

SEMICLASSICAL NONLINEAR SCHRÖDINGER EQUATIONS WITH POTENTIAL AND FOCUSING INITIAL DATA

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1. Introduction

We study the semi-classical limit $\varepsilon \rightarrow 0$ of solutions $\mathbf{u}^\varepsilon : (t, y) \in \mathbb{R} \times \mathbb{R}^n \rightarrow \mathbb{C}$ of the equation

$$i\varepsilon\partial_t \mathbf{u}^\varepsilon + \frac{1}{2}\varepsilon^2 \Delta \mathbf{u}^\varepsilon = V(y)\mathbf{u}^\varepsilon + \lambda|\mathbf{u}^\varepsilon|^{2\sigma} \mathbf{u}^\varepsilon ,$$

where $\lambda > 0$ (the nonlinearity is repulsive), with concentrating initial data

$$\mathbf{u}^\varepsilon(0, y) = R \left(\frac{y - y_0}{\varepsilon} \right) e^{i(y \cdot \eta_0 / \varepsilon)} .$$

Similar problems were studied for attractive nonlinearities ($\lambda < 0$), by Bronski and Jerrard ([1]), and Keraani ([15]). In that case, if the power is L^2 -subcritical ($\sigma < 2/n$) and R is the ground state solution of an associated scalar elliptic equation, then when V is smooth with $V \in W^{2,\infty}$, the following asymptotics holds in $X := L_{loc}^\infty(\mathbb{R}; L^2(\mathbb{R}^n))$,

$$(1.1) \quad \begin{aligned} \frac{1}{\varepsilon^{n/2}} \left\| \mathbf{u}^\varepsilon(t, y) - R \left(\frac{y - y(t) + \varepsilon y^\varepsilon(t)}{\varepsilon} \right) e^{i(y \cdot \eta(t) / \varepsilon) + i\theta^\varepsilon(t)} \right\|_X &= O(\sqrt{\varepsilon}) , \\ \frac{1}{\varepsilon^{n/2}} \left\| \varepsilon \nabla_y \left(\mathbf{u}^\varepsilon(t, y) - R \left(\frac{y - y(t) + \varepsilon y^\varepsilon(t)}{\varepsilon} \right) e^{i(y \cdot \eta(t) / \varepsilon) + i\theta^\varepsilon(t)} \right) \right\|_X &= O(\sqrt{\varepsilon}) , \end{aligned}$$

where $\theta^\varepsilon(t) \in [0, 2\pi[$, $y^\varepsilon : \mathbb{R} \rightarrow \mathbb{R}^n$ is locally uniformly bounded and $(y(t), \eta(t))$ are the integral curves associated to the classical Hamiltonian

$$(1.2) \quad p(t, y, \tau, \eta) = \tau + \frac{1}{2}|\eta|^2 + V(y),$$

with initial data (y_0, η_0) .

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In this paper, we address the case of a defocusing nonlinearity ($\lambda > 0$), when the potential is a polynomial of degree at most two.

In the case $\lambda > 0$, a different qualitative behaviour is expected. Intuitively, dispersive effects prevent the solution from keeping a concentrating aspect as in (1.1), for it is well known (see e.g. [5]) that the solutions to the nonlinear Schrödinger equation

$$(1.3) \quad i\partial_t \psi + \frac{1}{2}\Delta\psi = |\psi|^{2\sigma}\psi ,$$

have the same dispersive properties as the solutions to the linear Schrödinger equation, under suitable assumptions on σ and $\psi(0, y)$. In the case where the potential V is the harmonic potential, $V(y) = \omega^2|y|^2$, it was proved in [2] that when $y_0 = \eta_0 = 0$, the nonlinear term is relevant so long as the dispersive effects are not too strong. This is so in a boundary layer of size ε . Past this boundary layer, the nonlinear term becomes negligible, and the potential V imposes the dynamical behaviour of the solution. In the case of an *isotropic* potential,

$$(1.4) \quad V(y) = \frac{1}{2} \sum_{j=1}^n \omega_j^2 y_j^2,$$

where all the ω_j 's are equal, then focusing at the origin occurs at times $t = k\pi$, $k \in \mathbb{Z}$, and each focus crossing is described in terms of the Maslov index (this phenomenon is linear) and the nonlinear scattering operator associated to (1.3). The case $\eta_0 = 0$, $y_0 \in \mathbb{R}^n$, is also discussed, and we explain below how to infer the more general case $(y_0, \eta_0) \in \mathbb{R}^n \times \mathbb{R}^n$ (see (1.19)).

The case where the ω_j 's are (all positive) not necessarily equal is also discussed in [2]. The conclusion is that the nonlinear term is not relevant outside the initial boundary layer if and only if two of the ω_j 's are rationally independent. In the present paper, we consider the case of a generalized quadratic potential which excludes this case.

More precisely, we assume that the potential V is of the form

$$(1.5) \quad V(y) = \sum_{1 \leq j, k \leq n} \alpha_{jk} y_j y_k + \sum_{j=1}^n \beta_j y_j + \gamma,$$

where the constants α_{jk} , β_j and γ are real. We first notice that up to changing the origin and the basis, we can assume that the potential has a more rigid form.

Lemma 1.1. *Let V given by (1.5). There exist $\hat{y} \in \mathbb{R}^n$, and a family $f_1, \dots, f_n \in \mathbb{R}^n$ of orthogonal unit vectors such that, with \hat{y} as a new origin, the potential V*

writes, in the basis (f_1, \dots, f_n) ,

$$V(x) = \frac{1}{2} \sum_{j=1}^n \delta_j \omega_j^2 x_j^2 + \sum_{j=1}^n b_j x_j + c,$$

where $\omega_j > 0$, $\delta_j \in \{-1, 0, 1\}$, $b_j, c \in \mathbb{R}$ and for every j , $\delta_j b_j = 0$. The real numbers

$$\frac{1}{2} \delta_j \omega_j^2, \quad j = 1, \dots, n,$$

are the eigenvalues of the quadratic part of V .

Proof. Consider the quadratic part of the potential V ,

$$q(y) = \sum_{1 \leq j, k \leq n} \alpha_{jk} y_j y_k.$$

It is well-known that there exists a family $f_1, \dots, f_n \in \mathbb{R}^n$ of orthogonal unit vectors such that, in this new basis, q writes

$$q(\tilde{y}) = \frac{1}{2} \sum_{j=1}^n \delta_j \omega_j^2 \tilde{y}_j^2,$$

where $\omega_j > 0$, $\delta_j \in \{-1, 0, 1\}$. In this basis, V is of the form

$$V(\tilde{y}) = \frac{1}{2} \sum_{j=1}^n \delta_j \omega_j^2 \tilde{y}_j^2 + \sum_{j=1}^n \tilde{\beta}_j \tilde{y}_j + \gamma,$$

with $\tilde{\beta}_j \in \mathbb{R}$. If $\delta_j = 0$, we take $b_j = \tilde{\beta}_j$, and if $\delta_j \neq 0$, we use the one-dimensional formula,

$$x^2 + 2ax = (x + a)^2 - a^2.$$

The lemma follows. □

In these new coordinates, the Laplace operator is not changed, and the initial value problem we are interested in becomes

$$(1.6) \quad \begin{cases} i\varepsilon \partial_t \mathbf{u}^\varepsilon + \frac{1}{2} \varepsilon^2 \Delta \mathbf{u}^\varepsilon = V(x) \mathbf{u}^\varepsilon + \lambda |\mathbf{u}^\varepsilon|^{2\sigma} \mathbf{u}^\varepsilon, \\ \mathbf{u}^\varepsilon(0, x) = R \left(\frac{x - x_0}{\varepsilon} \right) e^{ix \cdot \xi_0 / \varepsilon} e^{i\kappa / \varepsilon}, \end{cases}$$

for some $x_0, \xi_0 \in \mathbb{R}^n$, $\kappa \in \mathbb{R}$. Notice that $\tilde{\mathbf{u}}^\varepsilon$, defined by $\tilde{\mathbf{u}}^\varepsilon(t, x) := \mathbf{u}^\varepsilon(t, x) e^{i(ct + \kappa) / \varepsilon}$,

solves

$$(1.7) \quad \begin{cases} i\varepsilon \partial_t \tilde{\mathbf{u}}^\varepsilon + \frac{1}{2} \varepsilon^2 \Delta \tilde{\mathbf{u}}^\varepsilon = (V(x) - c) \tilde{\mathbf{u}}^\varepsilon + \lambda |\tilde{\mathbf{u}}^\varepsilon|^{2\sigma} \tilde{\mathbf{u}}^\varepsilon , \\ \tilde{\mathbf{u}}^\varepsilon(0, x) = R \left(\frac{x - x_0}{\varepsilon} \right) e^{i(x \cdot \xi_0 / \varepsilon)} , \end{cases}$$

We can thus assume $c = 0$. We make an additional assumption on the potential.

ASSUMPTION 1.2. We suppose that the potential satisfies the following properties.

1. It is of the form

$$(1.8) \quad V(x) = \frac{1}{2} \sum_{j=1}^n \delta_j \omega_j^2 x_j^2 + \sum_{j=1}^n b_j x_j,$$

where $\omega_j > 0$, $\delta_j \in \{-1, 0, 1\}$, $b_j, c \in \mathbb{R}$ and for every j , $\delta_j b_j = 0$.

2. Either there exists j such that $\delta_j \neq 1$, or $\delta_j = 1$ for all j and the ω_j 's are not pairwise rationally dependent:

$$\exists j \neq k, \frac{\omega_j}{\omega_k} \notin \mathbb{Q}.$$

REMARK. We allow negative coefficients for the potential (case $\delta_j = -1$). In that case, the energy of \mathbf{u}^ε which is formally independent of time,

$$(1.9) \quad E^\varepsilon = \frac{1}{2} \|\varepsilon \nabla_x \mathbf{u}^\varepsilon(t)\|_{L^2}^2 + \frac{1}{\sigma + 1} \|\mathbf{u}^\varepsilon(t)\|_{L^{2\sigma+2}}^{2\sigma+2} + \int V(x) |\mathbf{u}^\varepsilon(t, x)|^2 dx ,$$

contains negative terms which are not controlled by the positive terms (in particular, by the H^1 -norm). Therefore, even the issue of global existence in H^1 is not obvious. We prove that for any $T > 0$, \mathbf{u}^ε cannot blow up for $|t| \leq T$, provided that ε is sufficiently small ($0 < \varepsilon \leq \varepsilon(T)$). Notice that in the case of an isotropic negative quadratic potential ($\delta_j = -1$ and $\omega_j = \omega$ for all j), global existence for fixed ε was proved in [3].

Assumption 1.2 has a simple geometric consequence. Forget the nonlinear term for a moment, and consider the classical Hamiltonian p given by (1.2). Because V is of the form given by (1.8), the bicharacteristic curves starting from any point $(x_0, \xi_0) \in \mathbb{R}^n \times \mathbb{R}^n$ can be computed explicitly. They solve the differential equation

$$\begin{cases} \dot{t} = 1 ; \dot{x}(t) = \xi(t) , \\ \dot{\tau} = 0 ; \dot{\xi}(t) = -\nabla V(x(t)) , \\ x(0) = x_0 ; \xi(0) = \xi_0 . \end{cases}$$

Introduce the auxiliary functions,

$$(1.10) \quad g_j(t) = \begin{cases} \frac{\sin(\omega_j t)}{\omega_j} , & \text{if } \delta_j = 1 , \\ t , & \text{if } \delta_j = 0 , \\ \frac{\sinh(\omega_j t)}{\omega_j} , & \text{if } \delta_j = -1 . \end{cases} \quad ; \quad h_j(t) = \begin{cases} \cos(\omega_j t) , & \text{if } \delta_j = 1 , \\ 1 , & \text{if } \delta_j = 0 , \\ \cosh(\omega_j t) , & \text{if } \delta_j = -1 . \end{cases}$$

Then the bicharacteristic curves are given by

$$(1.11) \quad x_j(t) = h_j(t)x_{0j} + g_j(t)\xi_{0j} - \frac{1}{2}b_j t^2 ; \quad \xi_j(t) = h_j(t)\xi_{0j} - \delta_j \omega_j^2 g_j(t)x_{0j} - b_j t.$$

As the analysis will prove later on, the second part of Assumption 1.2 implies that except at time $t = 0$, the energy is never concentrated at one point. Some new concentrations may happen for $t \neq 0$ (if $\delta_j = 1$ for at least one j), but on a vector space of dimension at least one, for which the nonlinear term turns out to be subcritical in the limit $\varepsilon \rightarrow 0$.

First, assume that $x_0 = \xi_0 = 0$. Taking $u^\varepsilon := \varepsilon^{-n/2} \lambda^{1/(2\sigma)} \tilde{u}^\varepsilon$ as a new unknown turns (1.7) into

$$(1.12) \quad \begin{cases} i\varepsilon \partial_t u^\varepsilon + \frac{1}{2} \varepsilon^2 \Delta u^\varepsilon = V(x) u^\varepsilon + \varepsilon^{n\sigma} |u^\varepsilon|^{2\sigma} u^\varepsilon , \\ u^\varepsilon|_{t=0} = \frac{1}{\varepsilon^{n/2}} \varphi \left(\frac{x}{\varepsilon} \right) , \end{cases}$$

where φ is given by $\varphi := \lambda^{1/(2\sigma)} R$. As we mentioned already, we expect the caustic crossing at time $t = 0$ to be described by the scattering operator associated to (1.3). For this operator to be well-defined, we make a second assumption, on the initial datum and the nonlinearity.

