ON THE NON-MINIMAL MARTIN BOUNDARY POINTS

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Dedicated to Professor KIYOSHI NOSHIRO on his 60th birthday

1. In a Green space[1] Ω we can introduce Martin's topology and make it the Martin space[2] $\hat{\Omega}$. Ω is a dense open subset of $\hat{\Omega}$ and the kernel

$$K(p, x) = \begin{cases} \dfrac{G(p, x)}{G(p, y_0)} & p \neq y_0 \\ 0 & p = y_0 \neq x \\ 1 & p = y_0 = x \end{cases}$$

can be extended continuously to $(p, x) \in \hat{\Omega} \times \Omega$, where $G(p, x)$ is a Green function in Ω and y_0 the fixed point of Ω. $\hat{\Omega}$ is a metric space. $\Delta = \hat{\Omega} - \Omega$ is divided into two disjoint subsets Δ_0, Δ_1 and $s \in \Delta_1$ is characterized by the fact that $K(s, x)$ is a minimal positive harmonic function[3] in $x \in \Omega$.

2. We shall show the following theorem:

THEOREM. *No point of Δ_0 is an isolated point.*

Proof. Let ω be an open subset of Ω, $\{x_n\}$ $(n = 1, 2, \ldots)$ be a sequence of points in ω such that $x_n \to x_0 \in \Delta$. If we denote by $\mathscr{V}$ the family of positive superharmonic functions in Ω, each of which dominates $K(x_n, y)$ on $\Omega - \omega$, then $\inf_{v \in \mathscr{V}} v(y)$ is equal to the positive superharmonic function except a polar set. We shall write this superharmonic function $\mathscr{E}^{\omega}_{K_n}(y)$.

In this case

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1) M. Brelot, G. Choquet, Espaces et lignes de Green. Annales Inst. Fourier 3 (1951), pp. 199-263.

2) M. Brelot, Le problème de Dirichlet. Axiomatique et frontière de Martin. Journal de Math. 35 (1956), pp. 297-335 (pp. 329-330). Cf. also R. S. Martin, Minimal positive harmonic functions, Trans. Amer. Math. Soc., 49 (1941), pp. 137-172. M. Parreau, Sur les moyennes des fonctions harmoniques et analytiques et la classification des surfaces de Riemann, Annales Inst. Fourier 3 (1952), pp. 103-197. L. Naïm, Sur le rôle de la frontière de R. S. Martin dans la théorie du potentiel, Annales Inst. Fourier 7 (1957), pp. 183-281.

3) R. S. Martin, loc. cit., p. 137.

$$\mathscr{E}^{\omega}_{K_n}(y) = \int K(x, y)\, d\mu_n(x)$$

where μ_n is a positive mass-distribution on $\overset{*}{\omega} \cap \Omega$ and the total mass of μ_n does not exceed 1, $\overset{*}{\omega}$ being the boundary of ω in $\hat{\Omega}$. By the theorem of choice, we can extract from $\{\mu_n\}$ the subsequence $\{\mu'_n\}$ such that μ'_n converges vaguely to μ and the carrier of μ is contained in $\overline{\overset{*}{\omega} \cap \Omega}$.

$$v(y) = \int K(x, y)\, d\mu(x)$$

is a positive superharmonic function in Ω, and we have

$$\mathscr{E}^{\omega}_{K_{x_0}}(y) \leq v(y).$$

In fact, for fixed $y \in \Omega$ and $r > 0$ we shall denote by ε'^{r}_{y} the mass-distribution which can be obtained after sweeping out the unit mass on y to the exterior of the sphere (circle) of radius r and with center y. Then

$$U^r(x) = \int K(x, z)\, d\varepsilon'^{r}_{y}(z)$$

is bounded and continuous on $\hat{\Omega}$. Therefore

$$\lim_{n\to\infty} \int U^r(x)\, d\mu'_n(x) = \int U^r(x)\, d\mu(x).$$

By reciprocal law

$$\lim_{n\to\infty} \int \mathscr{E}^{\omega}_{K_{x_{n'}}}(z)\, d\varepsilon'^{r}_{y}(z) = \int v(z)\, d\varepsilon'^{r}_{y}(z)$$

and by Fatou's lemma

$$\int \mathscr{E}^{\omega}_{K_{x_0}}(z)\, d\varepsilon'^{r}_{y}(z) \leq \varliminf_{n\to\infty} \int \mathscr{E}^{\omega}_{K_{x_{n'}}}(z)\, d\varepsilon'^{r}_{y}(z) = \int v(z)\, d\varepsilon'^{r}_{y}(z).$$

By making $r \to 0$ we can get for each $y \in \Omega$

$$\mathscr{E}^{\omega}_{K_{x_0}}(y) \leq v(y).$$

If μ_1 denotes the restriction of μ to Δ and μ_2 the restriction of μ to Ω, then

$$\begin{aligned} v(y) &= \int K(x, y)\, d\mu_1(x) + \int K(x, y)\, d\mu_2(x) \\ &= u(y) + w(y) \end{aligned}$$

where u is harmonic and w is a potential, and this is just the Riesz decomposition.

