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A MAXIMAL INEQUALITY FOR FILTRATION ON SOME FUNCTION SPACES

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1. Introduction

In this paper we consider a maximal inequality associated with filtration on Lorentz spaces and Orlicz spaces. Let (X, μ), (Y, ν) be arbitrary measure spaces and let T be a bounded linear operator from a function space defined on (Y, ν) to a function space on (X, μ). Let E_n be a sequence of measurable subsets of Y which are nested: $E_n \subset E_{n+1}$ for all n. Such a sequence is called a *filtration* of Y. Denote by χ_E the characteristic function of E. M. Christ and A. Kiselev in [3] considered the maximal operator

$$T^* f(x) = \sup_n |T(f\chi_{E_n})(x)|,$$

which was studied to obtain the a.e. convergence of an integral operator [4]. They obtained the following result.

Theorem 1.1. *Let $1 \le p,\ q < \infty$, and suppose that $T\colon L^p(Y) \to L^q(X)$ is a bounded linear operator. Then for any nested sequence of measurable subsets $\{E_n\} \subset Y$, the maximal operator T^* is a bounded operator from $L^p(Y)$ to $L^q(X)$ provided $p < q$. Moreover,*

$$\|T^*\|_{p,q} \le \left(1 - 2^{-\{(1/p)-(1/q)\}}\right)^{-1} \|T\|_{p,q}$$

where $\|T\|_{p,q}$ denotes the operator norm of T from $L^p(Y)$ to $L^q(X)$.

It should be noted that the phenomena for the maximal inequality occur because of the strict difference of convexity between two functions (t^p, t^q) generating the function spaces (L^p and L^q). Based on this fact, we extend the theorem above to some different function spaces which naturally contain the Lebesgue spaces. Especially, we thus show a version of Theorem 1.1 still holds on Lorentz spaces and Orlicz spaces reflecting the difference of convexity. For another reference concerning the Lorentz space, see the paper [5].

Let $L^{p,r}(X) = L^{p,r}(X, d\mu)$ denote the space of all measurable functions satisfying

$$\|f\|_{p,q} = \left(\frac{q}{p}\int_0^\infty \left[t^{1/p} f^*(t)\right]^q \frac{dt}{t}\right)^{1/q} < \infty$$

where f^* is the decreasing rearrangement of f (see [6]). Then we first have the following result:

Theorem 1.2. *Let $1 \le p \le r < s \le q < \infty$, and suppose $T\colon L^{p,r}(Y) \to L^{q,s}(X)$ is a bounded linear operator. Then T^* is bounded from $L^{p,r}(Y)$ to $L^{q,s}(X)$. Moreover,*

$$\|T^*\|_{L^{p,r}\to L^{q,s}} \le \left(1 - 2^{-\{(1/r)-(1/s)\}}\right)^{-1} \|T\|_{L^{p,r}\to L^{q,s}} \tag{1.1}$$

where $\|T\|_{L^{p,r}\to L^{q,s}}$ denotes the operator norm of T from $L^{p,r}$ to $L^{q,s}$.

Now we consider a generalization to Orlicz spaces. The Young function Φ is given by $\Phi(s) = \int_0^s \phi(t)\,dt$ for an increasing left continuous function ϕ with $\phi(0) = 0$. For the Young function, the Luxemburg norm is defined by

$$\rho^\Phi(f) = \inf\left\{k\colon \int \Phi\left(\frac{|f(y)|}{k}\right) d\nu(y) \le 1\right\}.$$

Then the Orlicz space $L^\Phi(Y) = L^\Phi(Y, d\nu)$ is the function space with the norm $\|\cdot\|_{L^\Phi} = \rho^\Phi(\cdot)$. For further details, see pp. 265–280 in [2].

Next, we consider a pair of Young functions Φ and Ψ. We impose several assumptions on Φ, Ψ. For any $s,\, t \ge 0$, let us assume

$$\Psi(st) \sim \Psi(s)\Psi(t). \tag{1.2}$$

Here $A \sim B$ means that there is a constant $C > 0$ such that

$$C^{-1}A \le B \le CA.$$

For the function Φ, we assume that there is a strictly convex function $\widetilde{\Phi}$ such that for any $\alpha \ge 1$,

$$\Phi(\alpha t) \le C\widetilde{\Phi}(\alpha)\Phi(t) \quad \text{and} \quad \widetilde{\Phi}(\alpha) \sim \widetilde{\Phi}(1/\alpha)^{-1}. \tag{1.3}$$

Then the second result is the following:

Theorem 1.3. *Let T be a bounded linear operator from $L^\Phi(Y)$ to $L^\Psi(X)$. Assume Φ and Ψ satisfy* (1.3) *and* (1.2), *respectively, and further assume*

$$\int_0^1 \Phi^{-1}(t)\Psi^{-1}(t^{-1})\frac{dt}{t} < \infty. \tag{1.4}$$

Then there is a constant C such that $\|T^* f\|_{L^\Psi} \le C\|f\|_{L^\Phi}$.