ASSUMPTION 1.3. The initial datum φ and the power σ are such that:

1. $\varphi \in \Sigma := \{f \in H^1(\mathbb{R}^n) ; |x|f \in L^2(\mathbb{R}^n)\}$, where Σ is equipped with the norm

$$\|f\|_\Sigma = \|f\|_{L^2} + \|\nabla f\|_{L^2} + \|xf\|_{L^2} .$$

2. $1 \leq n \leq 5$ and $\sigma > 1/2$, so that the nonlinearity $|z|^{2\sigma} z$ is twice differentiable.
3. If $n = 1$, we assume in addition $\sigma > 1$.
4. If $3 \leq n \leq 5$, we take $\sigma < 2/(n - 2)$.
5. If $n \leq 2$, we assume
 - Either $\sigma > (2 - n + \sqrt{n^2 + 12n + 4})/(4n)$,
 - Or $\|\varphi\|_\Sigma \leq \delta$ sufficiently small.

REMARK. i) The assumption $\varphi \in \Sigma$ makes the energy (1.9) well defined at time $t = 0$.

ii) The assumption $\sigma < 2/(n - 2)$ is needed for a complete H^1 theory on (1.3) to be available (see e.g. [5]). The assumption $\sigma > 1/2$, used later on for the nonlinearity to be twice differentiable, therefore imposes the restriction $n \leq 5$.

iii) The third and fifth points of the above assumption are here to insure the existence of a complete scattering theory for (1.3). When $n \geq 3$, this theory is available because $\sigma > 1/2$. Denote $U_0(t) = e^{i(t/2)\Delta}$ the free Schrödinger group. From [12] and [6], since $\varphi \in \Sigma$, there exist $\psi_{\pm} \in \Sigma$ such that the unique solution ψ to (1.3) such that $\psi|_{t=0} = \varphi$ satisfies

$$(1.13) \quad \lim_{t \rightarrow \pm\infty} \|U_0(-t)\psi(t) - \psi_{\pm}\|_{\Sigma} = 0 .$$

We can now state our main result in the case $x_0 = \xi_0 = 0$.

Theorem 1.4. *Suppose that Assumptions 1.2 and 1.3 are satisfied.*

1. *For any $T > 0$, there exists $\varepsilon(T) > 0$ such that for $0 < \varepsilon \leq \varepsilon(T)$, (1.12) has a unique solution $u^\varepsilon \in C([-T, T]; \Sigma)$.*
2. *This solution satisfies the following asymptotics.*
 - *For any $\Lambda > 0$,*

$$(1.14) \quad \limsup_{\varepsilon \rightarrow 0} \sup_{|t| \leq \Lambda\varepsilon} \left(\|u^\varepsilon(t) - v^\varepsilon(t)\|_{L^2} + \|\varepsilon \nabla_x u^\varepsilon(t) - \varepsilon \nabla_x v^\varepsilon(t)\|_{L^2} + \left\| \frac{x}{\varepsilon} u^\varepsilon(t) - \frac{x}{\varepsilon} v^\varepsilon(t) \right\|_{L^2} \right) = 0 ,$$

where

$$(1.15) \quad v^\varepsilon(t, x) = \frac{1}{\varepsilon^{n/2}} \psi \left(\frac{t}{\varepsilon}, \frac{x}{\varepsilon} \right) ,$$

and $\psi \in C(\mathbb{R}; \Sigma)$ is the solution to (1.3) such that $\psi|_{t=0} = \varphi$.

- *Beyond this boundary layer, we have*

$$(1.16) \quad \limsup_{\varepsilon \rightarrow 0} \sup_{\Lambda\varepsilon \leq \pm t \leq T} \left(\|u^\varepsilon(t) - u_{\pm}^\varepsilon(t)\|_{L^2} + \|\varepsilon \nabla_x u^\varepsilon(t) - \varepsilon \nabla_x u_{\pm}^\varepsilon(t)\|_{L^2} + \|xu^\varepsilon(t) - xu_{\pm}^\varepsilon(t)\|_{L^2} \right) \xrightarrow{\Lambda \rightarrow +\infty} 0 ,$$

where $u_{\pm}^\varepsilon \in C(\mathbb{R}; \Sigma)$ are the solutions to

$$(1.17) \quad \begin{cases} i\varepsilon \partial_t u_{\pm}^\varepsilon + \frac{1}{2} \varepsilon^2 \Delta u_{\pm}^\varepsilon = V(x) u_{\pm}^\varepsilon , \\ u_{\pm}^\varepsilon|_{t=0} = \frac{1}{\varepsilon^{n/2}} \psi_{\pm} \left(\frac{x}{\varepsilon} \right) , \end{cases}$$

and ψ_{\pm} are given by (1.13).

REMARK. This result can be viewed as a nonlinear analog to a result due to Nier. In [17] (see also [16]), the author studies the problem

$$(1.18) \quad \begin{cases} i\varepsilon\partial_t u^\varepsilon + \frac{1}{2}\varepsilon^2\Delta u^\varepsilon = V(x)u^\varepsilon + U\left(\frac{x}{\varepsilon}\right)u^\varepsilon, \\ u^\varepsilon|_{t=0} = \frac{1}{\varepsilon^{n/2}}\varphi\left(\frac{x}{\varepsilon}\right), \end{cases}$$

where U is a short range potential. The potential V in that case is bounded as well as all its derivatives. In that paper, the author proves that under suitable assumptions, the influence of U occurs near $t = 0$ and is localized near the origin, while only the value $V(0)$ of V at the origin is relevant in this régime. For times $\varepsilon \ll |t| < T_*$, the situation is different: the potential U becomes negligible, while V dictates the propagation. As in our paper, the transition between these two régimes is measured by the scattering operator associated to U .

Assumption 1.3 implies in particular $n\sigma > 1$, which makes the nonlinear term short range. With our scaling for the nonlinearity, this perturbation is relevant only near the focus, where the potential is negligible, while the opposite occurs for $\varepsilon \ll |t| \leq T$.

The case $x_0 = \xi_0 = 0$ turns out not to be so particular in the case of a potential V satisfying (1.8), when no linear term is present, that is $b_j = 0, \forall j$. Introduce the change of variables

$$(1.19) \quad \begin{aligned} u^\varepsilon(t, x) &= u^\varepsilon(t, x - x(t))e^{iS(t,x)/\varepsilon}, \\ \text{with } S(t, x) &= x \cdot \xi(t) - \frac{1}{2}(x(t) \cdot \xi(t) - x_0 \cdot \xi_0), \end{aligned}$$

where $x(t)$ and $\xi(t)$ are given by (1.11). It is easy to check that if u^ε solves (1.12) with $x_0 = \xi_0 = 0$, then u^ε solves

$$(1.20) \quad \begin{cases} i\varepsilon\partial_t u^\varepsilon + \frac{1}{2}\varepsilon^2\Delta u^\varepsilon = V(x)u^\varepsilon + \varepsilon^{n\sigma}|u^\varepsilon|^{2\sigma}u^\varepsilon, \\ u^\varepsilon|_{t=0} = \frac{1}{\varepsilon^{n/2}}\varphi\left(\frac{x - x_0}{\varepsilon}\right)e^{ix \cdot \xi_0/\varepsilon}. \end{cases}$$

Corollary 1.5. *Let $(x_0, \xi_0) \in \mathbb{R}^n \times \mathbb{R}^n$. Under Assumptions 1.2 and 1.3, with $b_j = 0, \forall j$, we have:*

1. *For any $T > 0$, there exists $\varepsilon(T) > 0$ such that for $0 < \varepsilon \leq \varepsilon(T)$, (1.20) has a unique solution $u^\varepsilon \in C([-T, T]; \Sigma)$.*
2. *This solution satisfies the following asymptotics.*

- For any $\Lambda > 0$,

$$(1.21) \quad \limsup_{\varepsilon \rightarrow 0} \sup_{|t| \leq \Lambda \varepsilon} \left(\|u^\varepsilon(t) - v^\varepsilon(t)\|_{L^2} + \|\varepsilon \nabla_x u^\varepsilon(t) - \varepsilon \nabla_x v^\varepsilon(t)\|_{L^2} + \left\| \frac{x - x(t)}{\varepsilon} u^\varepsilon(t) - \frac{x - x(t)}{\varepsilon} v^\varepsilon(t) \right\|_{L^2} \right) = 0 ,$$

where

$$v^\varepsilon(t, x) = \frac{1}{\varepsilon^{n/2}} \psi \left(\frac{t}{\varepsilon}, \frac{x - x(t)}{\varepsilon} \right) e^{iS(t,x)/\varepsilon} ,$$

$\psi \in C(\mathbb{R}; \Sigma)$ is the solution to (1.3) such that $\psi|_{t=0} = \varphi$ and S is given by (1.19).

- Beyond this boundary layer, we have

$$(1.22) \quad \limsup_{\varepsilon \rightarrow 0} \sup_{\Lambda \varepsilon \leq \pm t \leq T} \left(\|u^\varepsilon(t) - u_\pm^\varepsilon(t)\|_{L^2} + \|\varepsilon \nabla_x u^\varepsilon(t) - \varepsilon \nabla_x u_\pm^\varepsilon(t)\|_{L^2} + \|(x - x(t)) (u^\varepsilon(t) - u_\pm^\varepsilon(t))\|_{L^2} \right) \xrightarrow{\Lambda \rightarrow +\infty} 0 ,$$

where $u_\pm^\varepsilon \in C(\mathbb{R}; \Sigma)$ are the solutions to

$$\begin{cases} i\varepsilon \partial_t u_\pm^\varepsilon + \frac{1}{2} \varepsilon^2 \Delta u_\pm^\varepsilon = V(x) u_\pm^\varepsilon , \\ u_\pm^\varepsilon|_{t=0} = \frac{1}{\varepsilon^{n/2}} \psi_\pm \left(\frac{x - x_0}{\varepsilon} \right) e^{ix \cdot \xi_0 / \varepsilon} , \end{cases}$$

and ψ_\pm are given by (1.13).

REMARK. i) The functions u_\pm^ε are also given by $u_\pm^\varepsilon(t, x) = u_\pm^\varepsilon(t, x - x(t)) e^{iS(t,x)/\varepsilon}$.

ii) The change of variable (1.19) could also be used in the case of an isotropic (attractive) harmonic potential to generalize the results of [2].

iii) The above corollary shows in particular that the results stated in Theorem 1.4 are independent of the fact that the concentrating point is a critical point for the potential V .

iv) After this article was written, it was noticed that we can go further into reducing the assumptions. Denote $b = (b_1, \dots, b_n)$, and define u_\sharp^ε by

$$u_\sharp^\varepsilon(t, x) := u^\varepsilon \left(t, x - \frac{t^2}{2} b \right) \exp \left\{ i \left(tb \cdot x - \frac{t^3}{3} |b|^2 \right) / \varepsilon \right\} .$$

As noticed in [4], if u^ε solves (1.12), then u_\sharp^ε solves the same initial value problem, with V replaced by

$$V_\sharp(x) = V(x) - b \cdot x ,$$

which satisfies Assumption 1.2 and has no linear part. Therefore, the conclusions of Corollary 1.5 still hold without the assumption $b = 0$.

This paper is organized as follows. In Section 2, we study the linear equations (1.17). We introduce some tools which are relevant in the nonlinear setting, and prove that under Assumption 1.2, possible refocusing occurs with less intensity for $t \neq 0$ than for $t = 0$. In Section 3, we establish local existence results in Σ for (1.12) when ε is fixed, for general subquadratic potentials. In Section 4, we prove the first asymptotics of Theorem 1.4, and the proof of Theorem 1.4 is completed in Section 5. Finally, we examine in Section 6 the asymptotic behaviour of u^ε solution to (1.12) when V is a general subquadratic potential, not necessarily of the form (1.8).

2. The linear equation

In this section, we analyze some properties of solutions of the equation

$$(2.1) \quad i\varepsilon \partial_t u^\varepsilon + \frac{1}{2} \varepsilon^2 \Delta u^\varepsilon = V(x) u^\varepsilon .$$

Under Assumption 1.2, it turns out that some tools which are classical in a linear setting (Heisenberg observables) are very helpful to study nonlinear problems. Introduce the unitary group

$$(2.2) \quad U^\varepsilon(t) := \exp i \frac{t}{\varepsilon} \left(\frac{\varepsilon^2}{2} \Delta - V(x) \right) .$$

This group is well-defined for subquadratic potentials (see [19], p. 199), and in particular under our assumptions.

We consider the following Heisenberg observables (see e.g. [20]),

$$(2.3) \quad A_1^\varepsilon(t) := U^\varepsilon(t) \frac{x}{\varepsilon} U^\varepsilon(-t) \quad ; \quad A_2^\varepsilon(t) := U^\varepsilon(t) i \varepsilon \nabla_x U^\varepsilon(-t) .$$

They solve

$$\partial_t A_1^\varepsilon(t) = U^\varepsilon(t) i \nabla_x U^\varepsilon(-t) = \frac{1}{\varepsilon} A_2^\varepsilon(t) \quad ; \quad \partial_t A_2^\varepsilon(t) = -U^\varepsilon(t) \nabla_x V U^\varepsilon(-t) .$$

Therefore,

$$\begin{aligned} \partial_t^2 A_{1,j}^\varepsilon(t) &= -\frac{1}{\varepsilon} U^\varepsilon(t) \partial_j V U^\varepsilon(-t) \\ &= -\delta_j \omega_j^2 U^\varepsilon(t) \frac{x_j}{\varepsilon} U^\varepsilon(-t) - \frac{b_j}{\varepsilon} \\ &= -\delta_j \omega_j^2 A_{1,j}^\varepsilon(t) - \frac{b_j}{\varepsilon} . \end{aligned}$$

We thus have explicitly,

$$(2.4) \quad \begin{aligned} A_{1,j}^\varepsilon(t) &:= \frac{x_j}{\varepsilon} h_j(t) + i g_j(t) \partial_j - \frac{b_j}{2\varepsilon} t^2, \\ A_{2,j}^\varepsilon(t) &:= -\delta_j \omega_j^2 x_j g_j(t) + i h_j(t) \varepsilon \partial_j - b_j t. \end{aligned}$$

These operators inherit interesting properties which we list below.

