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REAL HYPERSURFACES IN NONFLAT COMPLEX SPACE FORMS ARE IRREDUCIBLE

Dedicated to President Koichi Ogiue on his retirement from Tokyo Metropolitan University

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1. Introduction

The study of real hypersurfaces in complex projective space CP^n and complex hyperbolic space CH^n has been an active field over the past three decades. Although these ambient spaces might be regarded as the simplest after the spaces of constant curvature, they impose significant restrictions on the geometry of their hypersurfaces. For instance, they do not admit totally umbilical hypersurfaces and Einstein hypersurfaces.

On the other hand, several important classes of real hypersurfaces in complex projective space have been constructed and investigated by many geometers. For instance, H.B. Lawson investigated real hypersurfaces of CP^n constructed by Clifford minimal hypersurfaces of S^{n+1} via Hopf fibration. R. Takagi [9] gave the list of homogeneous real hypersurfaces of CP^n. Many geometers then study the geometry from the list of Takagi and obtained various interesting geometric characterizations of homogeneous real hypersurfaces in CP^n.

Another important class of real hypersurfaces in CP^n which contains the list of R. Takagi is the class of Hopf hypersurfaces. Such hypersurfaces are real hypersurfaces whose structure vector $J\xi$ is a principal curvature vector, where J is the complex structure and ξ is the unit normal vector field. Examples and geometric characterizations of Hopf hypersurfaces have also been obtained by various geometers. It is known that in CP^n, M is a homogeneous real hypersurface if and only if M is a Hopf hypersurface with constant principal curvatures [6, 9].

The study of real hypersurfaces in complex hyperbolic space CH^n has followed developments in CP^n, often with similar results, but sometimes with differences (see [1, 7, 8] for more details).

It is well-known that real projective space and real hyperbolic space admit ample hypersurfaces which are the Riemannian products of some Riemannian manifolds. It is also well-known that CP^3 admits a complex hypersurface which is the Riemannian

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product of two complex projective lines. Moreover, it is proved in [3] that there exist infinitely many real hypersurfaces, both in complex projective space and in complex hyperbolic space, which are warped products of Riemannian manifolds.

In contrast with such properties we prove in this paper a fundamental general property on real hypersurfaces; namely, there do not exist real hypersurfaces which are Riemannian products of Riemannian manifolds, both in complex projective space and complex hyperbolic space. More precisely, we prove the following.

Theorem. *Every real hypersurface in a nonflat complex space form is irreducible.*

2. Preliminaries

If N is a Riemannian manifold isometrically immersed in a Kaehler manifold $\tilde{M}$ with complex structure J. Then the formulas of Gauss and Weingarten are given respectively by

$$\tilde{\nabla}_X Y = \nabla_X Y + \sigma(X, Y), \tag{2.1}$$

$$\tilde{\nabla}_X \xi = -A_\xi X + D_X \xi \tag{2.2}$$

for vector fields X, Y tangent to N and ξ normal to N, where $\tilde{\nabla}$ denotes the Riemannian connection on $\tilde{M}$, σ the second fundamental form, D the normal connection, and A the shape operator of N in $\tilde{M}$. The second fundamental form and the shape operator are related by $\langle A_\xi X, Y\rangle = \langle \sigma(X, Y), \xi\rangle$, where $\langle\ ,\ \rangle$ denotes the inner product on M as well as on $\tilde{M}$.

For a submanifold N of a Kaehler manifold $\tilde{M}$, the *equation of Gauss* is given by

$$\begin{aligned} R(X, Y; Z, W) =& \tilde{R}(X, Y; Z, W) + \langle \sigma(X, W), \sigma(Y, Z)\rangle \\ & - \langle \sigma(X, Z), \sigma(Y, W)\rangle \end{aligned} \tag{2.3}$$

for X, Y, Z, W tangent to M, where R and $\tilde{R}$ denote the curvature tensors of N and $\tilde{M}$, respectively.

For the second fundamental form σ, we define its covariant derivative $\bar{\nabla}\sigma$ with respect to the connection on $TM \oplus T^\perp M$ by

$$(\bar{\nabla}_X \sigma)(Y, Z) = D_X(\sigma(Y, Z)) - \sigma(\nabla_X Y, Z) - \sigma(Y, \nabla_X Z). \tag{2.4}$$

The *equation of Codazzi* is given by

$$(\tilde{R}(X, Y)Z)^\perp = (\bar{\nabla}_X \sigma)(Y, Z) - (\bar{\nabla}_Y \sigma)(X, Z), \tag{2.5}$$

where $(\tilde{R}(X, Y)Z)^\perp$ denotes the normal component of $\tilde{R}(X, Y)Z$.

The Riemann curvature tensor of $\tilde{M}^n(4c)$ satisfies

$$\tilde{R}(X, Y; Z, W) = c\Big\{ \langle X, W\rangle \langle Y, Z\rangle - \langle X, Z\rangle \langle Y, W\rangle + \langle JX, W\rangle \langle JY, Z\rangle - \langle JX, Z\rangle \langle JY, W\rangle + 2\langle X, JY\rangle \langle JZ, W\rangle \Big\}. \tag{2.6}$$

A submanifold N in a Kaehler manifold is called *purely real* if $J(T_xN) \cap T_xN = \{0\}$ for $x \in N$.

Here $\tilde{M}^n(4c)$ denotes a complex n-dimensional Kaehler manifold of constant holomorphic sectional curvature $4c$. Such Kaehler manifolds are called *complex space forms*. It is known that the universal covering of a complete complex space form $\tilde{M}^n(4c)$ is the complex projective n-space $CP^n(4c)$, the complex Euclidean n-space $\mathbf{C}^n$, or the complex hyperbolic space $CH^n(4c)$, according as $c > 0$, $c = 0$, or $c < 0$.

3. Proof of the theorem

Assume that M is a real hypersurface of a complex space form $M^n(4c)$ of constant holomorphic sectional curvature $4c$ with $c \neq 0$. Suppose that M is the Riemannian product of two Riemannian manifolds, namely, $M = N_1 \times N_2$ with $n_1 = \dim N_1 \geq 1$ and $n_2 = \dim N_2 \geq 1$.

For $x = (x_1, x_2) \in M = N_1 \times N_2$, we put $\mathcal{D}^j_{x_j} = T_{x_j}N_j \cap J(T_{x_j}N_j)$ for $j = 1, 2$. Let $U^j = \{x \in M : \dim \mathcal{D}^j_{x_j} > 0\}$.

We need the following.

Lemma 3.1. *Let $M = N_1 \times N_2$ be a real hypersurface of a nonflat complex space form $\tilde{M}^n(4c)$. Then exactly one of the following two cases occurs*:

(1) $\dim N_1 = 1$ *or* $\dim N_2 = 1$.