Compared with the result in [3] where $\Phi(t) = t^p$ and $\Psi(t) = t^q$, the result above is more general. For this particular example, the conditions (1.3) and (1.2) are satisfied and

$$\int_0^1 \Phi^{-1}(t)\Psi^{-1}(t^{-1})\frac{dt}{t} = \int_0^1 t^{(1/p)-(1/q)}\frac{dt}{t} < \infty,$$

provided $p < q$. We obtain another example if we set $\Psi(t) = t^q$, $\Phi(t) = t^p(\log(2+t))^\beta$ with $\beta > 0$. The condition (1.2) is clearly satisfied. It is easily verified that for any $\alpha \ge 1$, there exists $\varepsilon > 0$ such that $\Phi(\alpha t) \lesssim \alpha^{p_\varepsilon}\Phi(t)$ with $p_\varepsilon = p + \varepsilon\beta$. So if we set $\widetilde{\Phi}(t) = t^{p_\varepsilon}$, then (1.3) is satisfied and we can find ε so that $\widetilde{\Phi}$ satisfies the condition (1.4) for $p < q$.

The proof of these theorems follows the line of argument in [3]. But some technical difficulties arising in the consideration of Lorentz and Orlicz spaces will be settled by introducing several lemmas.

2. Proof of Theorem 1.2

We begin by proving an elementary but crucial lemma concerning Lorentz space.

Lemma 2.1. *Let F, G be disjoint measurable sets in Y and let f, g be measurable functions on X. If $r \le p < \infty$, then*

$$\|f\chi_F + g\chi_G\|_{p,r}^r \le \|f\chi_F\|_{p,r}^r + \|g\chi_G\|_{p,r}^r \tag{2.1}$$

and if $p \le r$, then

$$\|f\chi_F + g\chi_G\|_{p,r}^r \ge \|f\chi_F\|_{p,r}^r + \|g\chi_G\|_{p,r}^r. \tag{2.2}$$

Proof. By a limiting argument, we may assume that f and g are simple functions. Without loss of generality, we may write $f\chi_F$, $g\chi_G$ as $f\chi_F = \sum_{i=1}^n c_i\chi_{F_i}$, $g\chi_G = \sum_{i=1}^n c_i\chi_{G_i}$ respectively, where F_i, G_i are measurable sets contained in F, G respectively. We may also assume

$$|c_1| \ge |c_2| \ge \cdots \ge |c_i| \ge |c_{i+1}| \ge \cdots .$$

Set $a_i = \nu(F_i)$, $b_i = \nu(G_i)$. Also for $1 \le i \le n$, set $A_i = \sum_{k=1}^i a_k$, $B_i = \sum_{k=1}^i b_k$. Then the decreasing rearrangements of $f\chi_F$, $g\chi_G$ are given by

$$(f\chi_F)^*(t) = \begin{cases} |c_i| & \text{if } A_{i-1} \le t < A_i \\ 0 & \text{if } A_n \le t \end{cases},$$

$$(g\chi_G)^*(t) = \begin{cases} |c_i| & \text{if } B_{i-1} \le t < B_i \\ 0 & \text{if } B_n \le t. \end{cases}$$

Since the supports of f and g are disjoint, we have $f + g = \sum_i c_i \chi_{F_i \cup G_i}$. Thus we have

$$(f\chi_F + g\chi_G)^*(t) = \begin{cases} |c_i| & \text{if } A_{i-1} + B_{i-1} \le t < A_i + B_i \\ 0 & \text{if } A_n + B_n \le t. \end{cases}$$

Now for $i = 1, \dots, n$, let us set

$$\mathcal{S}_i = (A_i + B_i)^{r/p} - (A_{i-1} + B_{i-1})^{r/p} - A_i^{r/p} + A_{i-1}^{r/p} - B_i^{r/p} + B_{i-1}^{r/p}.$$