Lemma 2.1. *The operators $A_{l,j}^\varepsilon$ satisfy the following properties.*

- They commute with the linear part of (1.12),

$$(2.5) \quad \left[A_{l,j}^\varepsilon(t), i\varepsilon \partial_t + \frac{1}{2} \varepsilon^2 \Delta - V(x) \right] = 0, \quad \forall (l, j) \in \{1, 2\} \times \{1, \dots, n\}.$$

- Denote

$$\begin{aligned} \phi_1(t, x) &:= \frac{1}{2} \sum_{k=1}^n \left(\frac{h_k(t)}{g_k(t)} x_k^2 - b_k t x_k - \frac{t^3}{12} b_k^2 \right), \\ \phi_2(t, x) &:= -\frac{1}{2} \sum_{k=1}^n \left(\delta_k \omega_k^2 \frac{g_k(t)}{h_k(t)} x_k^2 + 2b_k t x_k + \frac{t^3}{3} b_k^2 \right). \end{aligned}$$

Then ϕ_1 and ϕ_2 are well-defined for almost every t , and

$$(2.6) \quad \begin{aligned} A_{1,j}^\varepsilon(t) &= i g_j(t) e^{i\phi_1(t,x)/\varepsilon} \partial_j \left(e^{-i\phi_1(t,x)/\varepsilon} \cdot \right), \\ A_{2,j}^\varepsilon(t) &= i \varepsilon h_j(t) e^{i\phi_2(t,x)/\varepsilon} \partial_j \left(e^{-i\phi_2(t,x)/\varepsilon} \cdot \right). \end{aligned}$$

- For $r \geq 2$, and $r < 2n/(n-2)$ if $n \geq 3$ ($r \leq \infty$ if $n = 1$), define $\delta(r)$ by

$$\delta(r) \equiv n \left(\frac{1}{2} - \frac{1}{r} \right).$$

Define $P^\varepsilon(t)$ by

$$P^\varepsilon(t) := \prod_{j=1}^n \left(|g_j(t)| + \varepsilon |h_j(t)| \right)^{1/n}.$$

There exists C_r such that, for any $f \in \Sigma$,

$$(2.7) \quad \|f\|_{L^r} \leq \frac{C_r}{P^\varepsilon(t)^{\delta(r)}} \|f\|_{L^2}^{1-\delta(r)} \max_{l,j} \|A_{l,j}^\varepsilon(t) f\|_{L^2}^{\delta(r)}.$$

- For any function $F \in C^1(\mathbb{C}, \mathbb{C})$ satisfying the gauge invariance condition

$$\exists G \in C(\mathbb{R}_+, \mathbb{R}), \quad F(z) = zG(|z|^2),$$

one has, for any $(l, j) \in \{1, 2\} \times \{1, \dots, n\}$ and almost all t ,

$$(2.8) \quad A_{l,j}^\varepsilon(t)F(w) = \partial_z F(w)A_{l,j}^\varepsilon(t)w - \partial_{\bar{z}} F(w)\overline{A_{l,j}^\varepsilon(t)w}.$$

Proof. The first point follows the definition of Heisenberg observables (Von Neumann equation). The second is straightforward computation. The third point is a consequence of the well-known Gagliardo-Nirenberg inequalities, and of (2.6). The last point is also a consequence of (2.6). \square

REMARK. i) In the definition of ϕ_1 (resp. ϕ_2), the factor $t^3 b_k^2/12$ (resp. $t^3 b_k^2/3$) may seem artificial, for it plays no role in the formula (2.6). We introduced these terms because their presence implies that ϕ_1 and ϕ_2 solve the eikonal equation

$$\partial_t \phi + \frac{1}{2} |\nabla_x \phi|^2 + V(x) = 0 .$$

This point is discussed further in details in Section 6.1.

ii) As noticed in [3], the fact that our operators enjoy the properties to be Heisenberg observables and factorized as in (2.6) is due to Assumption 1.2. We prove in Section 6 that other potentials cannot meet these two properties.

To conclude this section, we explain why the second point of Assumption 1.2 implies that there is no “strong” focusing outside $t = 0$ for (1.12). As we will see in the proof of Theorem 1.4, this is so because the solutions to (1.17) do not concentrate at one single point for $t \neq 0$.

Let $(l, j) \in \{1, 2\} \times \{1, \dots, n\}$. Because of (2.5), $A_{l,j}^\varepsilon u_\pm^\varepsilon$ solve (2.1), and

$$\|A_{l,j}^\varepsilon u_\pm^\varepsilon(t)\|_{L^2} = \|A_{l,j}^\varepsilon u_\pm^\varepsilon(0)\|_{L^2} = O(1) , \quad \text{as } \varepsilon \rightarrow 0.$$

Thus, for any r as in Lemma 2.1, there exists C independent of ε and t such that,

$$\|u_\pm^\varepsilon(t)\|_{L^r} \leq \frac{C}{P^\varepsilon(t)^{\delta(r)}} .$$

Notice that the concentration of u_\pm^ε is equivalent to the cancellation of the g_j 's. Assume that exactly p functions g_j 's cancel at time t_0 . For the corresponding h_j 's, we have $h_j(t_0) = 1$, and $P^\varepsilon(t_0)$ is of order exactly $\varepsilon^{p/n}$ as ε goes to zero. The functions u_\pm^ε concentrate on a space of dimension $n - p$.

At time $t = 0$, we have

$$(2.9) \quad \|u_\pm^\varepsilon(0)\|_{L^r}^r = \frac{1}{\varepsilon^{nr/2}} \int \left| \psi_\pm \left(\frac{x}{\varepsilon} \right) \right|^r dx = O \left(\varepsilon^{-r\delta(r)} \right).$$

From the second point of Assumption 1.2, if for $t_0 \neq 0$, p functions g_j 's cancel, then

necessarily, $p < n$, and

$$(2.10) \quad \|u_{\pm}^{\varepsilon}(t_0)\|_{L^r} = O\left(\varepsilon^{-\delta(r)p/n}\right).$$

Comparing (2.9) and (2.10) (recall that $p < n$) shows that the amplification of the L^r -norms cannot be so strong as at time $t = 0$. Since the scaling for the nonlinear term in (1.12) is critical for the concentration at one point, it is subcritical for any other concentration, this is why the nonlinear term is relevant only near the origin in the asymptotics stated in Theorem 1.4. This heuristic argument is made rigorous in Section 5, and uses the following lemma.

Lemma 2.2. *Let V satisfy Assumption 1.2, and denote $\underline{\omega} = \min \omega_j$. Let $\delta > 0$ and $k > 1$ such that $\delta k > 1$. Then*

$$\limsup_{\varepsilon \rightarrow 0} \varepsilon^{-(1/k)+\delta} \left(\int_{\Lambda\varepsilon}^{\pi/(2\underline{\omega})} \frac{dt}{P^{\varepsilon}(t)^{\delta k}} \right)^{1/k} \xrightarrow{\Lambda \rightarrow +\infty} 0.$$

Moreover, for any $T > 0$, there exists $C > 0$ independent of $\varepsilon \in]0, 1]$, such that

$$\left(\int_{\pi/(2\underline{\omega})}^T \frac{dt}{P^{\varepsilon}(t)^{\delta k}} \right)^{1/k} \leq C\varepsilon^{(1/k)-\delta+(\delta/n)}.$$

Sketch of the proof. The functions g_j 's may cancel at times $m\pi/\omega_j$, for $m \in \mathbb{Z}$. For $t \in [\Lambda\varepsilon, \pi/(2\underline{\omega})]$,

$$P^{\varepsilon}(t) \geq \frac{C}{t},$$

and the first part of the lemma follows. For the second part, split the considered integral into a sum of the form

$$\int_{\pi/(2\underline{\omega})}^{\pi/\underline{\omega}-\varepsilon} + \int_{\pi/\underline{\omega}-\varepsilon}^{\pi/\underline{\omega}+\varepsilon} + \int_{\pi/\underline{\omega}+\varepsilon}^{\pi/\omega_j-\varepsilon} + \dots + \int_{\pi/\omega_l+\varepsilon}^T.$$

We noticed that if at time $m\pi/\omega_j$, g_j cancels, then at most $n-1$ functions g_l 's cancel, and

$$P^{\varepsilon}(t) \geq C\varepsilon^{-1+1/n}, \quad \forall t \in \left[\frac{m\pi}{\omega_j} - \varepsilon, \frac{m\pi}{\omega_j} + \varepsilon \right].$$

This shows that integrals of the form

$$\int_{m\pi/\omega_j-\varepsilon}^{m\pi/\omega_j+\varepsilon}$$

yield the announced estimate. Other integrals are estimates in a similar fashion. □

3. Local existence results

In this section, we establish local existence results for nonlinear Schrödinger equations with a general subquadratic potential. This is a natural generalization of (1.12), and will be needed in Section 6. Consider a potential V satisfying the following properties.

- ASSUMPTION 3.1. The potential $V: \mathbb{R} \times \mathbb{R}^n \mapsto \mathbb{R}$ depends on t and x , and satisfies:
1. For fixed t , $V(t, \cdot) \in C^\infty(\mathbb{R}^n, \mathbb{R})$. We also assume that V is a measurable function of $(t, x) \in \mathbb{R} \times \mathbb{R}^n$.
 2. For $\alpha \in \mathbb{N}^n$, define

$$M_\alpha(t) = \sup_{x \in \mathbb{R}^n} |\partial_x^\alpha V(t, x)| + \sup_{x \leq 1} |V(t, x)|.$$

We assume that for any multi-index satisfying $|\alpha| \geq 2$, $M_\alpha \in L^\infty_{loc}(\mathbb{R})$.

Notice that the first point of Assumption 1.2 implies Assumption 3.1. Denote

$$(3.1) \quad U^\varepsilon(t) := \exp\left(i \frac{t}{\varepsilon} \left(\frac{\varepsilon^2}{2} \Delta - V\right)\right).$$

From [7], [8], there exists $\delta > 0$ independent of ε such that for $|t| \leq \delta$,

$$(3.2) \quad U^\varepsilon(t)f(x) = e^{-in(\pi/4)\text{sgn}t} \frac{1}{|2\pi\varepsilon t|^{n/2}} \int_{\mathbb{R}^n} k^\varepsilon(t, x, y) e^{iS(t,x,y)/\varepsilon} f(y) dy,$$

where S solves the eikonal equation

$$\partial_t S + \frac{1}{2} |\nabla_x S|^2 + V(t, x) = 0,$$

and k^ε is bounded as well as all its (x, y) -derivatives, uniformly for $\varepsilon \in]0, 1]$ and $|t| \leq \delta$.

The group U^ε is unitary on $L^2(\mathbb{R}^n)$, and there exist $\delta > 0$ and $C > 0$ independent of $\varepsilon \in]0, 1]$ such that for $|t| \leq \delta$,

$$\|U^\varepsilon(t)\|_{L^1 \rightarrow L^\infty} \leq \frac{C}{|\varepsilon t|^{n/2}}.$$

As noticed in [5] (see also [14]), this yields Strichartz type inequalities for U^ε .

DEFINITION 3.2. A pair (q, r) is *admissible* if $2 \leq r < 2n/(n-2)$ (resp. $2 \leq r \leq \infty$ if $n = 1$, $2 \leq r < \infty$ if $n = 2$) and

$$\frac{2}{q} = \delta(r) \equiv n \left(\frac{1}{2} - \frac{1}{r}\right).$$

The following proposition is a consequence of (3.2) and [14].

Proposition 3.3 (Strichartz inequalities). *The group $\mathbb{U}^\varepsilon(t)$ satisfies:*

1. *For any admissible pair (q, r) , any finite interval I , there exists $C_r(I)$ such that*

$$(3.3) \quad \varepsilon^{1/q} \|\mathbb{U}^\varepsilon(t)u\|_{L^q(I;L^r)} \leq C_r(I)\|u\|_{L^2}.$$

2. *For any admissible pairs (q_1, r_1) and (q_2, r_2) , and any finite interval I , there exists $C_{r_1,r_2}(I)$ such that*

$$(3.4) \quad \varepsilon^{(1/q_1)+(1/q_2)} \left\| \int_{I \cap \{s \leq t\}} \mathbb{U}^\varepsilon(t-s)F(s) ds \right\|_{L^{q_1}(I;L^{r_1})} \leq C_{r_1,r_2}(I) \|F\|_{L^{q_2'}(I;L^{r_2'})}.$$

The above constants are independent of ε .

For (q, r) an admissible pair and I a time interval, define

$$Y_r(I) := \left\{ \psi \in C(I; \Sigma); B\psi \in L^q(I; L^r) \cap L^\infty(I; L^2), \forall B \in \{\text{Id}, \nabla_x, |x|\} \right\}.$$

The main result of this section is the following.