(2) $n \geq 3$ *and either* $\dim N_1 = n$ *and* $\dim N_2 = n-1$ *or* $\dim N_1 = n-1$ *and* $\dim N_2 = n$.

Moreover, Case (2) *occurs only when, restricted to some open dense subset of M, N_1 and N_2 are both purely real submanifolds.*

Proof of Lemma 3.1. Since M is a real hypersurface of $\tilde{M}^n(4c)$, we have $n_1 + n_2 = 2n - 1$. Thus, we have $n_1 \geq n$ or $n_2 \geq n$. If $n_1 = n$ holds, then $n_2 = n - 1$. Similarly, if $n_2 = n$, then $n_1 = n - 1$.

Clearly, Case (1) occurs if $n = 2$. So, we may assume $n \geq 3$. Now, let us assume that $n_1 > n$. Then the dimension formula implies that $U^1 = M$. Thus, there is a nonempty connected open subset U of M on which the dimension of $\mathcal{D}^1$ is a positive constant. We shall work on U instead of M. Clearly, the restriction of $\mathcal{D}^1$ on U is a distribution on $U \times N_2$. For simplicity, we denote this distribution also by $\mathcal{D}^1$.

Let X be a unit vector field in $\mathcal{D}^1$ and Z a unit vector field in TN_2. Then we have $\nabla_Z X = \nabla_Z(JX) = 0$. Thus, by the formulas of Gauss and Weingarten, we obtain

$$\sigma(JX, Z) = \tilde{\nabla}_Z(JX) = J\tilde{\nabla}_Z X = J\sigma(X, Z). \tag{3.1}$$

Since $J\sigma(X, Z)$ is tangent to M, (3.1) gives

$$\sigma(X, Z) = \sigma(JX, Z) = 0, \quad X \in \mathcal{D}^1,\ Z \in TN_2. \tag{3.2}$$

Because the sectional curvature of M satisfies $K(X, Z) = 0$, (2.3) and (3.2) imply

$$0 = \tilde{K}(X, Z) + \langle \sigma(X, X), \sigma(Z, Z) \rangle\,, \tag{3.3}$$

where $\tilde{K}(X, Z)$ denotes the sectional curvature of the plane section $X \wedge Z$ on $\tilde{M}^n(4c)$. Since X, Z are orthonormal, they span a totally real plane, so that (2.6) and (3.3) give

$$\lambda(X)\mu(Z) = -c \neq 0 \tag{3.4}$$

for any unit vector $X \in \mathcal{D}^1$ and unit vector $Z \in TN_2$, where $\sigma(X, X) = \lambda(X)\xi$, $\sigma(Z, Z) = \mu(Z)\xi$ and ξ is the unit normal vector field.

It follows from (3.4) that $\lambda(X)$ and $\mu(Z)$ are independent of X and Z, respectively. Thus

$$\sigma(X, X) = \lambda \langle X, X \rangle \xi, \quad \sigma(Z, Z) = \mu \langle Z, Z \rangle \xi \tag{3.5}$$

for $X \in \mathcal{D}^1$ and $Z \in TN_2$. Since N_2 is totally geodesic in M, (3.5) implies that N_2 is totally umbilical in $\tilde{M}^n(4c)$. Hence, by applying a result of Chen and Ogiue [5], we have either (a) $\dim N_2 = 1$ or (b) N_2 is a real space form isometrically immersed in $\tilde{M}^n(4c)$ as a totally real submanifold whose mean curvature vector H_2 is perpendicular to $J(TN_2)$.

If $\dim N_2 \geq 2$, then the mean curvature vector of N_2 is parallel to ξ according to (3.5). Thus, $J(TN_2) \subset TN_1$. Hence, we obtain

$$\nabla_Z JW + \sigma(Z, JW) = J\nabla_Z W + J\sigma(Z, W) \tag{3.6}$$

for Z, W in TN_2. Since $J\nabla_Z W \in J(TN_2) \subset TN_1$, (3.6) yields

$$\sigma(Z, JW) = 0, \quad Z, W \in TN_2. \tag{3.7}$$

Therefore, by applying the equation of Gauss, we get

$$0 = K(Z, JW) = \tilde{K}(Z, JW) + \langle \sigma(Z, Z), \sigma(JW, JW) \rangle \tag{3.8}$$

for unit vectors Z, W in TN_2. Since Z, JW span a totally real plane for orthonormal vectors Z, W in TN_2, (3.5) and (3.8) imply

$$\lambda\mu = \langle \sigma(Z, Z), \sigma(JW, JW) \rangle = -c \neq 0. \tag{3.9}$$

On the other hand, (3.8) gives

$$\lambda\mu = \langle \sigma(Z, Z), \sigma(JZ, JZ) \rangle = -\tilde{K}(Z, JZ) = -4c. \tag{3.10}$$

Combining (3.9) and (3.10), we obtain $c = 0$ which is a contradiction. Therefore, if $n_1 > n$, we must have $n_2 = 1$. Consequently, exactly one of Case (1) or Case (2) occurs.

Next, assume that Case (2) occurs. Suppose that $\mathcal{D}^1$ contains a nonempty open subset V of M. Then, by applying exactly the same argument as in Case (i) to V instead of U, we conclude that $n_2 = 1$ which is a contradiction. Similarly, $\mathcal{D}^2$ does not contain any nonempty open subset of M. Consequently, when Case (2) occurs, then, restricted some open dense subset of M, N_1 and N_2 are purely real submanifolds. This completes the proof of Lemma 3.1. □

We consider the Case (1) and Case (2) of Lemma 3.1 separately.

CASE (1). $\dim N_2 = 1$.

First, assume that $J\xi \in TN_1$. If this occurs, we may choose an orthonormal frame $e_1, \ldots, e_{2n-1}$ on M in such a way that $e_1, \ldots, e_{2n-2}$ are tangent to N_1, e_{2n-1} tangent to N_2 and $e_1 = J\xi$, $e_3 = Je_2, \ldots, e_{2n-3} = Je_{2n-4}$, $e_{2n-1} = Je_{2n-2}$. Clearly, the distribution $\mathcal{D}^1$ is spanned by $e_3, \ldots, e_{2n-3}$.

Using the formulas of Gauss and Weingarten together with $\nabla_{e_{2n-1}} e_{2n-1} = \nabla_{e_{2n-1}} e_{2n-2} = 0$, we find $-\sigma(e_{2n-1}, e_{2n-2}) = J\tilde{\nabla}_{e_{2n-1}} e_{2n-1} = J\sigma(e_{2n-1}, e_{2n-1})$ which implies

$$\sigma(e_{2n-1}, e_{2n-2}) = \sigma(e_{2n-1}, e_{2n-1}) = 0. \tag{3.11}$$

Hence, by the equation of Gauss, we obtain $0 = K(e_{2n-1}, e_{2n-2}) = 4c$ which is a contradiction. Hence, we have $J\xi \notin TN_1$.