Then a simple computation shows that

$$\|f + g\|_{L^{p,r}}^r - \|f\|_{L^{p,r}}^r - \|g\|_{L^{p,r}}^r = \sum_i^n |c_i|^r \mathcal{S}_i.$$

Finally, we only need to observe that $\mathcal{S}_i \le 0$ if $0 < r/p \le 1$ and $\mathcal{S}_i \ge 0$ if $r/p \ge 1$. This completes the proof of Lemma 2.1. □

Now we prove Theorem 1.2. Fix p, r, q, s so that $1 \le p \le r < s \le q < \infty$. Without loss of generality, we may assume $\|f\|_{L^{p,r}(Y)} = 1$. Define a function $\mathcal{M}$ from measurable sets of (Y, ν) to $\mathbb{R}$ by

$$\mathcal{M}(S) = \|f\chi_S\|_{L^{p,r}(Y)}^r.$$

As mentioned in [3], we may assume that for $\lambda > 0$ and for any measurable set E, if $\lambda \le \mathcal{M}(E)$, then there is a measurable subset S such that $S \subset E$ and $\mathcal{M}(S) = \lambda$. This can be achieved by replacing Y by $Y \times [0, 1]$, ν by the product of ν and Lebesgue measure on $[0, 1]$, T by $T \circ \pi$ where $\pi f(y) = \int_0^1 f(y, s)\, ds$, and E_n by $E_n \times [0, 1]$. Then we see that the boundedness of T^* is implied by the boundedness of $(T \circ \pi)^*$. Indeed, assume that $(T \circ \pi)^*$ is bounded from $L^{p,r}(Y \times [0, 1])$ to $L^{q,s}(X)$ and (1.1) holds for $(T \circ \pi)^*$ instead of T^*. Given $f \in L^{p,r}(Y)$, apply the above assumption to $f \otimes \chi_{[0,1]}$. Since

$$(T \circ \pi)^* f \otimes \chi_{[0,1]} = \sup_n T\left(\int_0^1 \chi_{E_n \times [0,1]}(f \otimes \chi_{[0,1]})\, ds\right) = T^* f$$

and since $\|f \otimes \chi_{[0,1]}\|_{L^{p,r}(Y \times [0,1])} = \|f\|_{L^{p,r}(Y)}$, (1.1) follows.

We also need the following lemma which is a modification of the one in [3].

Lemma 2.2. *Let f be a measurable function with $\|f\|_{L^{p,r}(Y)} = 1$. Then there is a collection $\{B_k^l\}$ of measurable subsets of Y, with $l \in \{0, 1, 2, \dots\}$ and $1 \le k \le 2^l$,*

satisfying the following conditions.
1. $\{B_k^l : 1 \leq k \leq 2^l\}$ *is a partition of* Y *into disjoint measurable subsets.*
2. $\|\chi_{B_k^l} f\|^r_{L^{p,r}(Y)} \leq 2^{-l}$ *for* $1 \leq k \leq 2^l$.
3. *For each* n, E_n *can be decomposed as an empty, finite or countable union such that for some sequences* l_i^n, k_i^n,

$$E_n = \left(\bigcup_{i \geq 1} B_{k_i^n}^{l_i^n} \right) \bigcup D_n \quad with \quad l_1^n < l_2^n < l_3^n < \cdots$$

where D_n *is a measurable set for which* $\mathcal{M}(D_n) = 0$.

Proof of Lemma 2.2. Define for $1 \leq k \leq 2^l - 1$,

$$N_k^l = \min \left\{ n \in \mathbb{N} \colon \mathcal{M}(E_n) \geq 2^{-l} k \right\}.$$

By the divisibility assumption for $1 \leq k \leq 2^l - 1$, we can choose a subset A_k^l of $E_{N_k^l}$ in such a way that $\mathcal{M}(A_k^l) = k 2^{-l}$ and $A_{2^l}^l = Y$. Since E_n is increasing, we may assume that $A_i^l \subset A_{i+1}^l$ and $A_k^{l-1} = A_{2k}^l$. Now we define B_k^l by

$$\begin{aligned} B_1^l &= A_1^l, \\ B_2^l &= (A_2^l \setminus A_1^l), \ldots, B_k^l = (A_k^l \setminus A_{k-1}^l), \ldots, \\ B_{2^l}^l &= (A_{2^l}^l \setminus A_{2^l-1}^l). \end{aligned}$$