Proposition 3.4. *Let \mathbb{V} satisfying Assumption 3.1, φ and σ satisfying Assumption 1.3. There exist $T > 0$ and a unique solution $\psi \in Y_{2\sigma+2}(\cdot - T, T)$ to the initial value problem,*

$$(3.5) \quad \begin{cases} i\partial_t \psi + \frac{1}{2} \Delta \psi = \mathbb{V}(t, x)\psi + |\psi|^{2\sigma} \psi, \\ \psi|_{t=0} = \varphi. \end{cases}$$

This solution actually belongs to $Y(\cdot - T, T)$, where

$$Y(I) := \left\{ \psi \in C(I; \Sigma); B\psi \in L^q(I; L^r), \forall B \in \{\text{Id}, \nabla_x, |x|\}, \forall (q, r) \text{ admissible} \right\}.$$

If the potential \mathbb{V} does not depend on time, we have the following conservation laws:

- *Mass:* $\|\psi(t)\|_{L^2} = \|\varphi\|_{L^2}, \forall |t| < T.$
- *Energy:*

$$E(t) := \frac{1}{2} \|\nabla_x \psi(t)\|_{L^2}^2 + \frac{1}{\sigma+1} \|\psi(t)\|_{L^{2\sigma+2}}^{2\sigma+2} + \int \mathbb{V}(x)|\psi(t, x)|^2 dx \equiv E(0), \quad \forall |t| < T.$$

Proof. First, notice that Duhamel’s principle for (3.5) writes

$$(3.6) \quad \psi(t, x) = \mathbb{U}(t)\varphi - i \int_0^t \mathbb{U}(t-s) (|\psi|^{2\sigma} \psi)(s) ds,$$

where $U(t) := U^1(t)$. To estimate the nonlinear term, we use Gagliardo-Nirenberg inequalities, which demand estimates on $\nabla_x \psi$. We have,

$$\left[i\partial_t + \frac{1}{2}\Delta - V(t, x), \nabla_x \right] = \nabla_x V(t, x) ; \quad \left[i\partial_t + \frac{1}{2}\Delta - V(t, x), x \right] = \nabla_x .$$

Therefore, Duhamel’s principles for $\nabla_x \psi$ and $x\psi$ are, for $B \in \{\nabla_x, x\}$,

$$(3.7) \quad B\psi(t, x) = U(t)B\varphi - i \int_0^t U(t-s)B(|\psi|^{2\sigma}\psi)(s) ds + i \int_0^t U(t-s)h_B(s) ds,$$

with $h_{\nabla}(t, x) = \nabla_x V(t, x)\psi(t, x)$, $h_x(t, x) = \nabla_x \psi(t, x)$.

Recall from Assumption 3.1, the potential V is subquadratic, $\nabla_x V(t, x) = O(\langle x \rangle)$, locally in time. We formally have to solve a closed system of three equations with three unknowns. This is achieved thanks to Strichartz inequalities, provided by the case $\varepsilon = 1$ in Proposition 3.3. The method is classical, and we refer to [5] for a complete proof. □

4. Inside the boundary layer

In this section, we prove that for any $\Lambda > 0$, the solution u^ε to (1.12) is in $C([- \Lambda\varepsilon, \Lambda\varepsilon]; \Sigma)$ for ε sufficiently small, and satisfies the asymptotics (1.14).

Introduce the remainder $w^\varepsilon := u^\varepsilon - v^\varepsilon$. From Proposition 3.4, there exists $T^\varepsilon > 0$ such that $u^\varepsilon \in C([-T^\varepsilon, T^\varepsilon]; \Sigma)$. Recall that v^ε is given by (1.15), where ψ is the solution to

$$(4.1) \quad \begin{cases} i\partial_t \psi + \frac{1}{2}\Delta \psi = |\psi|^{2\sigma} \psi, \\ \psi|_{t=0} = \varphi(x). \end{cases}$$

It is well-known (see e.g. [5]) that if $\varphi \in \Sigma$, then $\psi \in C(\mathbb{R}, \Sigma)$, therefore $v^\varepsilon \in C(\mathbb{R}; \Sigma)$, and $w^\varepsilon \in C([-T^\varepsilon, T^\varepsilon]; \Sigma)$. This remainder solves

$$\begin{cases} i\varepsilon\partial_t w^\varepsilon + \frac{1}{2}\varepsilon^2\Delta w^\varepsilon = V(x)u^\varepsilon + \varepsilon^{n\sigma} (|u^\varepsilon|^{2\sigma}u^\varepsilon - |v^\varepsilon|^{2\sigma}v^\varepsilon) , \\ w^\varepsilon|_{t=0} = 0 . \end{cases}$$

We rewrite this problem as

$$(4.2) \quad \begin{cases} i\varepsilon\partial_t w^\varepsilon + \frac{1}{2}\varepsilon^2\Delta w^\varepsilon = V(x)w^\varepsilon + V(x)v^\varepsilon + \varepsilon^{n\sigma} (|u^\varepsilon|^{2\sigma}u^\varepsilon - |v^\varepsilon|^{2\sigma}v^\varepsilon) , \\ w^\varepsilon|_{t=0} = 0 . \end{cases}$$

We shall actually prove a more precise result than that stated in Theorem 1.4.

Proposition 4.1. *Suppose that Assumptions 1.2 and 1.3 are satisfied. Let $\Lambda > 0$. Then for $0 < \varepsilon \leq \varepsilon(\Lambda)$, $u^\varepsilon \in C([-\Lambda\varepsilon, \Lambda\varepsilon]; \Sigma)$ and*

$$\limsup_{\varepsilon \rightarrow 0} \sup_{|t| \leq \Lambda\varepsilon} \left(\|w^\varepsilon(t)\|_{L^2} + \|A_{l,j}^\varepsilon(t)w^\varepsilon\|_{L^2} \right) = 0, \quad \forall (l, j) \in \{1, 2\} \times \{1, \dots, n\}.$$

Recall that $U^\varepsilon(t)$ is the group associated to the linear part of (1.12), given by (2.2). It satisfies Strichartz inequalities stated in Proposition 3.3. Duhamel’s principle for (4.2) is

$$(4.3) \quad \begin{aligned} w^\varepsilon(t) = & -i\varepsilon^{n\sigma-1} \int_0^t U^\varepsilon(t-s) (|u^\varepsilon|^{2\sigma}u^\varepsilon - |v^\varepsilon|^{2\sigma}v^\varepsilon)(s) ds \\ & - i\varepsilon^{-1} \int_0^t U^\varepsilon(t-s)V(x)v^\varepsilon(s) ds. \end{aligned}$$

To apply the results of Proposition 3.3, we introduce special indexes in the following algebraic lemma, whose easy proof is left out.

Lemma 4.2. *Let σ as in Assumption 1.3. There exist \underline{q} , \underline{r} , \underline{s} and \underline{k} satisfying*

$$(4.4) \quad \begin{cases} \frac{1}{\underline{r}'} = \frac{1}{\underline{r}} + \frac{2\sigma}{\underline{s}}, \\ \frac{1}{\underline{q}'} = \frac{1}{\underline{q}} + \frac{2\sigma}{\underline{k}}, \end{cases}$$

and the additional conditions:

- The pair $(\underline{q}, \underline{r})$ is admissible,
- $0 < 1/\underline{k} < \delta(\underline{s}) < 1$.

If $n = 1$, we choose $(\underline{q}, \underline{r}) = (\infty, 2)$, $\underline{s} = \infty$ and $\underline{k} = 2\sigma$.

From Proposition 3.3 applied with the above indexes, and Hölder inequality, (4.3) yields, for $I^\varepsilon \ni 0$ a time interval contained in $[-T^\varepsilon, T^\varepsilon]$,

$$(4.5) \quad \begin{aligned} \|w^\varepsilon\|_{L^{\underline{q}}(I^\varepsilon; L^{\underline{r}})} & \lesssim \varepsilon^{n\sigma-1-2/\underline{q}} (\|u^\varepsilon\|_{L^{\underline{k}}(I^\varepsilon; L^{\underline{s}})}^{2\sigma} + \|v^\varepsilon\|_{L^{\underline{k}}(I^\varepsilon; L^{\underline{s}})}^{2\sigma}) \|w^\varepsilon\|_{L^{\underline{q}}(I^\varepsilon; L^{\underline{r}})} \\ & + \varepsilon^{-1-1/\underline{q}} \|Vv^\varepsilon\|_{L^1(I^\varepsilon; L^2)}. \end{aligned}$$

We now have two tasks:

- Estimate the source term $\|Vv^\varepsilon\|_{L^1(I^\varepsilon; L^2)}$.
- Control the factor $\|u^\varepsilon\|_{L^{\underline{k}}(I^\varepsilon; L^{\underline{s}})}^{2\sigma} + \|v^\varepsilon\|_{L^{\underline{k}}(I^\varepsilon; L^{\underline{s}})}^{2\sigma}$.

Recall that v^ε is given by (1.15), so

$$\|V(\cdot)v^\varepsilon(t, \cdot)\|_{L^2}^2 = \frac{1}{4} \sum_{j=1}^n \omega_j^4 \varepsilon^4 \|x_j^2 \psi(\varepsilon t, x_j)\|_{L^2}^2 + \sum_{j=1}^n b_j^2 \varepsilon^2 \|x_j \psi(\varepsilon t, x_j)\|_{L^2}^2.$$

If $\varphi \in \Sigma$, then $\psi \in C(\mathbb{R}, \Sigma)$, and the above quantities are infinite in general.

4.1. Further regularity for ψ when $\varphi \in \mathcal{S}(\mathbb{R}^n)$. If we assume that φ belongs to the Schwartz space $\mathcal{S}(\mathbb{R}^n)$, then we can prove additional regularity for ψ .

Lemma 4.3. *Let $\varphi \in \mathcal{S}(\mathbb{R}^n)$, and ψ be the solution of the initial value problem (4.1). Let σ satisfying Assumption 1.3, and $\Lambda > 0$. Then,*

$$\begin{aligned} |x|^k \psi &\in C([- \Lambda, \Lambda], L^2), \quad \forall k \leq 3, \\ |x|^k \nabla_x \psi &\in C([- \Lambda, \Lambda], L^2), \quad \forall k \leq 2. \end{aligned}$$

Proof. As mentioned above, it is well-known that $\psi \in C([- \Lambda, \Lambda], \Sigma)$. Using the simple remark,

$$\left[i\partial_t + \frac{1}{2}\Delta, x \right] = \nabla_x,$$

the function $x_j \psi$ solves, for $1 \leq j \leq n$,

$$(4.6) \quad \left(i\partial_t + \frac{1}{2}\Delta \right) x_j \psi = \partial_j \psi + |\psi|^{2\sigma} x_j \psi.$$

For $1 \leq k \leq n$, we have,

$$(4.7) \quad \left(i\partial_t + \frac{1}{2}\Delta \right) x_j x_k \psi = \partial_k (x_j \psi) + x_k \partial_j \psi + |\psi|^{2\sigma} x_j x_k \psi.$$

This shows that to know that $x_j x_k \psi \in C([- \Lambda, \Lambda], L^2)$, it is enough to prove that $x_l \nabla_x \psi \in C([- \Lambda, \Lambda], L^2)$, for any l . Differentiating (4.1) with respect to x yields,

$$(4.8) \quad \left(i\partial_t + \frac{1}{2}\Delta \right) \nabla_x \psi = (\sigma + 1)|\psi|^{2\sigma} \nabla_x \psi + \sigma |\psi|^{2\sigma-2} \psi^2 \overline{\nabla_x \psi}.$$

Therefore,

$$(4.9) \quad \left(i\partial_t + \frac{1}{2}\Delta \right) x_l \nabla_x \psi = \partial_l \nabla_x \psi + (\sigma + 1)|\psi|^{2\sigma} \nabla_x \psi + \sigma |\psi|^{2\sigma-2} \psi^2 \overline{\nabla_x \psi}.$$

This shows that it is enough to know that $\Delta \psi \in C([- \Lambda, \Lambda], L^2)$. This is well-known, from an idea due to Kato ([13], see also [5]). The idea consists in differentiating (4.1) with respect to time and proving that $\partial_t \psi \in C([- \Lambda, \Lambda], L^2)$ when $\varphi \in H^2(\mathbb{R}^n)$. Then from (4.1), we deduce that $\Delta \psi \in C([- \Lambda, \Lambda], L^2)$. Thus, $|x|^k \psi \in C([- \Lambda, \Lambda], L^2)$ for $k \leq 2$ and $|x|^k \nabla_x \psi \in C([- \Lambda, \Lambda], L^2)$ for $k \leq 1$.