Next, let us assume $J\xi \in TN_2$. Then N_1 is a holomorphic submanifold and N_2 is a totally real submanifold of $\tilde{M}^n(4c)$. Thus, in this case, M is a proper CR-product in the sense of [2]. But it was proved in [2] that there exist no proper CR-products of codimension one in any nonflat complex space form. Hence, we also have $J\xi \notin TN_2$. Consequently, we obtain

$$J\xi = \cos\alpha e_{2n-2} + \sin\alpha e_{2n-1}, \quad \sin\alpha\cos\alpha \neq 0, \tag{3.12}$$

where e_{2n-2} is a unit vector tangent to N_1 and e_{2n-1} a unit vector tangent to N_2. It follows from (3.12) that

$$Je_{2n-1} = -\sin\alpha\xi - \cos\alpha e_{2n-3} \tag{3.13}$$

for some unit vector $e_{2n-3} \in TN_1$ with $\langle e_{2n-3}, e_{2n-2}\rangle = 0$.

Since $\langle Je_{2n-3}, \xi\rangle = -\langle e_{2n-3}, J\xi\rangle = 0$, (3.13) gives

$$Je_{2n-3} = -\sin\alpha e_{2n-2} + \cos\alpha e_{2n-1}. \tag{3.14}$$

Using (3.12) and (3.14), we find

$$Je_{2n-2} = -\cos\alpha\xi - \sin\alpha e_{2n-3}. \tag{3.15}$$

Clearly, e_{2n-3}, e_{2n-2}, e_{2n-1}, ξ span a complex 2-plane H_x at each point $x \in M$ and $\mathcal{D}^1_x$ is the orthogonal complementary subspace of H_x in $T_x\tilde{M}^n(4c)$.

Since N_2 is totally geodesic in M, (3.13) and the formulas of Gauss and Weingarten imply

$$\begin{aligned} J\sigma(V, e_{2n-1}) &= J\tilde{\nabla}_V e_{2n-1} = \tilde{\nabla}_V Je_{2n-1} \\ &= -(\cos\alpha)\{(V\alpha)\xi + \nabla_V e_{2n-3} + \sigma(V, e_{2n-3})\} + \sin\alpha\{(V\alpha)e_{2n-3} + AV\}, \end{aligned} \tag{3.16}$$

for V tangent to M, where $A = A_\xi$ is the shape operator. Using (3.16), we obtain

$$(V\alpha) = -\langle AV, e_{2n-3}\rangle, \quad V \in TM. \tag{3.17}$$

Also, by taking the inner product of (3.16) with e_{2n-2}, we get

$$\langle\nabla_V e_{2n-3}, e_{2n-2}\rangle = \tan\alpha\langle AV, e_{2n-2}\rangle - \langle AV, e_{2n-1}\rangle, \quad V \in TM. \tag{3.18}$$

Moreover, by taking the inner product of (3.16) with $X \in \mathcal{D}^1$, we find

$$\langle\nabla_V e_{2n-3}, X\rangle = \tan\alpha\langle AV, X\rangle, \quad X \in \mathcal{D}^1,\ V \in TM. \tag{3.19}$$

In particular, if $V = e_{2n-1}$, (3.18) and (3.19) reduce respectively to

$$\sigma(e_{2n-1}, e_{2n-1}) = (\tan\alpha)\sigma(e_{2n-1}, e_{2n-2}), \tag{3.20}$$

$$\sigma(e_{2n-1}, X) = 0, \quad X \in \mathcal{D}^1. \tag{3.21}$$

We summarize the above results as the following.

Lemma 3.2. *Let $M = N_1^{n-2} \times N_2^1$ be a real hypersurface of a nonflat complex space form $M^n(4c)$. Then we have*

(i) $V\alpha = -\langle AV, e_{2n-3}\rangle,$

(ii) $\langle\nabla_V e_{2n-3}, e_{2n-2}\rangle = \tan\alpha\langle AV, e_{2n-2}\rangle - \langle AV, e_{2n-1}\rangle,$

(iii) $\langle\nabla_V e_{2n-3}, X\rangle = \tan\alpha\langle AV, X\rangle,$

(iv) $\sigma(e_{2n-1}, e_{2n-1}) = (\tan\alpha)\sigma(e_{2n-1}, e_{2n-2}),$

(v) $\sigma(e_{2n-1}, X) = 0$

for $X \in \mathcal{D}^1$, $V \in TM$.

For simplicity, we put $h_{AB} = \langle Ae_A, e_B\rangle$. Using (i), we find

$$\begin{aligned}(\bar{\nabla}_{e_{2n-2}}\sigma)&(e_{2n-3}, e_{2n-3})\\ &= -(e_{2n-2}e_{2n-3}\alpha)\xi - 2\sum_{j=1}^{2n-2}\omega^j_{2n-3}(e_{2n-2})h_{j\,2n-3}\xi,\end{aligned} \tag{3.22}$$

$$\begin{aligned}(\bar{\nabla}_{e_{2n-3}}\sigma)(e_{2n-2}, e_{2n-3}) &= -(e_{2n-3}e_{2n-2}\alpha)\xi\\ &- \sum_{j=1}^{2n-2}\omega^j_{2n-2}(e_{2n-3})h_{j\,2n-3}\xi - \sum_{j=1}^{2n-2}\omega^j_{2n-3}(e_{2n-3})h_{j\,2n-2}\xi.\end{aligned} \tag{3.23}$$

From (2.6), (3.14) and (3.15) we find

$$(\tilde{R}(e_{2n-2}, e_{2n-3})e_{2n-3})^{\perp} = 0. \tag{3.24}$$

Also, from (i) of Lemma 3.2, we also have

$$\begin{aligned}&e_{2n-2}e_{2n-3}\alpha - e_{2n-2}e_{2n-3}\alpha\\ &= \sum_{j=1}^{2n-2}\{\omega^j_{2n-2}(e_{2n-3}) - \omega^j_{2n-3}(e_{2n-2})\}h_{j\,2n-3}.\end{aligned} \tag{3.25}$$

Therefore, by applying the equation of Codazzi and (3.22)–(3.25), we obtain

$$\sum_{j=1}^{2n-2}\omega^j_{2n-3}(e_{2n-3})h_{j\,2n-2} = \sum_{j=1}^{2n-2}\omega^j_{2n-3}(e_{2n-2})h_{j\,2n-3}. \tag{3.26}$$

