ON DEVELOPMENT OF FORMAL SYSTEMS STARTING FROM PRIMITIVE LOGIC

KATUZI ONO

Introduction

It has been *my program* to develop fundamental theories of mathematics starting from TABOOS and standing on the *primitive logic* **LO**[1] at first instead of starting from AXIOMS and standing on the fairly brought up logic, the *lower classical logic* **LK**. This was proposed in my work [1].

To put this program into practice, however, we have to state TABOOS in **LO**. The *logical vocabulary* of **LO** is very scanty since it has originally no *logical constants* other than IMPLICATION and UNIVERSAL QUANTIFICATION.

In fact, we can *interpret faithfully* **LK** as well as the *intuitionistic predicate logic* **LJ** in **LO** as has been shown in my work [2], but we can not define logical constants such as CONJUNCTION, DISJUNCTION, and EXISTENTIAL QUANTIFICATION for propositions of every formal system. For most formal systems, I must admit that TABOOS are hard to be stated in terms of the scanty logical vocabulary of **LO** only.

Recently, I was able to show in my work [3] that, in any formal system having *just one primitive notion* and standing on the primitive logic **LO**, CONJUNCTION, DISJUNCTION, and EXISTENTIAL QUANTIFICATION can be so defined in terms of the original logical constants of **LO** that propositions of **LO** behave with respect to the newly defined logical constants together with the original ones just as propositions of the *positive logic* **LP**.[2] We can further define NEGATION so that propositions behave with respect to this and the above mentioned logical constants just as propositions of **LJ**.

In the present paper, I will show that also for any formal system standing on **LO**, we can introduce a *substitute system* having just one primitive notion

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1) The primitive logic **LO** was introduced in my work [1] as the *primiive system of positive logic*. The terminology PRIMITIVE LOGIC together with its reference notation **LO** was introduced in my work [2]. As for the TABOO notion, see my work [1].

2) As for **LP**, see my work [2] and [3]. See also LORENZEN [1] and CURRY [1].

which serves as a substitute for the *whole class of primitive notions* of the original system. Here we assume naturally that the number of primitive notions of any formal system is *finite*. I will illustrate the device *informally* in (**1**) and *formally* in (**2**).

In (**3**), I will explain that this method contributes to development of formal systems under my program, because it makes description of TABOOS in **LO** *easier*. This method enables us also to unify all the TABOOS of a TABOO system into a single TABOO, if the TABOO system can be stated in a finite number of TABOOS. AXIOM systems consisting of a single AXIOM show *no essential superiority* over systems consisting of a finite number of AXIOMS. However, single-TABOO TABOO-systems have an *essential superiority* over other TABOO systems. I will discuss the matter also in the same section.

(1) Informal illustration

As a typical example of formal systems, I will take up a theory of *natural numbers* having the primitive notions $\#$, $\emptyset$, and $\$$ ($\#$ is a predicate and $\#(x)$ means that x is a *natural number*, $\emptyset$ is also a predicate and $\emptyset(x)$ means that x is *zero*, and $\$$ is a binary relation and $x\$y$ means that y is the *successor* of x).

Let us now define a relation $\mathfrak{R}$ by

$$\mathfrak{R}(x, y, u, v) \fallingdotseq . \#(x) \vee \emptyset(y) \vee u\$v.$$

If we adopt $\mathfrak{R}$ as a *primitive notion* instead of the *class of* $\#$, $\emptyset$, *and* $\$$, then these notions can be defined in terms of $\mathfrak{R}$ by

$$\#(x) \fallingdotseq (y)(u)(v)\mathfrak{R}(x, y, u, v),$$
$$\emptyset(y) \fallingdotseq (x)(u)(v)\mathfrak{R}(x, y, u, v),$$
$$u\$v \fallingdotseq (x)(y)\mathfrak{R}(x, y, u, v).$$

Thus the single relation $\mathfrak{R}$ can be regarded as a substitute for the whole class of the original primitive notions $\#$, $\emptyset$, and $\$$.

In **LO**, we can not define the relation $\mathfrak{R}$ in terms of $\#$, $\emptyset$, and $\$$. However, the device above described would suggest a method of development of formal systems in **LO**.

(2) Formal description

Let us assume the case where a formal system having the primitive notions

$P_1, \ldots, P_m$ should be developed in **LO**. Let P_i be an $n(i)$-ary relation for every $i = 1, \ldots, m$. Put

$$s(k) \eqcirc n(1) + \cdots + n(k-1) \text{ for every } k = 1, \ldots, m+1.$$

For this formal system, we take the single $s(m+1)$-ary relation R as a substitute for the whole class of the m primitive notions. Now, I will define the substitutes of the m original primitive notions in terms of R by

$$\begin{aligned}&\mathfrak{P}_i(x_{s(i)+1}, \ldots, x_{s(i+1)}) \\ &\eqcirc (x_1) \cdots (x_{s(i)})(x_{s(i+1)+1}) \cdots (x_{s(m+1)})R(x_1, \ldots, x_{s(m+1)})\end{aligned}$$

for every $i = 1, \ldots, m$.