Now, we can apply Kato’s method to (4.8), and prove that if the nonlinearity $F(z) = |z|^{2\sigma} z$ is twice differentiable (hence the assumption $\sigma > 1/2$ in Assumption 1.3), then $\partial_t \nabla_x \psi \in C([- \Lambda, \Lambda], L^2)$. When using this information in (4.6), Kato’s

method proves that $x_j \partial_t \psi \in C([- \Lambda, \Lambda], L^2)$. Using the equation (4.6), we deduce that $x_j \Delta \psi \in C([- \Lambda, \Lambda], L^2)$. This information is enough to complete the proof of Lemma 4.3. Multiplying (4.7) by x_l yields,

$$\left(i \partial_t + \frac{1}{2} \Delta \right) x_j x_k x_l \psi = \partial_t (x_k \partial_j \psi) + x_l \partial_k (x_j \psi) + x_l x_k \partial_j \psi + |\psi|^{2\sigma} x_j x_k x_l \psi.$$

Reasoning as above, it is enough to know that $x \Delta \psi \in C([- \Lambda, \Lambda], L^2)$ and $x^\alpha \nabla_x \psi \in C([- \Lambda, \Lambda], L^2)$ for $|\alpha| \leq 2$. We saw how to prove the first point. We know that the second holds for $|\alpha| \leq 1$, thus we just have to multiply (4.9) by x_k ,

$$\begin{aligned} \left(i \partial_t + \frac{1}{2} \Delta \right) x_k x_l \nabla_x \psi &= \partial_k (x_l \nabla_x \psi) + x_k \partial_l \nabla_x \psi \\ &\quad + (\sigma + 1) |\psi|^{2\sigma} x_k \nabla_x \psi + \sigma |\psi|^{2\sigma - 2} \psi^2 \overline{x_k \nabla_x \psi}. \end{aligned}$$

Since $x \Delta \psi \in C([- \Lambda, \Lambda], L^2)$, we deduce that $|x|^2 \nabla_x \psi \in C([- \Lambda, \Lambda], L^2)$, which completes the proof. \square

REMARK. The assumption $\sigma > 1/2$ could be removed if we considered a smoother nonlinearity. Indeed, if we replaced $|\psi|^{2\sigma} \psi$ by $f(|\psi|^2) \psi$, with f smooth and

$$f(|\psi|^2) \lesssim |\psi|^{2\sigma} \text{ when } |\psi| \rightarrow 0,$$

we could prove Lemma 4.3 without the assumption $\sigma > 1/2$, and even more regularity for ψ (see for instance [10], [11]). This means, for (1.12), that we would replace $\varepsilon^{n\sigma} |u^\varepsilon|^{2\sigma} u^\varepsilon$ by $f(\varepsilon^n |u^\varepsilon|^2) u^\varepsilon$.

We apply Lemma 4.3 to study (1.12) thanks to the following result, which can be found for instance in [9], Proposition 3.5.

Proposition 4.4. *Let φ and σ satisfying Assumption 1.3. Let $\delta > 0$ and $\varphi_\delta \in \mathcal{S}(\mathbb{R}^n)$ such that $\|\varphi - \varphi_\delta\|_\Sigma \leq \delta$. If ψ_δ denotes the solution to (1.3) with initial datum φ_δ , then*

$$\|U_0(-t)(\psi(t) - \psi_\delta(t))\|_{L^\infty(\mathbb{R}; \Sigma)} \xrightarrow{\delta \rightarrow 0} 0,$$

and in particular, for every $\Lambda > 0$,

$$\|\psi - \psi_\delta\|_{L^\infty([- \Lambda, \Lambda]; \Sigma)} \xrightarrow{\delta \rightarrow 0} 0.$$

4.2. The coupling term. We want to estimate $\|u^\varepsilon\|_{L^k(I^\varepsilon; L^s)}^{2\sigma} + \|v^\varepsilon\|_{L^k(I^\varepsilon; L^s)}^{2\sigma}$. Gagliardo-Nirenberg inequalities and the property $\psi \in C(\mathbb{R}; \Sigma)$ yield

$$\|v^\varepsilon(t)\|_{L^s} = \varepsilon^{-\delta(\mathfrak{S})} \|\psi(\varepsilon t)\|_{L^s} \lesssim \varepsilon^{-\delta(\mathfrak{S})} \|\psi(\varepsilon t)\|_{L^2}^{1-\delta(\mathfrak{S})} \|\nabla_x \psi(\varepsilon t)\|_{L^2}^{\delta(\mathfrak{S})} \leq C_\Lambda \varepsilon^{-\delta(\mathfrak{S})},$$

for $|t| \leq \Lambda\varepsilon$, where C_Λ does not depend on ε . We expect a similar estimate to hold also for u^ε . From (2.7), it will be so if we know that u^ε is bounded in L^2 , as well as $A_{l,j}^\varepsilon u^\varepsilon$ for any l and j . The first point is easy: so long as u^ε is defined and sufficiently smooth, its L^2 -norm is constant (see Proposition 3.4). Showing the second is part of our proof. Since $\psi \in C(\mathbb{R}; \Sigma)$, it is easy to check that for any $\Lambda > 0$, there exists $C(\Lambda)$ independent of $\varepsilon \in]0, 1]$, such that

$$\|A_{l,j}^\varepsilon(t)v^\varepsilon(t, \cdot)\|_{L^2} \leq C(\Lambda) , \quad \forall (l, j) \in \{1, 2\} \times \{1, \dots, n\}, \quad \forall |t| \leq \Lambda\varepsilon .$$

Since $w^\varepsilon = 0$ at time $t = 0$ and $w^\varepsilon \in C([-T^\varepsilon, T^\varepsilon]; \Sigma)$ for some $T^\varepsilon > 0$, there exists $t^\varepsilon > 0$ such that for $|t| < t^\varepsilon$,

$$(4.10) \quad \|A_{l,j}^\varepsilon(t)w^\varepsilon(t, \cdot)\|_{L^2} \leq C(\Lambda) , \quad \forall (l, j) \in \{1, 2\} \times \{1, \dots, n\} .$$

So long as (4.10) holds, we can estimate $\|u^\varepsilon(t)\|_{L^{\underline{q}}}$ like $\|v^\varepsilon(t)\|_{L^{\underline{q}}}$, up to doubling the constants, but with the same power of ε .

Let $\eta > 0$ to be fixed later, and $I_\varepsilon \subset [-\eta\varepsilon, \eta\varepsilon]$ such that (4.10) holds on I_ε . If $\varphi \in \mathcal{S}(\mathbb{R}^n)$, Lemma 4.3 and (4.5) yield

$$(4.11) \quad \|w^\varepsilon\|_{L^{\underline{q}}(I_\varepsilon; L^{\underline{q}})} \lesssim \varepsilon^{n\sigma-1-2/\underline{q}-2\sigma\delta(\underline{y})+2\sigma/\underline{k}} \eta^{2\sigma/\underline{k}} \|w^\varepsilon\|_{L^{\underline{q}}(I_\varepsilon; L^{\underline{q}})} + \varepsilon^{1-1/\underline{q}} .$$

From Lemma 4.2,

$$n\sigma - 1 - \frac{2}{\underline{q}} - 2\sigma\delta(\underline{y}) + \frac{2\sigma}{\underline{k}} = 0 ,$$

and for $\eta > 0$ sufficiently small, the first term of the right hand side of (4.11) is absorbed by the left hand side,

$$(4.12) \quad \|w^\varepsilon\|_{L^{\underline{q}}(I_\varepsilon; L^{\underline{q}})} \lesssim \varepsilon^{1-1/\underline{q}} .$$

Apply Strichartz inequality (3.4) again, with now $r_1 = 2$ and $r_2 = \underline{q}$,

$$(4.13) \quad \|w^\varepsilon\|_{L^\infty(I_\varepsilon; L^2)} \lesssim \varepsilon^{n\sigma-1-1/\underline{q}-2\sigma\delta(\underline{y})+2\sigma/\underline{k}} \eta^{2\sigma/\underline{k}} \|w^\varepsilon\|_{L^{\underline{q}}(I_\varepsilon; L^{\underline{q}})} + \varepsilon \lesssim \varepsilon ,$$

from (4.12).

Assuming for a moment that we know that (4.10) holds for $|t| \leq \Lambda\varepsilon$, the above computation, repeated a finite number of times, yields an estimate of the form

$$(4.14) \quad \|w^\varepsilon\|_{L^\infty([- \Lambda\varepsilon, \Lambda\varepsilon]; L^2)} \leq C\varepsilon e^{C\Lambda} .$$

To prove that indeed (4.10) holds for $|t| \leq \Lambda\varepsilon$, we follow the same lines as above, replacing w^ε by $A_{l,j}^\varepsilon w^\varepsilon$. Since $A_{l,j}^\varepsilon$ commute with the linear part of (1.12) (see the

first point of Lemma 2.1), the analog of (4.3) for $A_{l,j}^\varepsilon w^\varepsilon$ is

$$(4.15) \quad \begin{aligned} A_{l,j}^\varepsilon(t)w^\varepsilon &= -i\varepsilon^{n\sigma-1} \int_0^t U^\varepsilon(t-s)A_{l,j}^\varepsilon(s) (|u^\varepsilon|^{2\sigma}u^\varepsilon - |v^\varepsilon|^{2\sigma}v^\varepsilon)(s) ds \\ &\quad - i\varepsilon^{-1} \int_0^t U^\varepsilon(t-s)A_{l,j}^\varepsilon(s)(V(x)v^\varepsilon(s)) ds. \end{aligned}$$

From Lemma 4.3, the source term (the last term in the above expression) is estimated as before. From (2.8) and (2.7), we can estimate the first term of the right hand side of (4.15) as above. This yields finally, so long as (4.10) holds and for $|t| \leq \Lambda\varepsilon$,

$$(4.16) \quad \|A_{l,j}^\varepsilon w^\varepsilon\|_{L^\infty(I^\varepsilon; L^2)} \leq C(l, j)\varepsilon e^{C(l,j)\Lambda} .$$

4.3. Conclusion. Let $\delta > 0$, and $\varphi_\delta \in \mathcal{S}(\mathbb{R}^n)$ such that $\|\varphi - \varphi_\delta\|_\Sigma \leq \delta$. Define ψ_δ as the solution to (1.3) with initial datum φ_δ , and v_δ^ε by

$$v_\delta^\varepsilon(t, x) = \frac{1}{\varepsilon^{n/2}}\psi_\delta\left(\frac{t}{\varepsilon}, \frac{x}{\varepsilon}\right) .$$

The remainder $w_\delta^\varepsilon := u^\varepsilon - v_\delta^\varepsilon$ solves

$$(4.17) \quad \begin{cases} i\varepsilon\partial_t w_\delta^\varepsilon + \frac{1}{2}\varepsilon^2\Delta w_\delta^\varepsilon = V(x)w_\delta^\varepsilon + V(x)v_\delta^\varepsilon + \varepsilon^{n\sigma} (|u^\varepsilon|^{2\sigma}u^\varepsilon - |v_\delta^\varepsilon|^{2\sigma}v_\delta^\varepsilon) , \\ w_\delta^\varepsilon(0, x) = \frac{1}{\varepsilon^{n/2}}(\varphi - \varphi_\delta)\left(\frac{x}{\varepsilon}\right) , \end{cases}$$

which is the analog of (4.2), with an initial datum which is nonzero, but arbitrarily small in Σ (as δ goes to zero).

Our method proves both the existence of u^ε in Σ up to time $\Lambda\varepsilon$ for ε sufficiently small, and the asymptotics (1.14). This approach is classical in geometrical optics (see e.g. [18]). From Proposition 3.4, it is well defined in Σ on the time interval $[-T^\varepsilon, T^\varepsilon]$ for some $T^\varepsilon > 0$. Since $v_\delta^\varepsilon \in C(\mathbb{R}; \Sigma)$, we want to prove that w_δ^ε exists in Σ up to time $\Lambda\varepsilon$ for ε sufficiently small, and is asymptotically small. By construction, we have

$$(4.18) \quad \|w_\delta^\varepsilon(0)\|_{L^2} + \sum_{(l,j) \in \{1,2\} \times \{1,\dots,n\}} \|A_{l,j}^\varepsilon(0)w_\delta^\varepsilon\|_{L^2} \leq \delta .$$

From Proposition 3.4, either w_δ^ε (hence u^ε) exists in Σ on the time interval $[-\Lambda\varepsilon, \Lambda\varepsilon]$, or the maximal solution belongs to $C([0, T^\varepsilon]; \Sigma)$ with $0 < T^\varepsilon < \Lambda\varepsilon$ and

$$\liminf_{t \rightarrow T^\varepsilon} \|w_\delta^\varepsilon(t)\|_\Sigma = \infty .$$

In the latter case, for any $\Gamma > 0$, there is a first time, T_Γ^ε such that

$$(4.19) \quad \|\tilde{w}^\varepsilon(T_\Gamma^\varepsilon)\|_{L^2} + \sum_{(l,j) \in \{1,2\} \times \{1,\dots,n\}} \|A_{l,j}^\varepsilon(T_\Gamma^\varepsilon)\tilde{w}^\varepsilon\|_{L^2} = \Gamma\delta .$$

We prove that there is $\Gamma > 0$ independent of Λ and ε , and a constant $C = C(\Lambda)$ independent of ε such that for $\varepsilon \leq 1$ and $t^\varepsilon \leq T_\Gamma^\varepsilon$,

$$(4.20) \quad \sup_{|t| \leq \Lambda\varepsilon} \left(\|w_\delta^\varepsilon(t)\|_{L^2} + \sum_{(l,j) \in \{1,2\} \times \{1,\dots,n\}} \|A_{l,j}^\varepsilon(t)w_\delta^\varepsilon\|_{L^2} \right) \leq \frac{\Gamma}{2}\delta + C\varepsilon .$$

Choosing ε sufficiently small so that $C\varepsilon < \Gamma/2$ contradicts (4.19). This proves that we can take $t^\varepsilon = \Lambda\varepsilon$ in (4.10).