On the other hand, from Lemma 3.2 (iii), we have

$$\begin{aligned}\omega^j_{2n-3}(e_{2n-2}) &= (\tan\alpha)h_{j\,2n-2},\\ \omega^j_{2n-3}(e_{2n-3}) &= (\tan\alpha)h_{j\,2n-3}.\end{aligned} \tag{3.27}$$

Substituting (3.27) into (3.26), we get

$$\omega^{2n-2}_{2n-3}(e_{2n-3})h_{2n-2\,2n-2} = \omega^{2n-2}_{2n-3}(e_{2n-2})h_{2n-3\,2n-2}. \tag{3.28}$$

We find from Lemma 3.2 (ii) and (3.28) that

$$h_{2n-3\,2n-1}h_{2n-2\,2n-2} = h_{2n-2\,2n-1}h_{2n-3\,2n-2}. \tag{3.29}$$

We also have from Lemma 3.2 (i)

$$(\bar{\nabla}_{e_{2n-1}}\sigma)(e_{2n-3}, e_{2n-3}) = -(e_{2n-1}e_{2n-3}\alpha)\xi, \tag{3.30}$$

$$(\bar{\nabla}_{e_{2n-3}}\sigma)(e_{2n-1}, e_{2n-3}) = -(e_{2n-3}e_{2n-1}\alpha)\xi - \sum_{j=1}^{2n-2} \omega_{2n-3}^{j}(e_{2n-3})h_{j\,2n-1}\xi. \tag{3.31}$$

From (2.6), (3.13) and (3.14) we find

$$\big(\tilde{R}(e_{2n-1}, e_{2n-3})e_{2n-3}\big)^{\perp} = 0. \tag{3.32}$$

Also, we have $e_{2n-1}e_{2n-3}\alpha - e_{2n-1}e_{2n-3}\alpha = [e_{2n-1}, e_{2n-3}]\alpha = 0$. Thus, by the equation of Codazzi and (3.30)–(3.32), we obtain

$$\sum_{j=1}^{2n-2} \omega_{2n-3}^{j}(e_{2n-3})h_{j\,2n-1} = 0. \tag{3.33}$$

Hence, by applying (v) of Lemma 3.2 and (3.33) we get

$$\omega_{2n-3}^{2n-2}(e_{2n-3})h_{2n-2\,2n-1} = 0 \tag{3.34}$$

which is equivalent to

$$h_{2n-2\,2n-1}\{h_{2n-3\,2n-1} - (\tan\alpha)h_{2n-3\,2n-2}\} = 0, \tag{3.35}$$

by (ii) of statement (ii) of Lemma 3.2.

It follows from (i) and (v) of Lemma 3.2 that

$$(\bar{\nabla}_{e_{2n-1}}\sigma)(e_{2n-3}, e_{2n-2}) = -(e_{2n-1}e_{2n-2}\alpha)\xi, \tag{3.36}$$

$$(\bar{\nabla}_{e_{2n-2}}\sigma)(e_{2n-1}, e_{2n-3}) = -(e_{2n-2}e_{2n-1}\alpha)\xi - \omega_{2n-3}^{2n-2}(e_{2n-2})h_{2n-2\,2n-1}\xi. \tag{3.37}$$

From (2.6), (3.13)–(3.15) we find

$$\big(\tilde{R}(e_{2n-1}, e_{2n-2})e_{2n-3}\big)^{\perp} = -c\,\xi. \tag{3.38}$$

Since $e_{2n-1}e_{2n-2}\alpha - e_{2n-1}e_{2n-3}\alpha = 0$, the equation of Codazzi and (3.36)–(3.38) imply

$$\varphi\omega_{2n-3}^{2n-2}(e_{2n-2}) = -c \neq 0, \quad \varphi = h_{2n-2\,2n-1}. \tag{3.39}$$

From (3.34), (3.35) and (3.39) we find $h_{2n-22n-1} \neq 0$ and

$$\omega_{2n-3}^{2n-2}(e_{2n-3}) = 0, \quad h_{2n-3\,2n-1} = (\tan\alpha)h_{2n-3\,2n-2}. \tag{3.40}$$

By substituting the second equation of (3.40) into (3.29), we find

$$h_{2n-3\,2n-2}\{\varphi - (\tan\alpha)h_{2n-2\,2n-2}\} = 0. \tag{3.41}$$

Since we have $\varphi - (\tan\alpha)h_{2n-2\,2n-2} = \omega_{2n-2}^{2n-3}(e_{2n-2}) \neq 0$ from (3.39), we obtain $h_{2n-3\,2n-2} = 0$ from (3.41). Thus, we also have $h_{2n-3\,2n-1} = 0$ by (3.40). Hence, by (i) of Lemma 3.2, we also have $e_{2n-1}\alpha = e_{2n-2}\alpha = 0$. Consequently, we have

$$h_{2n-3\,2n-2} = h_{2n-3\,2n-1} = e_{2n-1}\alpha = e_{2n-2}\alpha = 0. \tag{3.42}$$

We have from (3.36), (3.42) and Lemma 3.2 (i)

$$(\bar{\nabla}_{e_{2n-1}}\sigma)(e_{2n-3}, e_{2n-2}) = 0. \tag{3.43}$$

On the other hand, from (i) and (v) of Lemma 3.2, (3.40) and (3.42), we have

$$(\bar{\nabla}_{e_{2n-3}}\sigma)(e_{2n-1}, e_{2n-2}) = (e_{2n-3}\varphi)\xi. \tag{3.44}$$

From (2.6), (3.13)–(3.15) we find

$$\left(\tilde{R}(e_{2n-1}, e_{2n-3})e_{2n-2}\right)^{\perp} = c(1 - 3\cos^2\alpha)\xi. \tag{3.45}$$

Hence, by the equation of Codazzi and (3.43)–(3.45), we get

$$e_{2n-3}\varphi = c(3\cos^2\alpha - 1). \tag{3.46}$$

We get from (2.4)

$$(\bar{\nabla}_{e_{2n-3}}\sigma)(e_{2n-1}, e_{2n-1}) = (e_{2n-3}h_{2n-1\,2n-1})\xi. \tag{3.47}$$

On the other hand, from (i) of Lemma 3.2 and (3.42), we have

$$(\bar{\nabla}_{e_{2n-1}}\sigma)(e_{2n-3}, e_{2n-1}) = 0. \tag{3.48}$$

From (2.6), (3.13)–(3.15) we find

$$\left(\tilde{R}(e_{2n-3}, e_{2n-1})e_{2n-1}\right)^{\perp} = 3c\sin\alpha\cos\alpha\xi. \tag{3.49}$$

