \in G$.

Let $g \in G$ and $i \in I$. The construction of the summands $T_i$ go by induction as follows. Set $X_i^{(0)} := X_i$, so $R_g \otimes_A X_i^{(0)} \simeq X_{{}^g i}^{(0)}$. Assuming that $X_i^{(n-1)}$ and $X_{{}^g i}^{(n-1)}$ are constructed such that $R_g \otimes_A X_i^{(n-1)} \simeq X_{{}^g i}^{(n-1)}$, we shall construct $X_i^{(n)}$ and $X_{{}^g i}^{(n)}$ and maps such that the diagram

$$\text{(2.4.d)} \qquad \begin{array}{ccc} R_g \otimes_A X_i^{(n-1)} & \xrightarrow{\simeq} & X_{{}^g i}^{(n-1)} \\ {\scriptstyle R_g \otimes_A \phi_i^{(n-1)}} \downarrow & & \downarrow {\scriptstyle \phi_{{}^g i}^{(n-1)}} \\ R_g \otimes_A X_i^{(n)} & \xrightarrow{\simeq} & X_{{}^g i}^{(n)} \end{array}$$

is commutative.

For each $j \in I$ and $t < 0$, let $Z_i^{(n-1)}(j,t) := X_j[t] \otimes_k \mathrm{Hom}(X_j[t], X_i^{(n-1)})$. There is a map $\alpha_i^{(n-1)}(j,t)\colon Z_i^{(n-1)}(j,t) \to X_i^{(n-1)}$ obtained by choosing a basis $(\beta_l)$ of $\mathrm{Hom}(X_j[t], X_i^{(n-1)})$, and the restriction of $\alpha_i^{(n-1)}(j,t)$ to the direct summand $X_j[t]$ of $Z_i^{(n-1)}(j,t)$ corresponding to $\beta_l$ is, by definition, $\beta_l$; that is, $\alpha_i^{(n-1)}(j,t) = \sum_l \beta_l$. If we choose another basis $(\beta_l')$, then the transition matrix from $(\beta_l)$ to $(\beta_l')$ induces an automorphism $\tau$ of $Z_i^{(n-1)}(j,t)$ such that the diagram

$$\begin{tikzcd} Z_i^{(n-1)}(j,t) \arrow[r, "\sum_l \beta_l"] \arrow[d, "\tau"'] & X_i^{(n-1)} \arrow[d, equal] \\ Z_i^{(n-1)}(j,t) \arrow[r, "\sum_l \beta'_l"] & X_i^{(n-1)} \end{tikzcd}$$

is commutative. By assumption, and by the fact that $R_g \otimes_A \cdot$ is an equivalence with inverse $R_{g^{-1}} \otimes_A \cdot$, we obtain the isomorphisms

$$\begin{aligned} R_g \otimes_A Z_i^{(n-1)}(j,t) &\simeq R_g \otimes_A X_j[t] \otimes_k \operatorname{Hom}(X_j[t], X_i^{(n-1)}) \\ &\simeq X_{{}^g j}[t] \otimes_k \operatorname{Hom}(X_j[t], X_i^{(n-1)}) \\ &\simeq X_{{}^g j}[t] \otimes_k \operatorname{Hom}(R_{g^{-1}} \otimes_A X_{{}^g j}[t], X_i^{(n-1)}) \\ &\simeq X_{{}^g j}[t] \otimes_k \operatorname{Hom}(X_{{}^g j}[t], X_{{}^g i}^{(n-1)}) \\ &\simeq Z_{{}^g i}^{(n-1)}({}^g j, t). \end{aligned}$$

Consequently, by the above observation, we obtain the commutative diagram.

$$\begin{tikzcd} R_g \otimes_A Z_i^{(n-1)}(j,t) \arrow[r, "R_g \otimes_A \alpha_i^{(n-1)}(j,t)"] \arrow[d, "\simeq"'] & R_g \otimes_A X_i^{(n-1)} \arrow[d, "\simeq"] \\ Z_{{}^g i}^{(n-1)}({}^g j, t) \arrow[r, "\alpha_{{}^g i}^{(n-1)}({}^g j, t)"] & X_{{}^g i}^{(n-1)} \end{tikzcd}$$

