ON A CHARACTERISTIC FEATURE OF THE POSITIVE LOGICS

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In this short note, we would like to point out that the following property (called ASSUMPTION REMOVABILITY in the present paper) is characteristic of the positive logics, the primitive logic **LO**, the positive predicate logics **LP** (intuitionistic) and **LQ** (classical)[1]:

Assumption Removability. *If any proposition $\mathfrak{C}$ can be deduced from some assumption $\mathfrak{A}$ having no primitive notions in common with $\mathfrak{C}$, then $\mathfrak{C}$ is also provable without any assumption.*

This is surely a characteristic property of these positive logics, because the proposition does not hold in the logics **LJ** (intuitionistic predicate logic), **LK** (lower classical predicate logic), **LM** (minimal predicate logic, intuitionistic), and **LN** (minimal predicate logic, classical)[2]. One can realize this easily by the pair of example propositions $\neg(A \rightarrow A)$ and $\neg B$. Although $\neg B$ is surely deducible from $\neg(A \rightarrow A)$ in any one of these logics, and moreover, $\neg(A \rightarrow A)$ has no primitive notions in common with $\neg B$, we can never assert that $\neg B$ is provable without any assumption. On the other hand, the ASSUMPTION REMOVABILITY holds for the positive logics as shown later.

LO and **LP** can be formulated in Gentzen's manner as the sub-logic of Gentzen's **LJ** having logical constants $\rightarrow$ and ( ) only and the sub-logic of **LJ** having logical constants $\rightarrow$, $\wedge$, $\vee$, ( ), and ($\exists$), respectively. Also, **LQ** can be formulated in Gentzen's manner as the sub-logic of Gentzen's **LK** having logical constants $\rightarrow$, $\wedge$, $\vee$, ( ), and ($\exists$)[3]. We can easily see that any sequent provable in any one of these positive logics can be proved in the same logic

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[1] As for **LO**, **LP** and **LQ**, see Ono [6]. See also Curry [2] and Lorenzen [5]. Curry refers to **LP** and **LQ** by **LA** and **LC** in [2], respectively.

[2] As for **LM** and **LN**, see Johansson [4] (Minimalkalkül), Ono [6], and Curry [2]. Curry refers to **LN** by **LE** in [2].

[3] As for Gentzen's **LJ** and **LK**, see Gentzen [3].

by making use of sequents $\Gamma \vdash \Delta$[4] of non-vacant Δ only. We can also see that Gentzen's cut-elimination theorem[5] holds for any one of these logics.

Now we prove ASSUMPTION REMOVABILITY with respect to the positive logics. Namely, let **L** be any one of the positive logics **LO**, **LP**, or **LQ** formulated in Gentzen's manner, and let $\mathfrak{C}$ be a proposition deducible in the logic **L** from the assumption $\mathfrak{A}$ having no primitive notions in common with $\mathfrak{C}$. Then, the sequent $\mathfrak{A} \vdash \mathfrak{C}$ is also provable in **L**, and by virtue of Gentzen's cut-elimination theorem, $\mathfrak{A} \vdash \mathfrak{C}$ can be proved by a proof Π without making use of cuts.

For any sequent $\Gamma \vdash \Delta$ in Π, new sequents $\Gamma_{\mathfrak{A}} \vdash \Delta_{\mathfrak{A}}$ and $\Gamma_{\mathfrak{C}} \vdash \Delta_{\mathfrak{C}}$ are defined by the following:

$\Gamma_{\mathfrak{A}}$ (or $\Gamma_{\mathfrak{C}}$) is the sequence of all the propositions in Γ which have at least one primitive notion in common with $\mathfrak{A}$ (or with $\mathfrak{C}$). $\Delta_{\mathfrak{A}}$ as well as $\Delta_{\mathfrak{C}}$ is defined similarly.