Hence, by the equation of Codazzi and (3.47)–(3.49), we get

$$e_{2n-3}h_{2n-1\,2n-1} = 3c\sin\alpha\cos\alpha. \tag{3.50}$$

We obtain from (iv) of Lemma 3.2, (3.46) and (3.50)

$$3c\sin\alpha\cos\alpha = e_{2n-3}(\varphi\tan\alpha) = \varphi(\sec^2\alpha)e_{2n-3}\alpha + c(3\cos^2\alpha - 1)\tan\alpha$$

which implies

$$e_{2n-3}\alpha = \frac{c}{\varphi}\sin\alpha\cos\alpha. \tag{3.51}$$

Applying statement (i) of Lemma 3.2 and (3.51), we have

$$h_{2n-3\,2n-3} = -\frac{c}{\varphi}\sin\alpha\cos\alpha. \tag{3.52}$$

Therefore, by Lemma 3.2 (iv), (2.6), (3.13), (3.42), (3.52) and the equation of Gauss, we get

$$0 = K(e_{2n-3}, e_{2n-1}) = c(1 + 3\cos^2\alpha) + h_{2n-3\,2n-3}h_{2n-1\,2n-1} = 4c\cos^2\alpha,$$

which is a contradiction. Hence, Case (1) cannot occur.

CASE (2). $n \geq 3$, $n_1 = n$, and $n_2 = n-1$. From Lemma 3.1, we know that, restricted to an open dense subset $\hat{U}$ of M, N_1 and N_2 are purely real submanifolds of $\tilde{M}^n(4c)$. We shall only work on $\hat{U}$ to derive a contradiction. Without loss of generality, we may just simply assume that $\hat{U} = M$.

Since $\dim N_1 = n$ and N_1 is a purely real submanifold of $\tilde{M}^n(4c)$, $J\xi$ cannot be tangent to N_2 at every point $x \in M = N_1 \times N_2$. Thus

$$J\xi = \cos\alpha e_1 + \sin\alpha e_{n+1}, \quad \cos\alpha \neq 0 \tag{3.53}$$

for some unit vectors $e_1 \in TN_1$, $e_{n+1} \in TN_2$.

Let $\mathcal{H} = TM \cap J(TM)$ denote the maximal holomorphic subbundle of TM. Then $\mathcal{H}$ is the orthogonal complementary subbundle of the complex line bundle spanned by ξ, $J\xi$. Put

$$\mathcal{H}^j = \mathcal{H} \cap TN_j, \quad j = 1,\ 2. \tag{3.54}$$

Since $n \geq 3$, $\dim N_1 = n$ and $\dim N_2 = n-1$, we have $\operatorname{rank}(\mathcal{H}^1) = n-1$ and $\operatorname{rank}(\mathcal{H}^2) = n-1$ or $n-2$ according as $\sin\alpha = 0$ or $\sin\alpha \neq 0$, respectively.

We need the following.

Lemma 3.3. *In* Case (2) *we have the following.*

(a) $$\sigma(Z, JX) = 0, \quad Z \in TN_2,\ X \in \mathcal{H}^1,$$

(b) $$\sigma(Y, JW) = 0, \quad Y \in TN_1,\ W \in \mathcal{H}^2,$$

(c) $$\langle AV, JW\rangle = \sin\alpha\,\langle \nabla_V e_{n+1}, W\rangle, \quad W \in \mathcal{H}^2,$$

where V is a vector in TM.

Proof. For vector fields X in $\mathcal{H}^1$ and Z in TN_2, the formulas of Gauss and Weingarten give $\nabla_Z JX + \sigma(Z, JX) = J\sigma(Z, X)$, which implies formula (a). Similarly, we have formula (b).

For vector $V \in TM$, we have $-JAV = J\tilde{\nabla}_V \xi = \tilde{\nabla}_V J\xi$. Thus, from (3.53) we obtain formula (c). $\square$

Let $X \in TN_1$, $Z \in TN_2$ and U, V be any vectors in TM. Then we obtain from the equation of Gauss that

$$0 = \tilde{R}(Z, U, V, X) + \langle \sigma(Z, X), \sigma(U, V)\rangle - \langle \sigma(Z, V), \sigma(U, X)\rangle . \tag{3.55}$$

From (2.6) we have

$$\begin{aligned}\tilde{R}(Z, U, V, X) = c\{-\langle Z, V\rangle \langle U, X\rangle + \langle JU, V\rangle \langle JZ, X\rangle + \langle Z, JV\rangle \langle JU, X\rangle \\ + 2\langle Z, JU\rangle \langle JV, X\rangle\}.\end{aligned} \tag{3.56}$$

It follows from (a) and (b) of Lemma 3.3, (3.55) and (3.56) that

Lemma 3.4. *In* Case (2) *we have the following.*

$$\begin{aligned}\langle \sigma(Z, X), \sigma(V, JY)\rangle =c\{\langle X, V\rangle \langle Z, JY\rangle - \langle V, Y\rangle \langle JZ, X\rangle \\ + 2\langle X, Y\rangle \langle Z, JV\rangle\}\end{aligned} \tag{d}$$

for $X, V \in TN_1$, $Z \in TN_2$, $Y \in \mathcal{H}^1$.

$$\begin{aligned}\langle \sigma(X, Z), \sigma(W, JP)\rangle =c\{\langle Z, W\rangle \langle X, JP\rangle - \langle W, P\rangle \langle JX, Z\rangle \\ + 2\langle Z, P\rangle \langle X, JW\rangle\}\end{aligned} \tag{e}$$

for $X \in TN_1$, $Z, W \in TN_2$, $P \in \mathcal{H}^2$.

CASE (2-a). $J\xi = e_1 \in TN_1$.

In this case, we get $\sin\alpha = 0$ from (3.53) and

$$\mathcal{H} = \{V \in TM : \langle X, e_1\rangle = 0\}, \quad \mathcal{H}^2 = TN_2. \tag{3.57}$$

Hence we obtain from (c) of Lemma 3.3 that

$$\sigma(V, JW) = 0, \quad V \in TM,\ W \in TN_2. \tag{3.58}$$

If $\sigma(Z, e_1) = 0$ for all $Z \in TN_2$, then from the equation of Gauss we have

$$\langle \sigma(Z, Z), \sigma(e_1, e_1)\rangle = -c \tag{3.59}$$

for any unit vector $Z \in TN_2$. From (3.59) we obtain $h_{11} \neq 0$ and $\sigma(Z, Z) = \sigma(W, W)$ for any unit vectors $Z, W \in TN_2$. Since N_2 is totally geodesic in $M = N_1 \times N_2$, this implies that N_2 is totally umbilical in $\tilde{M}^n(4c)$. Because $\dim N_2 \geq 2$, a result of [5] implies that N_2 is a totally real submanifold in $\tilde{M}^n(4c)$ such that ξ is perpendicular to $J(TN_2)$. Hence, by applying (3.55) and the equation of Gauss, we obtain

$$\langle \sigma(Z,Z), \sigma(JZ,JZ)\rangle = -4c \quad \text{and} \quad \langle \sigma(W,W), \sigma(JZ,JZ)\rangle = -c$$

for orthonormal vectors Z, W in TN_2. Clearly, this is impossible, since $c \neq 0$ and $\sigma(Z,Z) = \sigma(W,W)$. Hence, $\sigma(Z, J\xi) \neq 0$ for some $Z \in TN_2$. Therefore, by applying (e) of Lemma 3.4, we obtain

$$\sigma(V, JY) = 0 \quad \text{for } V \in \mathcal{H},\ Y \in \mathcal{H}^1. \tag{3.60}$$