Resuming the computations of Section 4.2 yields the same estimates as (4.11), plus a term estimated by $\delta\varepsilon^{-1/q}$, due to the initial datum. This means that in (4.13), (4.14) and (4.16), we have to replace ε by $\varepsilon + \delta$ in the right hand sides; this yields (4.20). We infer,

$$\limsup_{\varepsilon \rightarrow 0} \sup_{|t| \leq \Lambda\varepsilon} \left(\|w_\delta^\varepsilon(t)\|_{L^2} + \sum_{(l,j) \in \{1,2\} \times \{1,\dots,n\}} \|A_{l,j}^\varepsilon(t)w_\delta^\varepsilon\|_{L^2} \right) \leq \frac{\Gamma}{2}\delta ,$$

where C does not depend on δ . Choosing δ arbitrarily small, the above estimate and Proposition 4.4 yield Proposition 4.1.

Finally, Proposition 4.1 implies the asymptotics (1.14). Rewrite the definition of $A_{l,j}^\varepsilon$,

$$\begin{pmatrix} A_{1,j}^\varepsilon \\ A_{2,j}^\varepsilon \end{pmatrix} = \begin{pmatrix} h_j & g_j/\varepsilon \\ -\varepsilon\delta_j\omega_j^2g_j & h_j \end{pmatrix} \begin{pmatrix} x_j/\varepsilon \\ i\varepsilon\partial_j \end{pmatrix} - b_j \begin{pmatrix} t^2/(2\varepsilon) \\ t \end{pmatrix} .$$

The determinant of the above matrix is

$$h_j^2 + \delta_j\omega_j^2g_j^2 \equiv 1,$$

and we have

$$(4.21) \quad \begin{aligned} \frac{x_j}{\varepsilon} &= h_j(t)A_{1,j}^\varepsilon(t) - \frac{g_j(t)}{\varepsilon}A_{2,j}^\varepsilon(t) + b_j \left(\frac{t^2}{2\varepsilon}h_j(t) - \frac{t}{\varepsilon}g_j(t) \right) , \\ i\varepsilon\partial_j &= \varepsilon\delta_j\omega_j^2g_j(t)A_{1,j}^\varepsilon(t) + h_j(t)A_{2,j}^\varepsilon(t) + b_j \left(\delta_j\omega_j^2\frac{t^2}{2}g_j(t) + th_j(t) \right) . \end{aligned}$$

Since $g_j(t) = O(t)$ as t goes to zero, it is clear that Proposition 4.1 implies the asymptotics (1.14).

5. Beyond the boundary layer

In this section, we complete the proof of Theorem 1.4. The end of the proof is divided into two parts; we first study the transition between the two régimes (1.14) and (1.16), then prove the existence of u^ε along with the asymptotics (1.16). Since

the proofs are similar for positive or negative times, we restrict to the case of positive times.

5.1. Matching the two régimes. In Proposition 4.1, Λ was a fixed parameter; in any boundary layer of size $\Lambda\varepsilon$ around the origin, the asymptotic behaviour of u^ε is given by v^ε . For $t \gg \varepsilon$, the behaviour of u^ε is asymptotically the same as that of u_+^ε . We now prove that the transition between these two régimes occurs in a boundary layer of size $\Lambda\varepsilon$, when Λ goes to infinity.

Proposition 5.1. *The function u_+^ε becomes an approximate solution of u^ε when t reaches $\Lambda\varepsilon$, for large Λ .*

$$\limsup_{\varepsilon \rightarrow 0} \left(\|u^\varepsilon(\Lambda\varepsilon) - u_+^\varepsilon(\Lambda\varepsilon)\|_{L^2} + \|A_{l,j}^\varepsilon(\Lambda\varepsilon) (u^\varepsilon - u_+^\varepsilon)\|_{L^2} \right) \xrightarrow{\Lambda \rightarrow +\infty} 0, \\ \forall (l, j) \in \{1, 2\} \times \{1, \dots, n\}.$$

Proof. From Proposition 4.1, we only have to prove the above limit when u^ε is replaced by v^ε . We proceed to another reduction of the problem, by noticing that for $|t| \leq \Lambda\varepsilon$, the role of the potential V is negligible not only for u^ε , but also for u_+^ε . Define v_+^ε by

$$(5.1) \quad \begin{cases} i\varepsilon\partial_t v_+^\varepsilon + \frac{1}{2}\varepsilon^2\Delta v_+^\varepsilon = 0, \\ v_+^\varepsilon|_{t=0} = \frac{1}{\varepsilon^{n/2}}\psi_+\left(\frac{x}{\varepsilon}\right). \end{cases}$$

By scaling, we have

$$v_+^\varepsilon(t, x) = \frac{1}{\varepsilon^{n/2}}\psi_+^0\left(\frac{t}{\varepsilon}, \frac{x}{\varepsilon}\right),$$

where $\psi_+^0(t, x) = \exp(it\Delta/2)\psi_+(x)$.

Lemma 5.2. *Let $\Lambda \geq 1$. The potential V is negligible for $0 \leq t \leq \Lambda\varepsilon$ in (1.17),*

$$\limsup_{\varepsilon \rightarrow 0} \sup_{0 \leq t \leq \Lambda\varepsilon} \left(\|u_+^\varepsilon(t) - v_+^\varepsilon(t)\|_{L^2} + \|A_{l,j}^\varepsilon(t) (u_+^\varepsilon - v_+^\varepsilon)\|_{L^2} \right) = 0, \\ \forall (l, j) \in \{1, 2\} \times \{1, \dots, n\}.$$

Proof of Lemma 5.2. Denote $w_+^\varepsilon = u_+^\varepsilon - v_+^\varepsilon$. We have,

$$\begin{cases} i\varepsilon\partial_t w_+^\varepsilon + \frac{1}{2}\varepsilon^2\Delta w_+^\varepsilon = V(x)w_+^\varepsilon + V(x)v_+^\varepsilon, \\ w_+^\varepsilon|_{t=0} = 0. \end{cases}$$

From the classical energy estimates (which are also a consequence of Strichartz inequalities),

$$\begin{aligned} \sup_{0 \leq t \leq \Lambda \varepsilon} \|w_+^\varepsilon(t)\|_{L^2} &\lesssim \varepsilon^{-1} \int_0^{\Lambda \varepsilon} \|V(\cdot)v_{+\text{app}}^\varepsilon(t, \cdot)\|_{L^2} dt \\ &\lesssim \int_0^\Lambda \|V(\varepsilon \cdot)\psi_+^0(t, \cdot)\|_{L^2} dt . \end{aligned}$$

By density (for ψ_+), we can assume that ψ_+^0 has the same smoothness as in Lemma 4.3 (the proof is even easier since we now consider *linear* problems). In that case we have

$$\sup_{0 \leq t \leq \Lambda \varepsilon} \|w_+^\varepsilon(t)\|_{L^2} = O(\varepsilon) .$$

The proof that $A_{i,j}^\varepsilon(t)w_+^\varepsilon$ satisfies the same property is straightforward. Finally, without the smoothness assumption of Lemma 4.3, $O(\varepsilon)$ is replaced by $o(1)$, and the proof of Lemma 5.2 is complete. \square

Recall that we have

$$\begin{aligned} v_+^\varepsilon(\Lambda \varepsilon, x) &= \frac{1}{\varepsilon^{n/2}} U_0(\Lambda)\psi_+\left(\frac{x}{\varepsilon}\right) , \quad v^\varepsilon(\Lambda \varepsilon, x) = \frac{1}{\varepsilon^{n/2}} \psi\left(\Lambda, \frac{x}{\varepsilon}\right) , \\ \lim_{t \rightarrow +\infty} \left\| U_0(-t)\psi(t) - \psi_+ \right\|_\Sigma &= 0 , \end{aligned}$$

where the last line is nothing but (1.13). This implies in particular, since U_0 is unitary on L^2 ,

$$\limsup_{\varepsilon \rightarrow 0} \|v^\varepsilon(\Lambda \varepsilon) - v_+^\varepsilon(\Lambda \varepsilon)\|_{L^2} \xrightarrow{\Lambda \rightarrow +\infty} 0 ,$$

which is the first asymptotics in Proposition 5.1.

To conclude the proof, the idea is that the operator appearing in (1.13) are close to the operators $A_{i,j}^\varepsilon(t)$ for $|t| \leq \Lambda \varepsilon$. Using the identity

$$U_0(t)xU_0(-t) = x + it\nabla_x ,$$

and the fact that the group U_0 is unitary on L^2 , we can rewrite (1.13) as

$$\begin{aligned} &\|\psi(t) - U_0(t)\psi_+\|_{L^2} + \|\nabla_x \psi(t) - U_0(t)\nabla_x \psi_+\|_{L^2} \\ &\quad + \|(x + it\nabla_x)(\psi(t) - U_0(t)\psi_+)\|_{L^2} \xrightarrow{t \rightarrow +\infty} 0 . \end{aligned}$$

From the definition of the function h_j 's and g_j 's, we have, as $t \rightarrow 0$,

$$h_j(t) = 1 + O(t) \quad ; \quad g_j(t) = t + O(t^2) .$$

Therefore, we have, in the case of v^ε ,

$$\begin{aligned} & \left\| \left(A_{1,j}^\varepsilon(\Lambda\varepsilon) - \frac{x_j}{\varepsilon} - i(\Lambda\varepsilon)\partial_j \right) v^\varepsilon(\Lambda\varepsilon, \cdot) \right\|_{L^2} \\ &= \left\| \left(\frac{x_j}{\varepsilon}(h_j(\Lambda\varepsilon) - 1) + i(g_j(\Lambda\varepsilon) - \Lambda\varepsilon)\partial_j - \frac{b_j}{2}\Lambda^2\varepsilon \right) v^\varepsilon(\Lambda\varepsilon, \cdot) \right\|_{L^2} \\ &= \left\| \left(x_j(h_j(\Lambda\varepsilon) - 1) + i \frac{g_j(\Lambda\varepsilon) - \Lambda\varepsilon}{\varepsilon} \partial_j - \frac{b_j}{2}\Lambda^2\varepsilon \right) \psi(\Lambda, \cdot) \right\|_{L^2} \\ &= O(\varepsilon), \end{aligned}$$

for any fixed $\Lambda \geq 1$, since $\psi \in C(\mathbb{R}; \Sigma)$. Similar computations hold with $A_{2,j}^\varepsilon$, and when v^ε is replaced by v_+^ε . The proof of Proposition 5.1 is complete. \square

5.2. The linear régime. We now complete the proof of Theorem 1.4. Fix $T > 0$. From (4.21), it is enough to prove that $u^\varepsilon(t)$, as well as $A_{l,j}^\varepsilon(t)u^\varepsilon$ for any l, j , remains bounded in L^2 , up to time T , provided that ε is sufficiently small. The relation (4.21) shows in addition that we can prove the asymptotics (1.16) when the operators $\varepsilon\nabla$ and x are replaced by the $A_{l,j}^\varepsilon(t)$'s.

Our method is the same as in Section 4. Introduce the remainder

$$\tilde{w}_+^\varepsilon = u^\varepsilon - u_+^\varepsilon.$$

From Proposition 4.1, it is well defined in Σ up to time $\Lambda\varepsilon$ for any $\Lambda > 0$, provided that ε is sufficiently small. It solves

$$i\varepsilon\partial_t\tilde{w}^\varepsilon + \frac{1}{2}\varepsilon^2\Delta\tilde{w}^\varepsilon = V(x)\tilde{w}^\varepsilon + \varepsilon^{n\sigma}|u^\varepsilon|^{2\sigma}u^\varepsilon.$$

Since $v_+^\varepsilon \in C(\mathbb{R}; \Sigma)$ (see in particular (2.5) and (4.21)), we want to prove that \tilde{w}^ε exists in Σ up to time T for ε sufficiently small, and is asymptotically small in the sense of (1.16). From Proposition 5.1,

$$\limsup_{\varepsilon \rightarrow 0} \left(\|\tilde{w}^\varepsilon(\Lambda\varepsilon)\|_{L^2} + \|A_{l,j}^\varepsilon(\Lambda\varepsilon)\tilde{w}^\varepsilon\|_{L^2} \right) \xrightarrow{\Lambda \rightarrow +\infty} 0, \quad \forall (l, j) \in \{1, 2\} \times \{1, \dots, n\}.$$

Let $\delta > 0$. From Proposition 5.1, there exist $\varepsilon_0 > 0$ and Λ_0 such that for $0 < \varepsilon \leq \varepsilon_0$ and $\Lambda \geq \Lambda_0$,

$$(5.2) \quad \|\tilde{w}^\varepsilon(\Lambda\varepsilon)\|_{L^2} + \sum_{(l,j) \in \{1,2\} \times \{1,\dots,n\}} \|A_{l,j}^\varepsilon(\Lambda\varepsilon)\tilde{w}^\varepsilon\|_{L^2} \leq \delta.$$

From Proposition 5.1 again, there exists $t^\varepsilon > \Lambda\varepsilon$ such that

$$(5.3) \quad \sup_{\Lambda\varepsilon \leq t \leq t^\varepsilon} \left(\|\tilde{w}^\varepsilon(t)\|_{L^2} + \sum_{(l,j) \in \{1,2\} \times \{1,\dots,n\}} \|A_{l,j}^\varepsilon(t)\tilde{w}^\varepsilon\|_{L^2} \right) \leq 2\delta.$$

Let $0 < \varepsilon \leq \varepsilon_0$ and $\Lambda \geq \Lambda_0$. From Proposition 3.4, either \tilde{w}^ε (hence u^ε) exists in Σ on the time interval $[0, T]$, or the maximal solution belongs to $C([0, T^\varepsilon]; \Sigma)$ with $0 < T^\varepsilon < T$ and

$$\liminf_{t \rightarrow T^\varepsilon} \|\tilde{w}^\varepsilon(t)\|_\Sigma = \infty .$$

From (4.21), in the latter case, there is a first time, T_0^ε such that

$$(5.4) \quad \|\tilde{w}^\varepsilon(T_0^\varepsilon)\|_{L^2} + \sum_{(l,j) \in \{1,2\} \times \{1,\dots,n\}} \|A_{l,j}^\varepsilon(T_0^\varepsilon)\tilde{w}^\varepsilon\|_{L^2} = 4\delta .$$

We prove that, up to choosing Λ even larger, there is a constant $C = C(T)$ independent of ε and Λ such that for $\varepsilon \leq 1$ and $t^\varepsilon \leq T_0^\varepsilon$,

$$(5.5) \quad \sup_{\Lambda\varepsilon \leq t \leq t^\varepsilon} \left(\|\tilde{w}^\varepsilon(t)\|_{L^2} + \sum_{(l,j) \in \{1,2\} \times \{1,\dots,n\}} \|A_{l,j}^\varepsilon(t)\tilde{w}^\varepsilon\|_{L^2} \right) \leq 3\delta + C\varepsilon^{2\sigma\delta(\varrho)/n} .$$

Choosing ε sufficiently small so that $C\varepsilon^{2\sigma\delta(\varrho)/n} < \delta$ contradicts (5.4). This proves that we can take $t^\varepsilon = T$ in (5.3), hence the first point of Theorem 1.4, along with the asymptotics (1.16), since $\delta > 0$ is arbitrary (recall that for any fixed $\delta > 0$, we have to choose ε small and Λ large, so that (5.2) holds).