Since $p \leq r$, by (2.2) in Lemma 2.1 $\mathcal{M}(S_1 \cup S_2) \geq \mathcal{M}(S_1) + \mathcal{M}(S_2)$ if S_1 and S_2 are disjoint. So for all $1 \leq k \leq 2^l$, we have

$$\mathcal{M}(B_k^l) = \mathcal{M}(A_k^l \setminus A_{k-1}^l) \leq \mathcal{M}(A_k^l) - \mathcal{M}(A_{k-1}^l) = 2^{-l}.$$

Form the construction, it follows that for each n, there are sequences $\{l_i^n\}, \{m_i^n\}$ so that

$$A_{m_i^n}^{l_i^n} \subset E_n, \quad A_{m_i^n}^{l_i^n} \subset A_{m_{i+1}^n}^{l_{i+1}^n}, \quad \lim_{i \to \infty} \mathcal{M}\left(A_{m_i^n}^{l_i^n}\right) = \mathcal{M}(E_n)$$

and l_i^n is strictly increasing as i increase. Indeed, using binary expansion, we can write $\mathcal{M}(E_n) = \sum_{j=1}^{\infty} 2^{-l_j^n}$ where l_j^n is strictly increasing as j increases. By our construction of the sets $\{A_k^l\}$, we see that for each $i \in \mathbb{N}$, there is a $A_{m_i^n}^{l_i^n}$ such that $A_{m_i^n}^{l_i^n} \subset E_n$ and $\mathcal{M}(A_{m_i^n}^{l_i^n}) = \sum_{j=1}^{i} 2^{-l_j^n}$. Since $A_i^l \subset A_{i+1}^l$ and $A_k^{l-1} = A_{2k}^l$, we have $A_{k_i^n}^{l_i^n} \subset A_{k_{i+1}^n}^{l_{i+1}^n}$.

Now observe $\left(A_{m_{i+1}^n}^{l_{i+1}^n} \setminus A_{m_i^n}^{l_i^n} \right) = B_{k_i^n}^{l_{i+1}^n}$ for some sequence $\{k_i^n\}$. Since $\bigcup_i A_{k_i^n}^{l_i^n} = \bigcup_i B_{k_i^n}^{l_i^n}$, by the monotone convergence theorem, we have $\mathcal{M}\left(E_n \setminus \bigcup_i B_{k_i^n}^{l_i^n} \right) = 0$. Now we set $D_n = E_n \setminus \bigcup_i B_{k_i^n}^{l_i^n}$. This completes the proof of Lemma 2.2. $\square$

Let $N\colon X \to \mathbb{Z}$ be a measurable function. Define an operator $T^N f(x) = T(f\chi_{E_{N(x)}})(x)$. To prove Theorem 1.2, it is sufficient to show that

$$\|T^N f\|_{L^{q,s}(X)} \le C\|f\|_{L^{p,r}(Y)}$$

where C is independent of N. Set $A_n = \{x\colon N(x) = n\}$ and define $R_{l,k}$ to be the index set $\{n\colon B_k^l$ appears in the decomposition of $E_n\}$. Define measurable sets D_j^l by $D_j^l = \bigcup_{n\in R_j^l} A_n$. Observe $D_i^l \cap D_j^l = \emptyset$ if $i \neq j$. Suppose not. Then there is an A_n such that $A_n \subset D_i^l \cap D_j^l$ because A_n is pair-wise disjoint. So B_i^l and B_j^l appear in the decomposition of E_n. But scale-2^l element is contained at most once in E_n. It is a contradiction.

Note $f\chi_{E_n} = \sum_{(l,j)\colon E_n=\bigcup B_j^l} f\chi_{B_j^l\cup D_n}$. We write

$$T^N f = \sum_n \chi_{A_n} T(f\chi_{E_n}) = \sum_n \sum_{(l,j)\colon E_n=\bigcup B_j^l} \chi_{A_n} T(f\chi_{B_j^l\cup D_n})$$
$$= \sum_l \sum_j \chi_{D_j^l} T(f\chi_{B_j^l\cup D_n}).$$

Since T is bounded from $L^{p,r}(Y)$ to $L^{q,s}(X)$, we may drop D_n in the above expression. Since $q > 1$, the Lorentz space $L^{q,s}$ is a Banach space (see 1.6 of [1]). Thus we have

$$\|T^N f\|_{q,s} \le \sum_{l=0}^{\infty} \left\| \sum_j \chi_{D_j^l} T(f\chi_{B_j^l}) \right\|_{q,s}.$$