Now, we call any sequent $\Gamma \vdash \Delta$ in Π an $\mathfrak{A}$-*sequent* (or a $\mathfrak{C}$-*sequent*) if and only if $\Gamma_{\mathfrak{A}} \vdash \Delta_{\mathfrak{A}}$ (or $\Gamma_{\mathfrak{C}} \vdash \Delta_{\mathfrak{C}}$) is provable in **L**. Evidently, any fundamental sequent of Π is an $\mathfrak{A}$-sequent or a $\mathfrak{C}$-sequent, and any sequent deduced from an $\mathfrak{A}$-sequent or a pair of $\mathfrak{A}$-sequents (a $\mathfrak{C}$-sequent or a pair of $\mathfrak{C}$-sequents) in Π is an $\mathfrak{A}$-sequent (a $\mathfrak{C}$-sequent).

Moreover, we can show easily that any sequent deduced from a pair of an $\mathfrak{A}$-sequent and a $\mathfrak{C}$-sequent in Π is also an $\mathfrak{A}$-sequent or a $\mathfrak{C}$-sequent[6]. To show this, we have only to check the following three kinds of inferences:

$$\frac{\Gamma \vdash \Delta, \mathfrak{F} \quad \Gamma \vdash \Delta, \mathfrak{G}}{\Gamma \vdash \Delta, \mathfrak{F} \wedge \mathfrak{G}},$$

$$\frac{\Gamma, \mathfrak{F} \vdash \Delta \quad \Gamma, \mathfrak{G} \vdash \Delta}{\Gamma, \mathfrak{F} \vee \mathfrak{G} \vdash \Delta},$$

$$\frac{\Gamma \vdash \Delta, \mathfrak{F} \quad \Gamma, \mathfrak{G} \vdash \Lambda}{\Gamma, \mathfrak{F} \rightarrow \mathfrak{G} \vdash \Delta, \Lambda}.$$

$\mathfrak{F} \wedge \mathfrak{G}$ in the first inference, as well as $\mathfrak{F} \vee \mathfrak{G}$ in the second inference, as well as $\mathfrak{F} \rightarrow \mathfrak{G}$ in the third inference, is either a proposition having no primitive notions in common with $\mathfrak{A}$ or a proposition having no primitive notions in

4) We employ the notation $\Gamma \vdash \Delta$ in place of Gentzen's notation $\Gamma \rightarrow \Delta$, because we use $\rightarrow$ as the logical constant IMPLICATION. In Gentzen [3], IMPLICATION is denoted by $\supset$.

5) The HAUPTSATZ of Gentzen [3].

6) As for the inference schemes for sequents, see Gentzen [3].

common with $\mathfrak{C}$.

First case: $\mathfrak{F} \wedge \mathfrak{G}$ in the first inference (or, $\mathfrak{F} \vee \mathfrak{G}$ in the second inference, or $\mathfrak{F} \rightarrow \mathfrak{G}$ in the third inference) have no primitive notion in common with $\mathfrak{A}$. By the supposition, either $\Gamma \vdash \Delta, \mathfrak{F}$ or $\Gamma \vdash \Delta, \mathfrak{G}$ in the first inference (or, either $\Gamma, \mathfrak{F} \vdash \Delta$ or $\Gamma, \mathfrak{G} \vdash \Delta$ in the second inference; or, either $\Gamma \vdash \Delta, \mathfrak{F}$ or $\Gamma, \mathfrak{G} \vdash \Lambda$ in the third inference) is an $\mathfrak{A}$-sequent. Hence, $\Gamma_{\mathfrak{A}} \vdash \Delta_{\mathfrak{A}}$ in the first inference ($\Gamma_{\mathfrak{A}} \vdash \Delta_{\mathfrak{A}}$ in the second inference, either $\Gamma_{\mathfrak{A}} \vdash \Delta_{\mathfrak{A}}$ or $\Gamma_{\mathfrak{A}} \vdash \Lambda_{\mathfrak{A}}$ in the third inference) must be provable in **L**. Accordingly, $\Gamma \vdash \Delta, \mathfrak{F} \wedge \mathfrak{G}$ in the first inference, as well as $\Gamma, \mathfrak{F} \vee \mathfrak{G} \vdash \Delta$ in the second inference, as well as $\Gamma, \mathfrak{F} \rightarrow \mathfrak{G} \vdash \Delta, \Lambda$ in the third inference (For **LO** and **LP**, Δ must be vacant, so $\Gamma_{\mathfrak{A}} \vdash \Delta_{\mathfrak{A}}$ can not be provable. Hence, $\Gamma_{\mathfrak{A}} \vdash \Lambda_{\mathfrak{A}}$ must be provable by assumption. For **LQ**, $\Gamma_{\mathfrak{A}} \vdash \Delta_{\mathfrak{A}}, \Lambda_{\mathfrak{A}}$ can be deduced from any one of $\Gamma_{\mathfrak{A}} \vdash \Delta_{\mathfrak{A}}$ and $\Gamma_{\mathfrak{A}} \vdash \Lambda_{\mathfrak{A}}$, and moreover, at least one of these sequents must be provable by assumption.) is an $\mathfrak{A}$-sequent.