Let $Z_{{}^g i}^{(n-1)} := \bigoplus_{j \in I,\, t<0} Z_i^{(n-1)}(j,t)$ and let

$$\alpha_i^{(n-1)} = \sum_{j \in I,\, t<0} \alpha_i^{(n-1)}(j,t) \colon Z_i^{(n-1)} \to X_i^{(n-1)}.$$

It follows that we have the commutative diagram.

$$\begin{tikzcd} R_g \otimes_A Z_i^{(n-1)} \arrow[r, "R_g \otimes_A \alpha_i^{(n-1)}"] \arrow[d, "\simeq"'] & R_g \otimes_A X_i^{(n-1)} \arrow[d, "\simeq"] \\ Z_{{}^g i}^{(n-1)} \arrow[r, "\alpha_{{}^g i}^{(n-1)}"] & X_{{}^g i}^{(n-1)} \end{tikzcd}$$

Since the map $\phi_i^{(n-1)} \colon X_i^{(n-1)} \to X_i^{(n)}$ is defined by forming the distinguished triangle

$$Z_i^{(n-1)} \xrightarrow{\alpha_i^{(n-1)}} X_i^{(n-1)} \to X_i^{(n)} \to Z_i^{(n-1)}[1],$$

we deduce the existence of the commutative diagram (2.4.d).

Finally, let $T_i := \operatorname{hocolim}(X_i)$. By the definition of the homotopy colimit (see [6, (4.1)]) it follows that (2.4.c) also holds. $\square$

In order to lift a stable equivalence to a graded derived equivalence by using Okuyama's strategy, in our general setting we need to assume that $p$ does not divide the order of $G$.

**Corollary 2.5.** *Let $R$ and $S$ be two $G$-graded symmetric crossed products, and denote $A = R_1$ and $B = S_1$. Assume that $G$ is a $p'$-group and $I$ is a $G$-set. Let $M$ be a $G$-graded $(R,S)$-bimodule inducing a Morita stable equivalence between $R$ and $S$, and let $\{S_i \mid i \in I\}$ be a set of representatives for the simple $B$-modules.*

*If there are objects $X_i \in \mathcal{D}^b(A\text{-mod})$, $i \in I$, satisfying the conditions* (2.3.a), (2.3.b), (2.3.c) *and* (2.4.a), *and such that $X_i$ is stably isomorphic to $M_1 \otimes_B S_i$, for all $i \in I$, then there is a $G$-graded derived equivalence between $R$ and $S$.*

Proof. By Theorem 2.4 there is a symmetric crossed product $R'$ and a $G$-graded derived equivalence between $R$ and $R'$, and hence a $G$-graded stable Morita equivalence between $R$ and $R'$ (see [5, Remark 3.4]). Consequently, we have a stable Morita equivalence between $R'$ and $S$ induced by a $G$-graded $(R',S)$-bimodule $M'$. Since simple $R'_1$-modules are sent to simple $S_1$-modules, by a theorem of Linckelmann, a direct $(R'_1 \otimes_k S_1^{\mathrm{op}})$-summand $N$ of $M'_1$ induces a Morita equivalence between $R'_1$ and $S_1$. Since $G$ is a $p'$-group, by the argument used in [4, Example 5.8], we have that $N$ is in fact a $\Delta(R' \otimes_k S^{\mathrm{op}})$-summand $N$ of $M'_1$, hence $(R' \otimes_k S^{\mathrm{op}}) \otimes_{\Delta(R'\otimes_k S^{\mathrm{op}})} N$ induces a $G$-graded Morita equivalence between $R'$ and $S$. Finally, compose this equivalence with the graded derived equivalence between $R$ and $R'$ to obtain a graded derived equivalence between $R$ and $S$. $\square$

## 3. Splendid stable and derived equivalences

**3.1.** Let $R$ and $S$ be symmetric $G$-graded crossed products over the algebraically closed field $k$. Denote $A = R_1$, $B = S_1$ and $\Delta = \Delta(R\otimes_k S^{\mathrm{op}}) = \bigoplus_{g\in G} R_g \otimes_k S_g^{\mathrm{op}}$.