Recall that u_+^ε solves the linear equation (1.17); its L^2 -norm is independent of time, and from (2.5), the same holds for $A_{l,j}^\varepsilon u_+^\varepsilon$, for any l and j . So long as (5.3) holds, we thus have an L^2 bound for u^ε and $A_{l,j}^\varepsilon u^\varepsilon$,

$$(5.6) \quad \sup_{\Lambda\varepsilon \leq t \leq t^\varepsilon} \left(\|u^\varepsilon(t)\|_{L^2} + \sum_{(l,j) \in \{1,2\} \times \{1,\dots,n\}} \|A_{l,j}^\varepsilon(t)u^\varepsilon\|_{L^2} \right) \leq C_* .$$

Denote $J^\varepsilon := [\Lambda\varepsilon, t^\varepsilon]$. From Strichartz inequalities and Lemma 4.2,

$$\|\tilde{w}^\varepsilon\|_{L^\infty(J^\varepsilon; L^2)} \leq \|\tilde{w}^\varepsilon(\Lambda\varepsilon)\|_{L^2} + C\varepsilon^{n\sigma-1-1/q} \|u^\varepsilon\|_{L^{\frac{2\sigma}{1-q}}(J^\varepsilon; L^2)} \|u^\varepsilon\|_{L^q(J^\varepsilon; L^2)} .$$

From (5.6), (2.7) and Lemma 2.2, we infer that if $t^\varepsilon \leq \pi/(2\omega)$,

$$\|\tilde{w}^\varepsilon\|_{L^\infty(J^\varepsilon; L^2)} \leq \|\tilde{w}^\varepsilon(\Lambda\varepsilon)\|_{L^2} + \rho(\Lambda)\varepsilon^{1/q} \|u^\varepsilon\|_{L^q(J^\varepsilon; L^2)} ,$$

where $\rho(\Lambda)$ is a function independent of ε that goes to zero as Λ goes to infinity. Using (5.6) and (2.7) again, we have

$$\varepsilon^{1/q} \|u^\varepsilon\|_{L^q(J^\varepsilon; L^2)} \leq C|J^\varepsilon|^{1/q} \leq CT^{1/q} .$$

Therefore,

$$\|\tilde{w}^\varepsilon\|_{L^\infty(J^\varepsilon; L^2)} \leq \|\tilde{w}^\varepsilon(\Lambda\varepsilon)\|_{L^2} + CT^{1/q}\rho(\Lambda) .$$

Taking Λ even larger if necessary, (5.2) implies that if $t^\varepsilon \leq \pi/(2\underline{\omega})$, then

$$\|\tilde{w}^\varepsilon\|_{L^\infty(J^\varepsilon;L^2)} \leq \|\tilde{w}^\varepsilon(\Lambda\varepsilon)\|_{L^2} + \delta .$$

For $t^\varepsilon \geq \pi/(2\underline{\omega})$, the second part of Lemma 2.2 implies

$$\begin{aligned} \|\tilde{w}^\varepsilon\|_{L^\infty(J^\varepsilon;L^2)} &\leq \|\tilde{w}^\varepsilon(\Lambda\varepsilon)\|_{L^2} + C\varepsilon^{n\sigma-1-1/q} \|u^\varepsilon\|_{L^k(J^\varepsilon;L^2)}^{2\sigma} \|u^\varepsilon\|_{L^q(J^\varepsilon;L^2)} \\ &\leq \|\tilde{w}^\varepsilon(\Lambda\varepsilon)\|_{L^2} + CT^{1/q} \varepsilon^{n\sigma-1-2/q} \|u^\varepsilon\|_{L^k(J^\varepsilon;L^2)}^{2\sigma} \\ (5.7) \qquad &\leq \|\tilde{w}^\varepsilon(\Lambda\varepsilon)\|_{L^2} + CT^{1/q} \varepsilon^{n\sigma-1-2/q} \|u^\varepsilon\|_{L^k(\{\Lambda\varepsilon, \pi/(2\underline{\omega})\};L^2)}^{2\sigma} \\ &\qquad\qquad\qquad + CT^{1/q} \varepsilon^{n\sigma-1-2/q} \|u^\varepsilon\|_{L^k(\{\pi/(2\underline{\omega}), t^\varepsilon\};L^2)}^{2\sigma} \\ &\leq \|\tilde{w}^\varepsilon(\Lambda\varepsilon)\|_{L^2} + \delta + C(T)\varepsilon^{2\sigma\delta(\underline{\omega})/n} . \end{aligned}$$

Computations for $A_{l,j}^\varepsilon(t)\tilde{w}^\varepsilon$ are similar. Since $A_{l,j}^\varepsilon$ acts like a derivative on the nonlinear term (Lemma 2.1), we have

$$\|A_{l,j}^\varepsilon\tilde{w}^\varepsilon\|_{L^\infty(J^\varepsilon;L^2)} \leq \|A_{l,j}^\varepsilon(\Lambda\varepsilon)\tilde{w}^\varepsilon\|_{L^2} + C\varepsilon^{n\sigma-1-1/q} \|u^\varepsilon\|_{L^k(J^\varepsilon;L^2)}^{2\sigma} \|A_{l,j}^\varepsilon u^\varepsilon\|_{L^q(J^\varepsilon;L^2)} .$$

Estimate (5.6), along with Proposition 3.4, implies that there exists $C(T)$ such that for $t^\varepsilon \leq T$,

$$\varepsilon^{1/q} \|A_{l,j}^\varepsilon u^\varepsilon\|_{L^q(J^\varepsilon;L^2)} \leq C(T) .$$

We thus have the same estimate as above, for Λ sufficiently large,

$$(5.8) \qquad \|A_{l,j}^\varepsilon\tilde{w}^\varepsilon\|_{L^\infty(J^\varepsilon;L^2)} \leq \|A_{l,j}^\varepsilon(\Lambda\varepsilon)\tilde{w}^\varepsilon\|_{L^2} + \frac{\delta}{2n} + C(T)\varepsilon^{2\sigma\delta(\underline{\omega})/n} .$$

Summing (5.7) and (5.8) yields (5.5), which completes the proof of Theorem 1.4.

6. Partial results for general subquadratic potentials

Intuitively, there is no reason why Theorem 1.4 should not be true for more general potentials than (1.5), in particular for potentials satisfying Assumption 3.1. We prove in particular that (1.14) still holds for this class of potentials. However, we cannot prove (1.16). From the technical point of view, this is due to the lack of operators such as $A_{l,j}^\varepsilon$. For the linear régime, these operators have three major advantages:

- They commute with the linear part of the equation, including the potential, see (2.5).
- They yield modified Gagliardo-Nirenberg inequalities, (2.7).
- They act on the nonlinear term like derivatives, (2.8).

As we mentioned in the proof of Lemma 2.1, the last two points follow from the formula (2.6). We first prove that there exists an operator satisfying a similar formula *and* commuting with the linear part of the equation, (2.5), if and only if the potential is of

the form we consider, (1.5). We then prove (1.14) for general potentials satisfying Assumption 3.1.

6.1. Lemma 2.1 holds only for potentials of the form (1.5). Let \mathbb{V} satisfying Assumption 3.1, independent of time ($\mathbb{V} = \mathbb{V}(x)$), and define an operator $A^\varepsilon(t)$ by

$$(6.1) \quad A_j^\varepsilon(t) = i f_j(t) e^{i\phi(t,x)/\varepsilon} \partial_j \left(e^{-i\phi(t,x)/\varepsilon} \cdot \right) = \frac{f_j(t)}{\varepsilon} \partial_j \phi(t, x) + i f_j(t) \partial_j ,$$

where f_j and ϕ are real-valued functions, to be determined. The operator A^ε is defined as $A^\varepsilon = (A_j^\varepsilon)_{1 \leq j \leq n}$. Note that the weight f_j may depend on the index j (anisotropy), but not ϕ . This is the generalization of (2.6). Such an operator formally satisfies (2.8) and an analog to (2.7). Notice that in (2.6), the phases ϕ_l ($l = 1$ or 2) solve the eikonal equation

$$(6.2) \quad \partial_t \phi + \frac{1}{2} |\nabla_x \phi|^2 + \mathbb{V}(x) = 0 .$$

Proposition 6.1. *Let $\phi \in C^4([0, T] \times \mathbb{R}^n; \mathbb{R})$ and $f_j \in C^1([0, T])$ for some $T > 0$. Assume that for any $1 \leq j \leq n$, f_j does not cancel on the interval $]0, T]$. Then A^ε , defined by (6.1), satisfies (2.5) if and only if \mathbb{V} is of the form (1.5).*

REMARK. i) We do not assume that ϕ solves the eikonal equation (6.2). However, we will see in the proof that it is essentially necessary.
 ii) Since from Von Neumann equation, Heisenberg observables always satisfy (2.5), the above proposition implies that such an observable can be written under the form (6.1), for some functions f and ϕ , if and only if the potential \mathbb{V} is of the form (1.5).

Proof. We now only have to prove the “only if” part. Computations yield

$$(6.3) \quad \left[i \varepsilon \partial_t + \frac{1}{2} \varepsilon^2 \Delta - \mathbb{V}(x), A_j^\varepsilon(t) \right] = f_j'(t) \partial_j \phi + f_j(t) \partial_{j_i}^2 \phi + f_j(t) \partial_j \mathbb{V} + \varepsilon \left(- f_j'(t) \partial_j + f_j(t) \nabla_x(\partial_j \phi) \cdot \nabla_x + \frac{1}{2} f_j(t) \Delta(\partial_j \phi) \right) .$$

This bracket is zero if and only if the terms in ε^0 and ε^1 are zero. The term in ε is the sum of an operator of order one and of an operator of order zero. It is zero if and only if both operators are zero. The operator of order one is zero if and only if

$$f_j(t) \partial_{j_j}^2 \phi = f_j'(t) , \quad \partial_{j_k}^2 \phi \equiv 0 \quad \text{if } j \neq k .$$

In particular, $\partial_{j_j}^2 \phi$ is a function of time only, independent of x , and we have

$$\frac{1}{2} f_j(t) \Delta(\partial_j \phi) \equiv 0 .$$

From the above computations, the first two terms in ε^0 also write

$$f'_j(t)\partial_j\phi + f_j(t)\partial_{j_t}^2\phi = \sum_{k=1}^n f_j(t)\partial_k\phi\partial_{jk}^2\phi + f_j(t)\partial_{j_t}^2\phi = f_j(t)\partial_j \left(\partial_t\phi + \frac{1}{2}|\nabla_x\phi|^2 \right).$$

Canceling the term in ε^0 in (6.3) therefore yields, since f_j is never zero on $]0, T]$,

$$(6.4) \quad \partial_j \left(\partial_t\phi + \frac{1}{2}|\nabla_x\phi|^2 + V(x) \right) = 0.$$

Differentiating the above equation with respect to x_k and x_l , all the terms with ϕ vanish, since we noticed that the derivatives of order at least three of ϕ are zero. We deduce that for any triplet (j, k, l) , $\partial_{jkl}^3V \equiv 0$, that is, V is of the form (1.5).

