ON REGULAR SEQUENCES

J. DIEUDONNÉ

In memory of TADASI NAKAYAMA

1. The concept of *regular sequence* of elements of a ring A (first introduced by Serre under the name of A-sequence [2]), has far-reaching uses in the theory of local rings and in algebraic geometry. It seems, however, that it loses much of its importance when A is not a noetherian ring, and in that case, it probably should be superseded by the concept of *quasi-regular sequence* [1].

One of the convenient properties of a regular sequence $t_1, \ldots, t_n$ in a noetherian local ring A, where the t_i's belong to the maximal ideal of A, is that the sequence remains regular after an arbitrary *permutation* of its terms. This is due to the fact that in such a case, the notions of regular sequence and of quasi-regular sequence coincide [1, 15. 1. 10], and the notion of quasi-regular sequence is independent of the order of the elements of the sequence.

I will give below an example of a non noetherian local ring A and of two elements t_1, t_2 of the maximal ideal of A, such that the sequence (t_1, t_2) is regular, whilst the sequence (t_2, t_1) is not. Such unpleasant phenomena greatly reduce the usefulness of the notion of regular sequence.

2. To construct our example, we start with the ring B of all germs of indefinitely differentiable functions of a real variable x in the neighborhood of 0. It is well known that B is a local ring, whose maximal ideal $\mathfrak{n}$ is generated by the germ i of the identity mapping $x \to x$; the intersection of all powers $\mathfrak{n}^k$ $(k = 1, 2, \ldots)$ is the ideal $\mathfrak{r} \neq 0$ consisting of all germs of functions whose derivatives all vanish at $x = 0$. Observe that the complement of $\mathfrak{r}$ in B consists of regular elements of B (i.e. elements which are not zero-divisors).

Now consider the ring of polynomials $B[T]$ in one indeterminate, and let C be the quotient of $B[T]$ by the ideal $\mathfrak{r}TB[T]$, consisting of all polynomials $r_1T + \cdots + r_mT^m$ having their coefficients in $\mathfrak{r}$. We prove that in C, the

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sequence consisting of the classes s_1, s_2 of i and T respectively, is regular. Indeeed we have $\mathfrak{r}TB[T]\subset iB[T]=\mathfrak{n}B[T]$; furthermore the relation $if\in\mathfrak{r}TB[T]$ for a polynomial $f\in B[T]$ means that if $f[T]=c_0+c_1T+\cdots+c_mT^m$, then $ic_0=0$ and $ic_k\in\mathfrak{r}$ for $k\geqslant 1$; but this implies that $c_0=0$, and $c_k\in\mathfrak{r}$ for $k\geqslant 1$, by definition of B. Thus we see that s_1 is a regular element of C; furthermore we have $C/s_1C=B[T]/iB[T]=\mathbf{R}[T]$, since $B/iB=B/\mathfrak{n}=\mathbf{R}$ (real field); as $\mathbf{R}[T]$ is an integral domain, the image of s_2 in C/s_1C, which is identified with T in $\mathbf{R}[T]$, is a regular element, and this shows that the sequence (s_1, s_2) is regular.

On the other hand, by definition s_2 is a zero-divisor in C, for the images of the elements of $\mathfrak{r}$ (other than 0) in C are $\neq 0$, but the images of their products by T are all 0. *A fortiori* the sequence (s_2, s_1) is not regular.

To form our example, note that the ideal s_1C+s_2C is a maximal ideal $\mathfrak{m}$; one has only to take for A the local ring $C_{\mathfrak{m}}$, for t_1 and t_2 the images in A of s_1 and s_2; it is readily verified that the elements of the complement of $\mathfrak{m}$ in C are not zero-divisors, hence t_2 is a zero-divisor in A; on the other hand, the sequence (t_1, t_2) is regular by flatness.

3. The same construction gives an example of a quasi-regular element of C which is not regular, namely the element s_2 (a regular sequence is quasi-regular, and a subsequence of a quasi-regular sequence is quasi-regular [1, 15. 1. 10 and 15. 1. 9]).

Bibliography

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University of Nice
Nice, France