By definition, the (cochain) complex $C$ of $G$-graded exact $(R,S)$-bimodules induces a $G$-graded stable equivalence between $R$ and $S$ if

$$C \otimes_S C^\vee \simeq R \oplus Z$$

in the bounded homotopy category of finitely generated $G$-graded $(R,R)$-bimodules, and

$$C^\vee \otimes_R C \simeq S \oplus W$$

in the bounded homotopy category of finitely generated $G$-graded $(S,S)$-bimodules, where $Z$ and $W$ are bounded complexes of projective bimodules. Note that by [4, Lemma 2.6], the above isomorphisms are equivalent to

$$C_1 \otimes_S C_1^\vee \simeq A \oplus Z_1 \quad \text{in} \quad \mathcal{H}^b(\Delta(R \otimes_k R^{\mathrm{op}})\text{-mod}),$$
$$C_1^\vee \otimes_R C_1 \simeq B \oplus W_1 \quad \text{in} \quad \mathcal{H}^b(\Delta(S \otimes_k S^{\mathrm{op}})\text{-mod}),$$

where $Z_1$ and $W_1$ are bounded complexes of projective $\Delta(R \otimes_k R^{\mathrm{op}})$-modules, respectively projective $\Delta(S \otimes_k S^{\mathrm{op}})$-modules.

By using arguments as in [5, 2.6 and Remark 3.4], one can easily adapt the proof of [2, Proposition 4.3] in order to deal with graded equivalences. We include the full proof for convenience.

**Proposition 3.2.** *Assume that $G$ is a $p'$-group, and let $C$ and $D$ be bounded complexes of $G$-graded $(R, S)$-bimodules such that $C$ induces a stable equivalence between $R$ and $S$, $D$ induces a derived equivalence between $R$ and $S$, and the stable equivalence between $A$ and $B$ induced by $D_1$ agrees on each simple module, up to isomorphism, with that induced by $C_1$.*

*Then there is a bounded complex $X$ of finitely generated $G$-graded $(R, S)$-bimodules such that*

1) *$X = C \oplus P$, where $P$ is a complex of $G$-graded projective bimodules*;
2) *$X$ induces a $G$-graded homotopy equivalence between $R$ and $S$*;
3) *In the derived category of $G$-graded $(R, S)$-bimodules, $X$ is isomorphic to the composition between $D$ and a $G$-graded Morita autoequivalence of $R$.*

Proof. By using well-known results of Rickard and [5, Remark 3.4], we may assume that

$$D_1 = (\cdots \to 0 \to N \to Y_1^{-n+1} \to Y_1^{-n+2} \to \cdots),$$

where ${}_AN$, $N_B$ and ${}_\Delta Y_1^i$ are projective, and $\Omega^{-n}(N)$ (which is again a $\Delta$-module) induces a stable Morita equivalence between $A$ and $B$.

Similarly, by truncating a projective $\Delta$-module resolution of $C_1$, we obtain a complex

$$T_1 = (\cdots \to 0 \to M \to T_1^{-m+1} \to T_1^{-m+2} \to \cdots)$$

of $\Delta$-modules, with ${}_AM$, $M_B$ and ${}_\Delta T_1^i$ projective, and $\Omega^{-m}(M)$ induces a stable Morita equivalence between $A$ and $B$ isomorphic to that induced by $C_1$.

We choose $n = m$ sufficiently large such that $\Omega^{-n}(M) \otimes_B \Omega^n(N^\vee)$ is a $\Delta(R \otimes_k R^{\mathrm{op}})$-module inducing a stable autoequivalence of $A$ sending simple $A$-modules to isomorphic copies of themselves. Linckelmann's theorem implies that this is in fact a Morita autoequivalence. Composing its indecomposable non-projective $\Delta$-module summand with $D_1$, we obtain the complex

$$L_1 = (\cdots \to 0 \to M \to L_1^{-n+1} \to L_1^{-n+2} \to \cdots)$$

of $\Delta$-modules, with $L_1^i$ projective. Denote

$$\widetilde{T}_1 = (\cdots \to 0 \to T_1^{-m+1} \to T_1^{-m+2} \to \cdots),$$

so we have maps of complexes $\phi\colon T_1 \to C_1$ and $\iota\colon \widetilde{T}_1 \to T_1$, and a map of triangles