Corresponding to the original formal system, we can thus introduce another formal system having just one primitive notion R and behaving quite similarly as the original system with respect to the substitutes of the original primitive notions. This corresponding system will be called SUBSTITUTE SYSTEM.

Since the substitute system has just one primitive notion R, we can define by virtue of my work [3], CONJUNCTION, DISJUNCTION, and EXISTENTIAL QUANTIFICATION in such a way that propositions in the substitute system behave with respect to these newly defined logical constants together with the original ones just as propositions in the positive logic **LP**. Hence, for the substitute system, we can start as if standing on **LP** from the beginning.

Considering in the *inverse direction*, we can make for any formal system standing on **LP** its substitute system standing of **LO** by a natural correspondence of elementary formulas to their substitutes of the form $(u) \cdots (v)R(x, \ldots, z)$, logical constants of **LP** correspond *partly* to the original logical constants and *partly* to the newly defined logical constants of **LO**. This gives an embedding of any **LP**-system into an **LO**-system.[3] If we further define NEGATION by

$$\neg \mathfrak{A} \equiv . \mathfrak{A} \rightarrow (x) \cdots (z)R(x, \ldots, z),$$

then we would have an *embedding* of any **LJ**-*system* into an **LO**-*system*. We can prove these facts by making use of the device mentioned above and the result

[3] I call any formal system $\mathfrak{L}$-SYSTEM (***logical system standing on*** $\mathfrak{L}$) if and only if it has a definite class of its primitive notions and standing on the logic $\mathfrak{L}$ but assuming nothing more. As for embedding of a logical system into another, see my work [2].

of my work [3].

Remark. I have occasionally employed $R(p, x, \ldots, z, d, \ldots, d)$ in my work [2] as a substitute for $P(x, \ldots, z)$. If we make substitute system by this method, we are forced to use some variables (such as p in the above example) as denoting object constants. Our device introduced here has a strong point that it does not force to do anything of this kind. In reality, my former proof of *faithful interpretability* of **J**- and **K**-*series logics*[4] in **LO** can be improved by the device of the present paper and the result of my recent work [3].

(3) On description of TABOOS

According to my program proposed in my work [1], we have to describe TABOOS of formal systems in **LO**. The logical vocabulary of **LO** looks like too scanty to do this.

By adopting the device described in (2) for any formal system, however, we can define for its substitute system, CONJUNCTION, DISJUNCTION, and EXISTENTIAL QUANTIFICATION so as to satisfy all the inference rules of **LP**. Hence, for the substitute system, we have only to describe TABOOS in **LP**. As the logical vocabulary of **LP** is much richer than that of **LO**, it looks like much easier to state TABOOS for the substitute system than to state them for the original system in **LO**.

If the TABOO system of the substitute system consists of a finite number of TABOOS $\mathfrak{T}_1, \ldots, \mathfrak{T}_n$, we can further unify them into a single TABOO $\mathfrak{T}_1 \vee \cdots \vee \mathfrak{T}_n$. Although AXIOM systems consisting of a single AXIOM show *no essential superiority* over AXIOM systems consisting of a finite number of AXIOMS, TABOO systems consisting of a single TABOO has an *essential superiority* over other TABOO systems.

If any TABOO system contains more than one TABOOS, we are forced to add the assumption (as has been done in my work [1]) that *members in the TABOO system are mutually equivalent*, when we wish to develop *nicely* a formal theory standing on a brought up logic such as **LJ** or **LK**. Since such assumption can never be called agreeable, we are eager to develop fundamental theories of mathematics without assuming such kind of things. Only if we could have

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[4] The notion of **J**- and **K**-*series logics* is introduced in my work [2].

a single-TABOO TABOO-system for each theory, we would surely be able to get rid of such assumption. It must be an interesting task to find out a fundamental theory of mathematics such as set theory which is developable from a single-TABOO TABOO-system, if any.

The Bernays-Gödel set-thery[5] having an AXIOM system consisting of a finite number of AXIOMS seems to suggest that such pursuit could be successful. Naturally, we can not expect to obtain a *single unified TABOO* $\mathfrak{T}_1 \vee \cdots \vee \mathfrak{T}_n$ trivially by taking the substitutes $\mathfrak{T}_i$ of $\rightarrow \mathfrak{A}_i$ of the finite AXIOM system $\mathfrak{A}_1, \ldots, \mathfrak{A}_n$ of the Bernays-Gödel set-theory, for example, because NEGATION can only be interpreted by TABOOS. Only by assuming that *TABOOS are equivalent to* $(x) \cdots (z) R(x, \ldots, z)$, we can describe TABOOS of the substitute system in **LO** as if we describe them in **LJ**. Accordingly, $(x) \cdots (z) R(x, \ldots, z)$ should be regarded as an additional TABOO, unless the substitute system has a TABOO logically equivalent to it.

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Mathematical Institute,
Nagoya University

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[5] As for Bernays-Gödel AXIOM-system of set theory, see BERNAYS [1] and GÖDEL [1].