Notice that since (6.4) holds for any $j \in \{1, \dots, n\}$, there exists a function Ξ of time only such that

$$\partial_t\phi + \frac{1}{2}|\nabla_x\phi|^2 + V(x) = \Xi(t).$$

This means that ϕ is almost a solution to the eikonal equation (6.2). Replacing ϕ by $\tilde{\phi}(t, x) := \phi(t, x) - \int_0^t \Xi(s) ds$ does not affect (6.1), and $\tilde{\phi}$ solves (6.2). \square

6.2. Heisenberg observables for general subquadratic potentials. We now suppose that $V = V(t, x)$ satisfies Assumption 3.1. Define the Heisenberg observable

$$A^\varepsilon(t) = U^\varepsilon(t) \frac{x}{\varepsilon} U^\varepsilon(-t),$$

where the group U^ε is defined by (3.1). The latter is in general not a differential operator, but a pseudo-differential operator (Egorov theorem, see e.g. [20]). We saw that if V satisfies Assumption 1.2 however, then it is explicit. The drawback of this approach is that we cannot assess the action of this operator on nonlinear terms in general. The operator A^ε satisfies two of the three properties we use to study the nonlinear problem:

Lemma 6.2. *The operator $A^\varepsilon(t)$ satisfies the following properties.*

- *The commutation,*

$$\left[A^\varepsilon(t), i\varepsilon\partial_t + \frac{1}{2}\varepsilon^2\Delta - V(t, x) \right] = 0.$$

- *The modified Sobolev inequality. If $v \in \Sigma$, then for $2 \leq r \leq 2n/(n-2)$, there exists C_r such that, for $|t| \leq \delta$,*

$$\|v\|_{L^r} \leq \frac{C_r}{|t|^{\delta(r)}} \|v\|_{L^2}^{1-\delta(r)} \|A^\varepsilon(t)v\|_{L^2}^{\delta(r)}.$$

Proof. The first point stems from the definition of $\mathbb{A}^\varepsilon(t)$. For the second, let $g^\varepsilon(t, x) = \mathbb{U}^\varepsilon(-t)v(x)$. We know that for any $f \in L^2 \cap L^1$,

$$\|f\|_{L^2} = \|\mathbb{U}^\varepsilon(t)f\|_{L^2},$$

and for $|t| \leq \delta$, from (3.2),

$$\|\mathbb{U}^\varepsilon(t)f\|_{L^\infty} \lesssim |\varepsilon t|^{-n/2} \|f\|_{L^1}.$$

Interpolating these two estimates yields,

$$\|\mathbb{U}^\varepsilon(t)f\|_{L^r} \lesssim |\varepsilon t|^{-\delta(r')} \|f\|_{L^{r'}},$$

therefore,

$$\|\mathbb{U}^\varepsilon(t)g^\varepsilon(t)\|_{L^r} \lesssim |\varepsilon t|^{-\delta(r')} \|g^\varepsilon(t)\|_{L^{r'}}.$$

Let $\lambda > 0$, and write,

$$\|g^\varepsilon(t)\|_{L^{r'}}^{r'} = \int_{|x| \leq \lambda} |g^\varepsilon(t, x)|^{r'} dx + \int_{|x| > \lambda} |g^\varepsilon(t, x)|^{r'} dx.$$

Estimate the first term by Hölder's inequality,

$$\int_{|x| \leq \lambda} |g^\varepsilon(t, x)|^{r'} dx \lesssim \lambda^{n/p'} \left(\int_{|x| \leq \lambda} |g^\varepsilon(t, x)|^{r'p} dx \right)^{1/p},$$

and choose $p = 2/r' (\geq 1)$. Estimate the second term by the same Hölder's inequality, after inserting the factor x as follows,

$$\begin{aligned} \int_{|x| > \lambda} |g^\varepsilon(t, x)|^{r'} dx &= \int_{|x| > \lambda} |x|^{-r'} |x|^{r'} |g^\varepsilon(t, x)|^{r'} dx \\ &\leq \left(\int_{|x| > \lambda} |x|^{-r'p'} dx \right)^{1/p'} \left(\int_{|x| > \lambda} |xg^\varepsilon(t, x)|^2 dx \right)^{1/p} \\ &\lesssim \lambda^{n/p' - r'} \|xg^\varepsilon(t, x)\|_{L^2}^{2/p}. \end{aligned}$$

In summary, we have the following estimate, for any $\lambda > 0$,

$$(6.5) \quad \|g^\varepsilon(t)\|_{L^{r'}} \lesssim \lambda^{n/(p'r')} \|g^\varepsilon(t)\|_{L^2} + \lambda^{n/(p'r')-1} \|xg^\varepsilon(t, x)\|_{L^2}.$$

Notice that $n/(p'r') = \delta(r)$, and equalize both terms of the right hand side of (6.5),

$$\lambda = \frac{\|xg^\varepsilon(t, x)\|_{L^2}}{\|g^\varepsilon(t)\|_{L^2}}.$$

This yields,

$$\|g^\varepsilon(t)\|_{L^{r'}} \lesssim \|g^\varepsilon(t)\|_{L^2}^{1-\delta(r)} \|xg^\varepsilon(t, x)\|_{L^2}^{\delta(r)}.$$

Therefore,

$$\begin{aligned} \|\mathbb{U}^\varepsilon(t)g^\varepsilon(t)\|_{L^r} &\lesssim |\varepsilon t|^{-\delta(r)} \|g^\varepsilon(t)\|_{L^2}^{1-\delta(r)} \|xg^\varepsilon(t, x)\|_{L^2}^{\delta(r)} \\ &\lesssim |t|^{-\delta(r)} \|g^\varepsilon(t)\|_{L^2}^{1-\delta(r)} \left\| \frac{x}{\varepsilon} g^\varepsilon(t, x) \right\|_{L^2}^{\delta(r)}. \end{aligned}$$

Back to v , this completes the proof of the lemma, since $\mathbb{U}^\varepsilon(t)$ is unitary on L^2 . □

6.3. A partial result for general subquadratic potentials. To conclude, we prove that the asymptotics (1.14) still holds if V satisfies Assumption 3.1.

Proposition 6.3. *Let V satisfying Assumption 3.1, such that V is continuous at $(t, x) = (0, 0)$, with $V(0, 0) = 0$. Suppose that Assumption 1.3 is satisfied. Then for any $\Lambda > 0$, the following holds:*

1. *There exists $\varepsilon(\Lambda) > 0$ such that for $0 < \varepsilon \leq \varepsilon(\Lambda)$, the initial value problem*

$$(6.6) \quad \begin{cases} i\varepsilon \partial_t u^\varepsilon + \frac{1}{2}\varepsilon^2 \Delta u^\varepsilon = V(t, x)u^\varepsilon + \varepsilon^{n\sigma} |u^\varepsilon|^{2\sigma} u^\varepsilon, \\ u^\varepsilon|_{t=0} = \frac{1}{\varepsilon^{n/2}} \varphi\left(\frac{x}{\varepsilon}\right), \end{cases}$$

has a unique solution $u^\varepsilon \in C([-\Lambda\varepsilon, \Lambda\varepsilon]; \Sigma)$.

2. *This solution satisfies the following asymptotics,*

$$(6.7) \quad \limsup_{\varepsilon \rightarrow 0} \sup_{|t| \leq \Lambda\varepsilon} \left(\|u^\varepsilon(t) - v^\varepsilon(t)\|_{L^2} + \|\varepsilon \nabla_x u^\varepsilon(t) - \varepsilon \nabla_x v^\varepsilon(t)\|_{L^2} + \left\| \frac{x}{\varepsilon} u^\varepsilon(t) - \frac{x}{\varepsilon} v^\varepsilon(t) \right\|_{L^2} \right) = 0,$$

where v^ε is given by (1.15).

Proof. The proof mimics the approach used in Section 4, except that we do not use intermediary operators such as $A_{l,j}^\varepsilon$. Denote $w^\varepsilon = u^\varepsilon - v^\varepsilon$. It solves

$$(6.8) \quad \begin{cases} i\varepsilon \partial_t w^\varepsilon + \frac{1}{2}\Delta w^\varepsilon = V(t, x)w^\varepsilon + V(t, x)v^\varepsilon + \varepsilon^{n\sigma} (|u^\varepsilon|^{2\sigma} u^\varepsilon - |v^\varepsilon|^{2\sigma} v^\varepsilon), \\ w^\varepsilon|_{t=0} = 0. \end{cases}$$

Obviously,

$$(6.9) \quad \left| |u^\varepsilon|^{2\sigma} u^\varepsilon - |v^\varepsilon|^{2\sigma} v^\varepsilon \right| \lesssim (|v^\varepsilon|^{2\sigma} + |w^\varepsilon|^{2\sigma}) |w^\varepsilon|.$$

We know that there exists C_0 such that for any t ,

$$\|\varepsilon \nabla_x v^\varepsilon(t)\|_{L^2} \leq C_0 .$$

Since $w^\varepsilon|_{t=0} = 0$ and $w^\varepsilon \in C(0, t^\varepsilon; \Sigma)$ for some $t^\varepsilon > 0$ (Proposition 3.4), we have

$$(6.10) \quad \|\varepsilon \nabla_x w^\varepsilon(t)\|_{L^2} \leq C_0 ,$$

for t in some interval $[0, t_1^\varepsilon]$. So long as (6.10) holds, we can get energy estimates from (6.8), proceeding as in Section 4 and using the Gagliardo-Nirenberg inequality

$$\|f\|_{L^x} \leq C \varepsilon^{-\delta(\mathcal{L})} \|f\|_{L^2}^{1-\delta(\mathcal{L})} \|\varepsilon \nabla_x f\|_{L^2}^{\delta(\mathcal{L})} .$$

Notice that we have,

$$\left[i\varepsilon \partial_t + \frac{1}{2} \varepsilon^2 \Delta - \mathcal{V}(t, x), \varepsilon \nabla_x \right] = \varepsilon \nabla_x \mathcal{V}(t, x) ; \quad \left[i\varepsilon \partial_t + \frac{1}{2} \varepsilon^2 \Delta - \mathcal{V}(t, x), \frac{x}{\varepsilon} \right] = \varepsilon \nabla_x .$$

Proceeding as in Section 4 yields,

$$(6.11) \quad \|w^\varepsilon\|_{L^\infty(0,t;L^2)} \leq C(\Lambda) \varepsilon^{-1} \|\mathcal{V}(s, x) v^\varepsilon\|_{L^1(0,t;L^2)} ,$$

along with

$$\begin{aligned} \|\varepsilon \nabla_x w^\varepsilon\|_{L^\infty(0,t;L^2)} &\leq C(\Lambda) \left(\|\nabla_x \mathcal{V}(s, x) w^\varepsilon\|_{L^1(0,t;L^2)} + \|\nabla_x (\mathcal{V}(s, x) v^\varepsilon)\|_{L^1(0,t;L^2)} \right), \\ \left\| \frac{x}{\varepsilon} w^\varepsilon \right\|_{L^\infty(0,t;L^2)} &\leq C(\Lambda) \left(\|\nabla_x w^\varepsilon\|_{L^1(0,t;L^2)} + \varepsilon^{-2} \|x \mathcal{V}(s, x) v^\varepsilon\|_{L^1(0,t;L^2)} \right). \end{aligned}$$

In particular, so long as (6.10) holds, with $|t| \leq \Lambda \varepsilon$,

$$\begin{aligned} \|w^\varepsilon\|_{L^\infty(0,t;L^2)} + \|\varepsilon \nabla_x w^\varepsilon\|_{L^\infty(0,t;L^2)} + \left\| \frac{x}{\varepsilon} w^\varepsilon \right\|_{L^\infty(0,t;L^2)} \\ \leq e^{C(\Lambda)\Lambda} \left(\varepsilon^{-1} \|\mathcal{V}(s, x) v^\varepsilon\|_{L^1(0,t;L^2)} + \|\nabla_x (\mathcal{V}(s, x) v^\varepsilon)\|_{L^1(0,t;L^2)} \right. \\ \left. + \varepsilon^{-2} \|x \mathcal{V}(s, x) v^\varepsilon\|_{L^1(0,t;L^2)} \right). \end{aligned}$$

Now,

$$\begin{aligned} \varepsilon^{-1} \|\mathcal{V}(s, x) v^\varepsilon\|_{L^1(0,t;L^2)} &= \varepsilon^{-1} \left\| \mathcal{V}(s, x) \frac{1}{\varepsilon^{n/2}} \psi \left(\frac{s}{\varepsilon}, \frac{x}{\varepsilon} \right) \right\|_{L^1(0,t;L^2)} \\ &= \varepsilon^{-1} \left\| \mathcal{V}(s, \varepsilon x) \psi \left(\frac{s}{\varepsilon}, x \right) \right\|_{L^1(0,t;L^2)} \\ &= \|\mathcal{V}(\varepsilon s, \varepsilon x) \psi(s, x)\|_{L^1(0,t/\varepsilon;L^2)} \\ &\leq \|\mathcal{V}(\varepsilon s, \varepsilon x) \psi(s, x)\|_{L^1(0,\Lambda;L^2)} . \end{aligned}$$

Notice that for $|t| \leq \delta$,

$$|\mathbb{V}(t, x)| \lesssim 1 + x^2.$$

From Lebesgue's dominated convergence theorem (\mathbb{V} is continuous at the origin and $\mathbb{V}(0, 0) = 0$) and Lemma 4.3, it follows, up to approximating φ in $\mathcal{S}(\mathbb{R}^n)$ as in Section 4,

$$\|\mathbb{V}(\varepsilon s, \varepsilon x)\psi(s, x)\|_{L^1(0, \Lambda\varepsilon; L^2)} \xrightarrow{\varepsilon \rightarrow 0} 0.$$

Similarly,

$$\|\nabla_x (\mathbb{V}(s, x)v^\varepsilon)\|_{L^1(0, \Lambda\varepsilon; L^2)} + \varepsilon^{-2} \|x\mathbb{V}(s, x)v^\varepsilon\|_{L^1(0, \Lambda\varepsilon; L^2)} \xrightarrow{\varepsilon \rightarrow 0} 0.$$

Therefore (6.10) remains valid up to time $t = \Lambda\varepsilon$, provided that ε is sufficiently small ($0 < \varepsilon \leq \varepsilon(\Lambda)$). This completes the proof of the proposition. \square

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