Now fix l and note that $q \ge s$ and $\{D_j^l\}$ are disjoint. By (2.1) in Lemma 2.1, we have the following.

$$\left\| \sum_j \chi_{D_j^l} T(f\chi_{B_j^l}) \right\|_{q,s}^s \le \sum_j \left\| \chi_{D_j^l} T(f\chi_{B_j^l}) \right\|_{q,s}^s$$
$$\le \sum_j \left(\|T\|_{L^{p,r}\to L^{q,r}}\right)^s \left\| f\chi_{B_j^l} \right\|_{p,r}^s.$$

The second inequality is trivial. By the decomposition in Lemma 2.2, the last in the above inequality is bounded by

$$\sum_j \left(\|T\|_{L^{p,r}\to L^{q,r}}\right)^s 2^{-\{(s/r)-1\}l} \|f\chi_{B_j^l}\|_{p,r}^r.$$

Since $p \le r$ and for each l, B_j^l are disjoint, another application of Lemma 2.1 implies

$\sum_j \|f\chi_{B_j^l}\|_{p,r}^r \le \|f\|_{p,r}^r$. Putting all things together, we have

$$\begin{aligned}\|T^N f\|_{q,s} &\le \sum_{l=0}^{\infty} 2^{-l(r^{-1}-s^{-1})}(\|T\|_{L^{p,r}\to L^{q,r}})\|f\|_{p,r} \\ &\le \left(1-2^{-\{(1/r)-(1/s)\}}\right)^{-1}(\|T\|_{L^{p,r}\to L^{q,r}})\end{aligned}$$

since $r < s$ and $\|f\|_{L^{p,r}} = 1$. This completes the proof of Theorem 1.2.

3. Proof of Theorem 1.3

We begin with making several observations. Since Ψ is strictly increasing, its inverse Ψ^{-1} satisfies

$$\Psi^{-1}(s)\Psi^{-1}(t) \le \Psi^{-1}(Cst), \qquad \Psi^{-1}\left(\frac{st}{C}\right) \le \Psi^{-1}(s)\Psi^{-1}(t). \tag{3.1}$$

Let L^Ω be an Orlicz space with Young's function Ω. If $\Omega(st) \ge C\Omega(s)\Omega(t)$ for some C, then by the definition of Orlicz space norm, we have $\int \Omega(|f(x)|/\|f\|_{L^\Omega})\,dx = 1$. The condition on Ω implies $1 \le C\int \Omega(|f(x)|)/\Omega(\|f\|_{L^\Omega})\,dx$ and hence $\Omega(\|f\|_{L^\Omega}) \le C\int \Omega(|f(x)|)\,dx$. Conversely if we assume $\Omega(st) \le C\Omega(s)\Omega(t)$ for some C, then we have $\Omega(\|f\|_{L^\Omega}) \ge C\int \Omega(|f(x)|)\,dx$. By the assumptions (1.2) on Ψ we have

$$\Psi(\|f\|_{L^\Psi}) \sim \int \Psi(|f(x)|)\,dx.$$

In the similar way it is easy to see that for f satisfying $\|f\|_{L^\Phi} \le 1$,

$$\widetilde{\Phi}(\|f\|_{L^\Phi}) \le C\int \widetilde{\Phi}(|f(x)|)\,dx.$$

As before, it is sufficient to show for all measurable $N\colon X \to \mathbb{Z}$, the operator T^N given by

$$T^N f(x) = T(f\chi_{E_{N(x)}})(x)$$

is bounded from L^Φ to L^Ψ. Without loss of generality we may assume $\|f\|_{L^\Phi} = 1$.

Now we introduce a decomposition for functions which is similar to Lemma 2.2.

Lemma 3.1. *Let f be a measurable function with $\|f\|_{L^\Phi} = 1$. Then there is a collection $\{B_j^l\}$ of measurable sets in X, indexed by $l \in \{0, 1, 2, \ldots\}$ and $1 \le j \le 2^l$, satisfying the following conditions*:

1. *$\{B_j^l\colon 1 \le j \le 2^l\}$ is a partition of X into disjoint measurable subsets.*
2. *$\int \Phi(|f|\chi_{B_j^l})\,dx = 2^{-l}$ for all $1 \le j \le 2^l$.*

3. *For each n, E_n can be decomposed as an empty, finite or countable union such that for some sequences l_i^n, k_i^n,*

$$E_n = \left(\bigcup_{i \geq 1} B_{k_i^n}^{l_i^n} \right) \bigcup D_n \quad with \quad l_1^n < l_2^n < l_3^n < \cdots ,$$

where $\mathcal{M}(D_n) = 0$.