Let e_2 be an unit vector in $\mathcal{H}^1$. Then there exist a $\theta \in \mathbf{R}$ and unit vectors $e_3 \in \mathcal{H}^1$, $e_{n+1} \in TN_2$ with $\langle e_2, e_3\rangle = 0$ such that

$$Je_2 = \cos\theta e_3 + \sin\theta e_{n+1}, \qquad \sin\theta \neq 0. \tag{3.61}$$

When $n = 3$, (3.61) gives $\langle Je_2, e_{n+1}\rangle = \langle Je_2, e_1\rangle = 0$. Thus, (3.61) implies that

$$Je_3 = -\cos\theta e_2 + \sin\theta\, \eta, \qquad \sin\theta \neq 0, \tag{3.62}$$

where $\eta = e_{n+2}$ is a unit vector in TN_2 with $\langle e_{n+2}, e_{n+1}\rangle = 0$.

When $n \geq 4$, (3.61) implies

$$Je_3 = -\cos\theta e_2 + \sin\theta\, \eta, \tag{3.63}$$

where $\eta = \cos\gamma e_4 + \sin\gamma e_{n+2}$, $\gamma \in \mathbf{R}$ with $\sin\gamma \neq 0$, e_4 is a unit vector in $\mathcal{H}^1$ with $\langle e_4, e_2\rangle = \langle e_4, e_3\rangle = 0$ and e_{n+2} is a unit vector in TN_2 with $\langle e_{n+2}, e_{n+1}\rangle = 0$.

From (3.61), (3.62) and (3.63), we get

$$J\eta = -\sin\theta e_3 + \cos\theta e_{n+1}, \tag{3.64}$$

$$Je_{n+1} = -\sin\theta e_2 - \cos\theta\, \eta. \tag{3.65}$$

Applying (3.60) with $V = e_j$, $j \in \{2, \ldots, n\}$ and $Y = e_3$ and (3.62)–(3.63), we have

$$\cos\theta h_{2j} - \sin\theta(\cos\gamma h_{4j} + \sin\gamma h_{jn+2}) = 0. \tag{3.66}$$

Notice that $\cos\gamma = 0$ and $\sin\gamma = 1$ when $n = 3$.

On the other hand, from Lemma 3.3 (b) with $Y = e_j$, $j \in \{2, \ldots, n\}$ and $W = e_{n+1}$, we find

$$\sin\theta h_{2j} + \cos\theta(\cos\gamma h_{4j} + \sin\gamma h_{jn+2}) = 0. \tag{3.67}$$

Combining (3.66) and (3.67), we obtain $h_{22} = \cdots = h_{2n} = 0$.

Also, from (3.60) with $V = Y = e_2$ and (3.61), we get $\cos\theta h_{23} + \sin\theta h_{2n+1} = 0$. Therefore, $h_{2n+1} = 0$. Hence, by applying the equation of Gauss again, we obtain $0 = K(e_2, e_{n+1}) = \tilde{K}(e_2, e_{n+1}) = c(1 + 3\sin^2\theta)$ which is a contradiction, since $c \neq 0$.

CASE (2-b). $J\xi = \cos\alpha e_1 + \sin\alpha e_{n+1}$, $\sin\alpha\cos\alpha \neq 0$.

Since Je_1 is perpendicular to e_1 and e_{n+1}, there exist $\gamma \in \mathbf{R}$, unit vectors $e_2 \in \mathcal{H}^1$ and $e_{n+2} \in \mathcal{H}^2$ such that

$$Je_1 = -\cos\alpha\xi + \sin\alpha\,\eta, \quad \eta = \cos\gamma e_2 + \sin\gamma e_{n+2}. \tag{3.68}$$

From (3.68) we find

$$J\eta = -\sin\alpha e_1 + \cos\alpha e_{n+1}, \quad Je_{n+1} = -\sin\alpha\xi - \cos\alpha\,\eta. \tag{3.69}$$

Clearly, ξ, e_1, e_{n+1}, η span a complex vector subbundle $\mathcal{L}$ of rank 2. It is easy to verify that $\zeta = -\sin\gamma e_2 + \cos\gamma e_{n+2}$ is a unit vector perpendicular to $\mathcal{L}$. Moreover, it is easy to see that

$$\mathcal{H}^1 = \{X \in TN_1 : \langle X, e_1\rangle = 0\}, \quad \mathcal{H}^2 = \{Z \in TN_2 : \langle Z, e_{n+1}\rangle = 0\}. \tag{3.70}$$

Assume $\sin\gamma = 0$. Then we may choose e_2 such that

$$\begin{aligned} Je_1 &= -\cos\alpha\xi + \sin\alpha e_2, \\ Je_2 &= -\sin\alpha e_1 + \cos\alpha e_{n+1}, \\ Je_{n+1} &= -\sin\alpha\xi - \cos\alpha e_2. \end{aligned} \tag{3.71}$$

We get, from Lemma 3.3 (a) with $X = e_2$ and $Z = e_{n+1}$, e_{n+2}, and (3.71), that

$$h_{n+1n+1} = (\tan\alpha)h_{1n+1}, \quad h_{n+1n+2} = (\tan\alpha)h_{1n+2}. \tag{3.72}$$

Also from Lemma 3.4 (d) , we get

$$h_{1n+1}\sigma(e_1, Je_2) \neq 0, \quad h_{2n+1}\sigma(e_1, Je_2) = h_{1n+2}\sigma(e_1, Je_2) = 0 \tag{3.73}$$

which imply $h_{2n+1} = h_{1n+2} = 0$. Hence, by applying (3.72), we get $h_{n+1n+2} = 0$. Therefore, by applying the equation of Gauss, we find

$$0 = \tilde{R}(e_1, e_{n+2}, e_{n+1}, e_{n+2}) = h_{1n+1}h_{n+2n+2}.$$

If $h_{1n+1} = 0$, then (3.72) yields $h_{n+1n+1} = 0$. Hence, by the equation of Gauss, we get $0 = K(e_1, e_{n+1}) = \tilde{K}(e_1, e_{n+1}) = c$, which is a contradiction.