$$\begin{tikzcd}
\widetilde{T}_1 \arrow[r, "\iota"] \arrow[d, equal] & T_1 \arrow[r] \arrow[d, "\phi"] & M[n] \arrow[r] \arrow[d, "\chi"] & \widetilde{T}_1[1] \arrow[d, equal] \\
\widetilde{T}_1 \arrow[r, "\phi\iota"] & C_1 \arrow[r] & K_1 \arrow[r] & \widetilde{T}_1[1]
\end{tikzcd}$$

where $K_1$ is the cone of $\phi\iota$. Denote

$$\widetilde{L}_1 = (\cdots \to 0 \to L_1^{-n+1} \to L_1^{-n+2} \to \cdots).$$

The $\mathrm{id}_M$ lifts to a map $\rho\colon K_1[-1] \to \widetilde{L}_1$ in $\mathcal{H}^b(\Delta)$. We obtain a triangle

$$K_1[-1] \xrightarrow{\rho} \widetilde{L}_1 \to X_1 \to K_1$$

in $\mathcal{H}^b(\Delta)$. Then $X_1 \simeq L_1$ in $\mathcal{D}^b(\Delta)$, and the required complex of $G$-graded $(R, S)$-bimodules is $X := (R \otimes_k S^{\mathrm{op}}) \otimes_\Delta X_1$. $\square$

**3.3.** For the remaining part of the paper, let $S = kHb$, $B = kKb$, $R = kN_H(D)c$ and $A = N_K(D)$, where $H$, $K$, $D$, $b$ and $c$ are as in the introduction. Denote also $H' = N_H(D)$ and $K' = N_K(D)$, and assume that $G = H/K$ is a $p'$-group.

Recall that the bounded complex $C$ of $(R, S)$-bimodules is *splendid*, if the indecomposable summands of its terms $C^i$ are relatively $\delta(D)$-projective $p$-permutation $k(H' \times H)$-modules. Note that the truncation of a projective resolution of $C$ as in the proof of Proposition 3.2 does not lead in general to a splendid complex.

By Proposition 3.2 and Corollary 2.5 we immediately get:

**Corollary 3.4.** *Assume that $G$ is a $p'$-group, $I$ is a $G$-set, and that $C$ is a splendid complex of $G$-graded $(R, S)$-bimodules inducing a stable equivalence $R$ and $S$. Let $\{S_i \mid i \in I\}$ be a set of representatives for the simple $B$-modules, and let $X_i \in \mathcal{D}^b(A\text{-mod})$, $i \in I$, be objects satisfying the conditions* (2.3.a), (2.3.b), (2.3.c) *and* (2.4.a), *such that $X_i$ is stably isomorphic to $C_1 \otimes_B S_i$ for all $i \in I$.*

*Then there is a complex $X$ of $G$-graded $(R, S)$-bimodules such that*:

1) *the image of $X_1$ in $\Delta$-stmod $\simeq \mathcal{D}^b(\Delta\text{-mod})/\mathcal{H}^b(\Delta\text{-proj})$ is isomorphic to $C_1$;*
2) *$X$ induces a splendid derived equivalence between $R$ and $S$;*
3) *$X_1 \otimes_B S_i \simeq X_i$ in $\mathcal{D}^b(A\text{-mod})$, for all $i \in I$.*

EXAMPLE 3.5. a) A case in which Corollary 3.4 applies is the so called T.I. situation, that is, when ${}^xD \cap D = 1$ for all $x \in K \setminus N_K(D)$. Then the bimodule $M_1 = {}_AA_B$ induces a stable Morita equivalence between $A$ and $B$, and $M_1$ is clearly a $\Delta$-module.

This is the situation which occurs in [1] and [6]. An inspection of the examples discussed in [6, Section 7] and of the proof of the main result of [1] shows that the objects $X_i$ satisfy the condition (2.4.a). Note that here $I$ becomes a $G$-set by letting $S_g \otimes_B S_i \simeq S_{{}^g i}$ for $g \in G$, $i \in I$.
b) Assume that $D$ is elementary abelian of order $p^2$, and that $b$ is the principal block of $\mathcal{O}K$. Then, by [7, Theorem 6.3], there is a splendid complex of $(A, B)$-bimodules inducing a stable equivalence between $A$ and $B$. We show here that there is even a complex of $\Delta$-modules.