The proof of the above lemma can be obtained by following the same line of argument as in [3]. So we omit the detailed proof. According to Lemma 3.1, we decompose f with the same notations for A_n, R_j^l, D_j^l as in the proof of Theorem 1.2. We write

$$\begin{aligned} T^N f(x) &= \sum_{n=1}^{\infty} T(f\chi_{E_n})(x)\chi_{A_n}(x) \\ &= \sum_{n=1} \sum_{j,l} T(f\chi_{B_j^l \cup D_n})(x)\chi_{A_n}(x) = \sum_{j,l} T(f_{j,l})(x)\chi_{D_j^l}(x), \end{aligned}$$

where $f_{j,l} = f\chi_{B_j^l}$. By the condition (1.2) on Ψ and the fact that D_i^l are mutually disjoint for each fixed l, we have

$$\Psi\left(\left\| \sum_j T(f_{j,l})\chi_{D_j^l} \right\|_{L^\Psi} \right) \leq C \sum_j \int \Psi\left(|T(f_{j,l})(x)|\chi_{D_j^l}(x) \right) dx.$$

On the other hand, using the boundedness of T from L^Φ to L^Ψ, we have

$$\Psi\left(\|f_{j,l}\|_{L^\Phi} \right) \geq \Psi\left(\|Tf_{j,l}\|_{L^\Psi} \right) \sim \int \Psi\left(|Tf_{j,l}| \right) dx.$$

By the decomposition and the condition (1.3) on Φ, we see that

$$\widetilde{\Phi}(\|f_{j,l}\|_{L^\Phi}) \leq \int \Phi(|f_{j,l}|)\, dx \sim 2^{-l}.$$

Hence we have

$$\begin{aligned} \Psi\left(\left\| \sum_j T(f_{j,l})\chi_{D_j^l} \right\|_{L^\Psi} \right) &\leq C \sum_j \Psi(\|Tf_{j,l}\|_{L^\Psi}) \\ &\leq C \sum_j \Psi\left(\widetilde{\Phi}(2^{-l}) \right) \leq C2^l \Psi\left(\widetilde{\Phi}(2^{-l}) \right) \end{aligned}$$

since the number of j is not greater than 2^l for each l. By the triangle inequality, we have

$$\|T^N f\|_{L^\Psi} \le \sum_l \left\| \sum_j T(f_{j,l})\chi_{D_j^l} \right\|_{L^\Psi}.$$

Summing with respect to l we get

$$\|T^N f\|_{L^\Psi} \le C \sum_l \widetilde{\Phi}(2^{-l})\ \Psi(2^l). \tag{3.2}$$

Finally, (1.4) implies the left hand side of the above is finite. This completes the proof of Theorem 1.3.

REMARK 1. In Theorem 1.3, if we set $\Phi(t) = t^p(\log(2+t))^\beta$ ($\beta > 0$) and $\Psi(t) = t^q$, then the inequality (3.2) can be expressed as

$$\|T^N f\|_{L^\Psi} \le C \sum_l 2^{-(1/p_\varepsilon - 1/q)l} = C\left(1 - 2^{-(1/p_\varepsilon - 1/q)}\right)^{-1}.$$

Thus we have the similar result as in Theorem 1.1. It is interesting to prove Theorem 1.3 for the case $\Psi(t) = t^p(\log(2+t))^\beta$ and $\Phi(t) = t^p$ where the convexity difference between Ψ and Φ is logarithmic. But the lack of convexity difference causes a difficulty in controlling the inequality (3.2).

REMARK 2. Theorem 1.1 can be easily extended to the vector valued function spaces (e.g. L^p_B where B is a Banach space). For example, if T is a linear operator from $L^p_A(Y, d\nu)$ to $L^q_B(X, d\mu)$ with $1 \le p,\ q \le \infty$ and $\{E_n\}$ is a nested set sequence, then the maximal operator T^* defined by

$$T^* F = \sup_n ||T(F\chi_{E_n})||_B$$

satisfies the same inequality as in Theorem 1.1.

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