Second case: $\mathfrak{F} \wedge \mathfrak{G}$ in the first inference (or, $\mathfrak{F} \vee \mathfrak{G}$ in the second inference; or, $\mathfrak{F} \rightarrow \mathfrak{G}$ in the third inference) have no primitive notions in common with $\mathfrak{C}$. Also in this case, we can prove quite similary as in the first case that $\Gamma \vdash \Delta, \mathfrak{F} \wedge \mathfrak{G}$ in the first inference, as well as $\Gamma, \mathfrak{F} \vee \mathfrak{G} \vdash \Delta$ in the second inference, as well as $\Gamma, \mathfrak{F} \rightarrow \mathfrak{G} \vdash \Delta, \Lambda$ in the third inference is a $\mathfrak{C}$-sequent.

Accordingly, we can conclude that $\mathfrak{A} \vdash \mathfrak{C}$ is also an $\mathfrak{A}$-sequent or a $\mathfrak{C}$-sequent. However, $\mathfrak{A} \vdash$ can never be proved in **L** as having been remarked, so $\vdash \mathfrak{C}$ must be provable in **L**.

Remark. According to the interpolation theorem of Craig (for the lower classical predicate logic) and Schütte (for the intuitionistic predicate logic)[7], either $\neg \mathfrak{A}$ or $\mathfrak{C}$ must be provable in any one of these logics as far as $\mathfrak{A} \rightarrow \mathfrak{C}$ is provable in it for any pair of propositions $\mathfrak{A}$ and $\mathfrak{C}$ containing no primitive notions in common. According to our assertion for positive logics, we can say further that $\mathfrak{C}$ must be provable in the corresponding case of any one of positive logics.

References

[1] Craig, W.: Linear reasoning. A new form of the Herbrand-Gentzen theorem, *J. Symb. Log.*, **22** (1957), 250-268.

[2] Curry, H. B.: Foundations of mathematical logic, New York (1963).

[7] See Craig [1] and Schütte [7].

[3] Gentzen, G.: Untersuchungen über das logische Schliessen, *Math. Z.*, **39** (1934), 176–210, 405–431.

[4] Johansson, I.: Der Minimalkalkül, ein reduzierter intuitionistischer Formalismus, *Compositio Math.*, **4** (1936), 119–136.

[5] Lorenzen, P.: Einführung in die operative Logik und Mathematik, Berlin-Göttingen-Heidelberg, (1955).

[6] Ono, K.: On universal character of the primitive logic, *Nagoya Math. J.*, **27-I** (1966), 331–353.

[7] Schütte, K.: Der Interpolationssatz der intuitionistischen Prädikatenlogik, *Math. Ann.*, **148** (1962), 192–200.

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