From now on let x_0 be a point of Δ_0 and x_0 be isolated. Let

$$K(x_0, y) = \int_{\Delta_1} K(x, y)\, d\nu(x)$$

be the canonical representation[4] of $K(x_0, y)$. Then we can find a neighbourhood $\hat{\delta}$ of x_0 such that

$$(1) \qquad \bar{\hat{\delta}} \cap \Delta_0 = \{x_0\}$$

and

$$(2) \qquad \nu(\Delta_1 - \bar{\hat{\delta}}) > 0.$$

If we set $\omega = \hat{\delta} \cap \Omega$, then

$$\begin{aligned} \mathscr{E}^{\omega}_{K_{x_0}}(y) &= \int_{\Delta_1} \mathscr{E}^{\omega}_{K_x}(y)\, d\nu(x) \\ &= \int_{\Delta_1 - \bar{\hat{\delta}}} \mathscr{E}^{\omega}_{K_x}(y)\, d\nu(x) + \int_{\bar{\hat{\delta}} \cap \Delta_1} \mathscr{E}^{\omega}_{K_x}(y)\, d\nu(x) \end{aligned}$$

The first term of the last side is harmonic, because ω is thin at each point of $\Delta_1 - \bar{\delta}$[5] and therefore we can get $\mathscr{E}^{\omega}_{K_x}(y) \equiv K(x, y)$.

We note that μ_1 is the restriction of the mass-distribution μ to $(\overline{\hat{\delta} \cap \Omega}) \cap \Delta$, which is contained in $\bar{\hat{\delta}} \cap \Delta$ and does not contain the point x_0. By (1) we can get $\mu_1(\Delta_0) = 0$, that is, μ_1 is the canonical mass-distribution of u, and by (2)

$$u_1(y) = \int_{\Delta_1 - \bar{\hat{\delta}}} K(x, y)\, d\nu(x) > 0.$$

Since

$$\begin{aligned} v(y) &= u(y) + w(y) \\ &\geq \mathscr{E}^{\omega}_{K_{x_0}}(y) \geq \int_{\Delta_1 - \bar{\hat{\delta}}} \mathscr{E}^{\omega}_{K_x}(y)\, d\nu(x) = \int_{\Delta_1 - \bar{\hat{\delta}}} K(x, y)\, d\nu(x) = u_1(y) \end{aligned}$$

and u is the greatest harmonic minorant of v, we have

$$u \geq u_1,$$

but the canonical mass-distribution of u has the carrier in $\Delta_1 \cap \bar{\hat{\delta}}$, whereas the canonical mass-distribution of u_1 has positive mass only in $\Delta_1 - \bar{\hat{\delta}}$. As $u_1 > 0$

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[4] R. S. Martin, loc. cit., p. 157.

[5] L. Naïm, loc. cit., p. 203 (théorème 3) and p. 205 (théorème 5).

the canonical mass-distribution of u has positive mass in $\Delta_1 - \bar{\hat{\delta}}$; this is a contradiction. Q.E.D.

COROLLARY. *If $\Delta_0 \neq \phi$ then Δ_0 contains at least countable points.*

Remark. In the above consideration we rely upon the following argument: we have always $u \geq u_1$, and, if $u_1 > 0$, then μ_1 is not the canonical mass-distribution of u.

Mr. K. Matsumoto has kindly pointed out the following result:

Let x_0 be a point of Δ_0 and ν be the canonical mass-distribution of $K_{x_0}(y)$, then the common part of the carrier of ν with Δ_1 is contained in $\bar{\Delta}_0$.

The proof follows from the preceding remark; let E be the carrier of ν. If $E \cap \Delta_1 \not\subset \bar{\Delta}_0$, then there exist a point z_0 and a set A satisfying the following conditions:

1) A is an open neighbourhood of z_0 in Δ,

2) $\nu(A) > 0$,

3) $A \subset \Delta_1$.

We can construct an open set (in $\hat{\Omega}$) $G : G \cap \Delta = A$. In this case, there exist two positive numbers $0 < \rho_1 < \rho$ such that:

i) dist $(z_0, x_0) > \rho$,

ii) $\overline{U_\rho(z_0)} \subset G$[6],

iii) $\nu(U_{\rho_1}(z_0) \cap \Delta) > 0$.

If we set $\omega = \Omega - \overline{U_\rho(z_0)}$, under the same notations as in the proof of the above theorem, we see from i), $x_0 \in \bar{\omega}$ and, as $\overline{(\mathring{\omega} \cap \Omega)} \cap \Delta \subset \overline{U_\rho(z_0)} \cap \Delta \subset G \cap \Delta = A \subset \Delta_1$, μ_1 is canonical and from $\Delta - \bar{\omega} \supset U_{\rho_1}(z_0) \cap \Delta$, $u_1 > 0$, this is a contradiction.

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[6] We denote $U_\rho(z_0) = \{x \in \hat{\Omega};\ \text{dist}\,(x, z_0) < \rho\}$, where the metric dist (x, z_0) is the Martin's metric.