Similarly, if $h_{n+2n+2} = 0$, then the equation of Gauss gives $0 = K(e_1, e_{n+2}) = \tilde{K}(e_1, e_{n+2}) = c$, which is also a contradiction. Consequently, we obtain $\sin\gamma \neq 0$.

Next, we assume $\cos\gamma = 0$. Then we may choose e_{n+2} such that

$$
\begin{aligned}
Je_1 &= -\cos\alpha\xi + \sin\alpha e_{n+2}, \\
Je_{n+1} &= -\sin\alpha\xi - \cos\alpha e_{n+2}, \\
Je_{n+2} &= -\sin\alpha e_1 + \cos\alpha e_{n+1}.
\end{aligned} \tag{3.74}
$$

Using (b) of Lemma 3.3 with $Y = e_j$ and $W = e_{n+2}$ and (3.74), we get

$$h_{jn+1} = (\tan\alpha)h_{1j}, \quad j = 1, \ldots, n. \tag{3.75}$$

Also from Lemma 3.4 (e) and (3.74), we get

$$
\begin{aligned}
&h_{1n+1}\sigma(e_{n+1}, Je_{n+2}) \neq 0, \\
&h_{2n+1}\sigma(e_{n+1}, Je_{n+2}) = h_{1n+2}\sigma(e_{n+1}, Je_{n+2}) = 0,
\end{aligned} \tag{3.76}
$$

which imply $h_{2n+1} = h_{1n+2} = 0$. Hence, by applying (3.75), we get $h_{12} = 0$. Thus, by the equation of Gauss, we find $0 = \tilde{R}(e_{n+1}, e_2, e_1, e_2) = h_{1n+1}h_{22}$.

If $h_{1n+1} = 0$, then (3.75) yields $h_{11} = 0$. Hence, by the equation of Gauss, we get $0 = K(e_1, e_{n+1}) = \tilde{K}(e_1, e_{n+1}) = c$, which is a contradiction.

Similarly, if $h_{22} = 0$, then by $h_{2n+1} = 0$ and the equation of Gauss we get $0 = \tilde{K}(e_2, e_{n+1}) = c$, which is also a contradiction. Consequently, we obtain $\cos\gamma \neq 0$. Consequently, in Case (2-b), we have $\sin\gamma\cos\gamma\sin\alpha\cos\alpha \neq 0$.

CASE (2-b-i). $n = 3$. In this case, for each unit vector e_3 in $\mathcal{H}^1$ perpendicular to e_2, e_3 is perpendicular to both $\mathcal{L}$ and ζ. Since e_3, ζ are orthonormal vectors, they span the orthogonal complementary complex distribution $\mathcal{L}^\perp$ of $\mathcal{L}$, so that we may thus choose e_3 such that

$$Je_3 = -\sin\gamma e_2 + \cos\gamma e_5, \quad \cos\gamma \neq 0. \tag{3.77}$$

Hence, we also have

$$Je_5 = -\sin\gamma\sin\alpha e_1 - \cos\gamma e_3 + \sin\gamma\cos\alpha e_4. \tag{3.78}$$

From (3.68), (3.69), and (3.78) we get

$$Je_2 = -\cos\gamma\sin\alpha e_1 + \sin\gamma e_3 + \cos\gamma\cos\alpha e_4. \tag{3.79}$$

Applying (a) of Lemma 3.3, (3.77) and (3.79), we have

$$(\sin\alpha)h_{1t} - (\tan\gamma)h_{3t} - (\cos\alpha)h_{4t} = 0, \tag{3.80}$$

$$h_{55} = -(\tan\gamma)h_{25}. \tag{3.81}$$

Similarly, from (b) of Lemma 3.3 with $W = e_5$ and (3.78), we find

$$(\tan\gamma\sin\alpha)h_{1j} + h_{j3} - (\tan\gamma\cos\alpha)h_{j4} = 0, \quad j = 1, 2, 3. \tag{3.82}$$

We have from (d) of Lemma 3.4

$$h_{it}\langle\sigma(e_j, Je_k), \xi\rangle = c\{\delta_{ij}\langle e_t, Je_k\rangle - \delta_{jk}\langle e_i, Je_t\rangle + 2\delta_{ik}\langle e_t, Je_j\rangle\} \tag{3.83}$$

for $i, j = 1, 2, 3$; $k = 2, 3$; $t = 4, 5$.

We find from (3.68), (3.69), (3.77)–(3.79) and (3.83), that

$$\begin{aligned} &h_{14}\sigma(e_1, Je_2) = c\cos\gamma\cos\alpha\xi \neq 0,\\ &h_{25}\sigma(e_1, Je_2) = 2c\sin\gamma\sin\alpha\xi \neq 0,\\ &h_{14}\sigma(e_1, Je_3) = h_{14}\sigma(e_2, Je_3) = 0,\\ &h_{15}\sigma(e_1, Je_3) = h_{24}\sigma(e_2, Je_3) = h_{35}\sigma(e_1, Je_3) = 0. \end{aligned} \tag{3.84}$$

From the first two equations of (3.84), we get $h_{14}, h_{25} \neq 0$ and

$$h_{25} = 2(\tan\gamma\tan\alpha)h_{14}. \tag{3.85}$$

Moreover, from the remaining equations of (3.84) we get

$$h_{15} = h_{24} = h_{34} = h_{35} = 0, \tag{3.86}$$

$$\sigma(e_1, Je_3) = \sigma(e_2, Je_3) = 0. \tag{3.87}$$

Applying $\sigma(e_1, Je_3) = 0$, (3.86) and (3.82) with $j = 2$, we find

$$h_{12} = h_{23} = 0. \tag{3.88}$$

Using $\sigma(e_2, Je_3) = 0$, we find

$$h_{25} = (\tan\gamma)h_{22}. \tag{3.89}$$

By (3.89) and the equation of Gauss, we get $0 = K(e_2, e_5) = c + h_{22}h_{55} - h_{25}^2$. Thus, by applying (3.81) and (3.89), we obtain $2h_{25}^2 = c$. Hence, from (3.81), (3.89) and the second equation of (3.84), we find

$$h_{25} = \sqrt{\frac{c}{2}}, \quad h_{22} = \sqrt{\frac{c}{2}}\cot\gamma, \quad h_{55} = -\sqrt{\frac{c}{2}}\tan\gamma, \quad c > 0. \tag{3.90}$$