Let $P$ be a subgroup of order $p$ of $D$, and let $e_P$ be the principal block of $C_K(P)$ and $f_P$ the principal block of $C_{K'}(P)$. Denote $G_P = N_H(P)/C_K(P)$. Then $G_P$ is a $p'$-group, and $G_P \simeq N_{H'}(P)/C_{K'}(P)$. Furthermore, denote

$$\delta(G) = \{(h', h) \in H' \times H \mid h^{-1}h' \in K\}$$

and

$$\delta(G_P) = \{(h', h) \in N_{H'}(P) \times N_H(P) \mid h^{-1}h' \in C_K(P)\}.$$

The principal blocks of $C_K(P)/P$ and $C_{K'}(P)/P$ have cyclic defect group $D/P$. By [7, Theorem 6.2], there is a complex

$$\widetilde{C}_P = \left(\cdots \to 0 \to N_P \xrightarrow{\phi_P} f_P k C_K(P) e_P \to 0 \to \cdots\right)$$

of $k\delta(G_P)$-modules (with $f_P k C_K(P) e_P$ in degree 0) inducing a splendid derived equivalence between $f_P k C_{K'}(P)$ and $e_P k C_K(P)$; here $N_P$ is a projective $k(\delta(G_P)/\delta(P))$-module regarded also as a $k\delta(G_P)$-module via inflation. Denote $V_P = \operatorname{Ind}_{\delta(G_P)}^{\delta(G)} N_P$. We have that $ckKb$ is a $k\delta(G)$-module, and the obvious map $f_p k C_K(P) e_P \to \operatorname{Res}_{\delta(G_P)}^{\delta(G)} ckKb$ induces by adjunction the $k\delta(G)$-linear map

$$\alpha_P \colon \operatorname{Ind}_{\delta(G_P)}^{\delta(G)} f_p k C_K(P) e_P \to ckKb.$$

We obtain the map $\psi_P = \alpha_P \circ \operatorname{Ind}_{\delta(G_P)}^{\delta(G)} \phi_P \colon V_P \to ckKb$ and the complex

$$C_1 := \left(\cdots \to 0 \to \bigoplus_Q V_Q \xrightarrow{\sum_Q \psi_Q} ckKb \to 0 \to \cdots\right)$$

of $\Delta$-modules, where $Q$ runs over the subgroups of order $p$ of $D$ up to $H$-conjugacy. It follows by [7, 4.1.2] that $\mathrm{Br}_{\delta(P)} C_1 \simeq \widetilde{C}_P$, hence by [7, Theorem 5.6], $C_1$ induces a splendid stable equivalence between $A$ and $B$.

This construction applies to the examples considered in [2]. It is not difficult to verify that in those cases condition (2.4.a) also holds.

ACKNOWLEDGEMENTS. This paper was written during a visit at the Chiba University, where the author was supported by a JSPS Invitation Fellowship. He is grateful to Shigeo Koshitani for all his help and hospitality, and to the referee for observations and suggestions.

## References

[1] J. Chuang: *Derived equivalence in* $\mathrm{SL}_2(p^2)$, Trans. Amer. Math. Soc. **353** (2001), 2897–2913.

[2] M. Holloway: *Broué's conjecture for the Hall-Janko group and its double cover*, Proc. London Math. Soc. **86** (2003), 109–130.

[3] S. König and A. Zimmermann: Derived Equivalences for Group Rings, Lecture Notes in Mathematics, **1685**, Springer-Verlag, Berlin, 1998.

[4] A. Marcus: *Equivalences induced by graded bimodules*, Comm. Algebra **26** (1998), 713–731.

[5] A. Marcus: *Tilting complexes for group graded algebras*, J. Group Theory **6** (2003), 175–193.

[6] J. Rickard: *Equivalences of derived categories for symmetric algebras*, J. Algebra **257** (2002), 460–481.

[7] R. Rouquier: *Block theory via stable and Rickard equivalences*; in Modular Representation Theory of Finite Groups (Charlottesville, VA, 1998) de Gruyter, Berlin (2001) 101–146.

[8] Y. Usami and N. Yoshida: Preprint (2003).

[9] A. Zimmermann: *Tilted symmetric orders are symmetric orders*, Arch. Math. **73** (1999), 15–17.

Babeş-Bolyai University
Faculty of Mathematics and Computer Science
Str. Mihail Kogălniceanu nr. 1
RO-400084 Cluj-Napoca, Romania
e-mail: marcus@math.ubbcluj.ro