We get from (3.85) and (3.90)

$$h_{14} = \frac{\sqrt{c}}{2\sqrt{2}}\cot\gamma\cot\alpha. \tag{3.91}$$

Using $h_{35} = 0$, (3.90), and the equation of Gauss for $K(e_3, e_5)$, we find

$$h_{33} = \sqrt{2c}\,(1 + 3\cos^2\gamma)\cot\gamma. \tag{3.92}$$

It follows from (3.79) with $j = 3$ and (3.86) that $h_{13} = -(\cot\gamma\csc\alpha)h_{33}$. Hence, by (3.92) we obtain

$$h_{13} = -\sqrt{2c}\,(1 + 3\cos^2\gamma)\cot^2\gamma\csc\alpha. \tag{3.93}$$

Applying (3.79) and the first equation in (3.84), we get

$$h_{14}(-\cos\gamma\sin\alpha h_{11} + \sin\gamma h_{13} + \cos\gamma\cos\alpha h_{14}) = c\cos^2\gamma\cos\alpha. \tag{3.94}$$

Combining (3.94) with (3.82) with $j = 1$, we find

$$h_{14}(-\sin\alpha h_{11} + \cos\alpha h_{14}) = c\cos\gamma\cos\alpha. \tag{3.95}$$

Substituting (3.91) into (3.95), we obtain

$$h_{11} = \frac{\sqrt{c}}{2\sqrt{2}}\,(\cot^2\alpha\cot\gamma - 8\sin\gamma\cos\gamma). \tag{3.96}$$

Substituting (3.91), (3.93) and (3.96) into (3.82) with $j = 1$, we find

$$\big((1 + 3\cos^2\gamma)\cot^2\gamma + 2\sin^2\alpha\sin^2\gamma\big)c = 0$$

which is a contradiction. Consequently, we have proved that every real hypersurface in a nonflat space form $\tilde{M}^n(4c)$ is irreducible if $n \le 3$.

CASE (2-b-ii). $n \ge 4$.

In this case, we have

$$Je_1 = -\cos\alpha\xi + \sin\alpha(\cos\gamma e_2 + \sin\gamma e_{n+2}), \tag{3.97}$$

$$Je_{n+1} = -\sin\alpha\xi - \cos\alpha(\cos\gamma e_2 + \sin\gamma e_{n+2}), \tag{3.98}$$

where $\sin\alpha\cos\alpha\sin\gamma\cos\gamma \neq 0$ and $e_2 \in \mathcal{H}^1$, $e_{n+2} \in \mathcal{H}^2$. Moreover, at each point $x \in M$, the vectors ξ, e_1, e_{n+1}, $\eta = \cos\gamma e_2 + \sin\gamma e_{n+2}$ span a complex 2-plane $\mathcal{L}_x \subset T_x\tilde{M}^n(4c)$.

Since Je_2 is perpendicular to ξ, e_2, e_{n+2}, we obtain from (3.97) and (3.98) that

$$Je_2 = -\cos\gamma\sin\alpha e_1 + \sin\gamma\cos\delta e_3 + \cos\gamma\cos\alpha e_{n+1} + \sin\gamma\sin\delta e_{n+3} \tag{3.99}$$

for some $\delta \in \mathbf{R}$, unit vector $e_3 \in \mathcal{H}^1$ with $\langle e_2, e_3\rangle = 0$, and unit vector $e_{n+3} \in \mathcal{H}^2$ with $\langle e_{n+2}, e_{n+3}\rangle = 0$. From (3.97)–(3.99) we get

$$Je_{n+2} = -\sin\gamma\sin\alpha e_1 - \cos\gamma\cos\delta e_3 + \sin\gamma\cos\alpha e_{n+1} - \cos\gamma\sin\delta e_{n+3}. \tag{3.100}$$

If $\sin\delta = 0$, then (3.99) and (3.100) reduce to (3.79) and (3.78), respectively. In this case, we also have $Je_3 = -\sin\gamma e_2 + \cos\gamma e_{n+2}$ with $\cos\gamma \neq 0$ from (3.97), (3.98), and (3.99). Hence, in this case the exact same argument as in Case (2-b-i) yields a contradiction. Thus, we have $\sin\delta \neq 0$.

If $\cos\delta = 0$, then (3.97)–(3.100) reduce to

$$Je_2 = -\cos\gamma\sin\alpha e_1 + \cos\gamma\cos\alpha e_{n+1} + \sin\gamma e_{n+3}, \tag{3.101}$$

$$Je_{n+2} = -\sin\gamma\sin\alpha e_1 + \sin\gamma\cos\alpha e_{n+1} - \cos\gamma e_{n+3}. \tag{3.102}$$

Hence, by (3.97), (3.98), and (3.101), we find

$$Je_{n+3} = -\sin\gamma e_2 + \cos\gamma e_{n+2}. \tag{3.103}$$

Using (3.97), (3.98), and Lemma 3.4 (d) with $X = e_j \in TN_1$, $V = e_1$, $Y = e_2$, and $Z = e_{n+1}, e_{n+2}$, we find

$$\begin{aligned} h_{jn+1}\sigma(e_1, Je_2) &= c\delta_{1j}\cos\gamma\cos\alpha\,\xi, \\ h_{jn+3}\sigma(e_1, Je_2) &= c\delta_{1j}\sin\gamma\,\xi, \quad j = 1,\ 2. \end{aligned} \tag{3.104}$$

Equations of (3.104) imply $\sigma(e_1, Je_2) \neq 0$ and $h_{2n+1} \neq 0$, $h_{2n+3} \neq 0$. Hence, by the equation of Gauss, we get $0 = \tilde{R}(e_2, e_{n+2}; e_{n+1}, e_{n+3})$. On the other hand, from (2.6), (3.100), (3.101), and (3.102) we get

$$\tilde{R}(e_2, e_{n+2}; e_{n+1}, e_{n+3}) = c\cos\alpha \neq 0,$$

which is a contradiction. Hence, we must have $\sin\delta\cos\delta \neq 0$ also.

Finally, from (3.97), (3.99), and Lemma 3.4 (d), we get

$$\begin{aligned} h_{jn+2}\sigma(e_1, Je_2) &= 2c\delta_{2j}\sin\gamma\sin\alpha\,\xi, \\ h_{jn+3}\sigma(e_1, Je_2) &= c\delta_{1j}\sin\gamma\sin\alpha\,\xi \end{aligned} \tag{3.105}$$

for $j = 1, 2, 3$. From (3.105) we obtain $h_{3n+2} = h_{3n+3} = 0$. Hence, by the equation of Gauss, we get $0 = \tilde{R}(e_3, e_{n+2}; e_{n+3}, e_{n+2})$.

On the other hand, from (2.6), (3.98), and (3.100), we get

$$\tilde{R}(e_3, e_{n+2}; e_{n+3}, e_{n+2}) = 3c\cos^2\gamma\cos\delta\sin\delta \neq 0,$$

which is a contradiction. Therefore, Case (2-b) is also impossible. Consequently, the real hypersurface must be irreducible. $\square$

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