A LOCALIZATION PRINCIPLE FOR A CLASS OF ANALYTIC FUNCTIONS

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It has been shown by Kiyoshi Noshiro [8; p. 35] that a bounded analytic function $w=f(z)$ in $|z|<1$ having radial limit values of modulus one almost everywhere satisfies a localization principle of the following type. Let (c) be any circular disk: $|w-\alpha|<\rho$ lying inside $|w|<1$ whose periphery may be tangent to the circumference $|w|=1$. Denote by $\varDelta$ any component of the inverse image of (c) under $w=f(z)$ and by $z=z(\xi)$ a function which maps $|\xi|<1$ onto the simply connected domain $\varDelta$ in a one-to-one conformal manner. Then, the function

$$W=F(\xi)=\frac{1}{\rho}[f(z(\xi))-\alpha]$$

is also a bounded analytic function in $|\xi|<1$ with radial limits of modulus one almost everywhere.

Maurice Heins [2; p. 455], [3] has established a localization principle for conformal mappings of type-Bl between Riemann surfaces.

The object of this note is to prove an extension of Noshiro's theorem.

Definition 1.

A function $w=f(z)$ which is bounded and analytic in $|z|<1$ and whose radial limit values $\lim_{r\to 1} f(re^{i\theta})=f^*(e^{i\theta})$ exist and are of modulus one for all points $e^{i\theta}$ on $|z|=1$ except for at most a set S of points $e^{i\theta}$ of linear measure zero will be called of *class* (U) *in* $|z|<1$. If the possible exceptional set S is of logarithmic capacity zero, cap $(S)=0$, $w=f(z)$ will be said to be of *class* (U^*) *in* $|z|<1$.

For the sake of completeness we prove the following lemma.

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LEMMA 1.

If $w=f(z)$ is an analytic function in $|z|<1$, then every component $G(\alpha,\rho,k)$ of the open set $G(\alpha,\rho)=\{z\,|\,|z|<1,\ |f(z)-\alpha|<\rho\}$ is a simply connected domain.

Proof: If $G(\alpha,\rho,k)$ is not simply connected, there exists a point z_0 in the complement of $G(\alpha,\rho,k)$ and a simple closed polygonal path Γ contained in $G(\alpha,\rho,k)$ for which the winding number of Γ with respect to z_0 is one, $n(\Gamma,z_0)=+1$. Now $|f(z)-\alpha|$ has a maximum value on Γ, and $\max_{z\in\Gamma}|f(z)-\alpha|=M<\rho$ since $\Gamma\subset G(\alpha,\rho,k)$. Therefore by the maximum modulus theorem, for all z in the interior of Γ, the relation $|f(z)-\alpha|\leq M<\rho$ is satisfied. This contradicts the assumption that $z_0\notin G(\alpha,\rho,k)$ so that $|f(z_0)-\alpha|\geq\rho$.

Definition 2.

An analytic function $w=f(z)$ defined in $|z|<1$ is said to be of *class* (U^*) *at a point* α if there exists a disk $|w-\alpha|<\rho$ such that for each component $G(\alpha,\rho,k)$ of $G(\alpha,\rho)=\{z\,|\,|z|<1,\ |f(z)-\alpha|<\rho\}$ the function

$$W=F(\xi)=\frac{1}{\rho}[f(z(\xi))-\alpha]$$

is of class (U^*) in $|\xi|<1$ where $z=z(\xi)$ is any one-to-one conformal mapping of $|\xi|<1$ onto the simply connected domain $G(\alpha,\rho,k)$. If $w=f(z)$ is of class (U^*) for every α in a domain G, then $w=f(z)$ is said to be *locally of class* (U^*) *in* G.

The structure of domains of the type of $G(\alpha,\rho,k)$ and of their frontiers $Fr(G(\alpha,\rho,k))$ for various classes of functions has been the object of extensive study by many authors. See for example the work of Lohwater [4], [5], [6], Noshiro [7], [8], Tsuji [11] and the author [10]. We now prove a lemma concerning the structure of $G(\alpha,\rho,k)$ for functions of class (U^*). The proof is a modification of a technique used in [5] and [10].

LEMMA 2.

Let $w=f(z)$ be a non-constant function of class (U^*). Assume that for each $e^{i\beta}$, cap $\{e^{i\theta}|f^*(e^{i\theta})=e^{i\beta}\}=0$. *Then for any α, $|\alpha|<1$ and any ρ, $0<\rho\leq 1-|\alpha|$ the frontier $\Gamma=Fr(G(\alpha,\rho,k))$ is a Jordan curve whose intersection with $|z|=1$ is of logarithmic capacity zero.*

Proof. By Lemma 1, $G(\alpha,\rho,k)$ is a simply connected domain and we now show that Γ is locally connected at each of its points; i.e. every neighborhood U of a point $p \in \Gamma$ contains a neighborhood V of p such that every point of $V \cap \Gamma$ lies in that component of $U \cap \Gamma$ which contains p. As a consequence, each point of Γ is then [12, p. 111] arcwise accessible from $G(\alpha, \rho, k)$. Now, at each point of Γ lying interior to $|z|<1$, Γ is locally connected, since it is part of a piecewise analytic arc, namely a level curve of $\log|f(z)-\alpha|$. Let $E=\Gamma \cap \{|z| =1\}$. If $p \in E$ and if Γ is not locally connected at p, then, by an elementary theorem [12; p. 18], there exists a non-degenerate subcontinuum N of Γ containing p and such that Γ is not locally connected at any point of N. Since N must lie on $|z|=1$, it is clear that N is an arc of $|z|=1$. Furthermore, there must exist [12; p. 18] a circular neighborhood V of p and a sequence of mutually disjoint components $N_1, N_2, \ldots$ of $\bar{V} \cap \Gamma$ converging to a non-degenerate limiting arc $N_0 \subset N$ containing p. Thus if q is any interior point of N_0, every radius of $|z|<1$ drawn to q must cross infinitely many of the components N_j arbitrarily close to q. Along such a radius of $|z|<1$, if $f(re^{i\theta})$ tends to a limit of modulus one, this limit must be $\frac{\alpha}{|\alpha|}$, since $|f(z)-\alpha|=\rho \leq 1-|\alpha|$ at all points of N_j. Since this occurs at every interior point of N_0 with at most the exception of a set of logarithmic capacity zero, we violate the hypothesis that for every $e^{i\beta}$, cap $\{e^{i\theta} | f^*(e^{i\theta})=e^{i\beta}\}=0$. Therefore $\Gamma=Fr(G(\alpha\ \rho, k))$ must be locally connected.

We show next that the set $E=\Gamma \cap \{|z|=1\}$ is of logarithmic capacity zero. Let M be the set of points on $|z|=1$ for which the radial limit values are of modulus one. We let $M=\{|z|=1\}-\tilde{M}$ and observe that cap $(\tilde{M})=0$. Because of the decomposition $E=(E \cap M) \cup (E \cap \tilde{M})$ it will suffice to prove that $E \cap M$ is of logarithmic capacity zero.

We divide M into two sets M_1 and M_2 in the following manner. Let $M_1 = \{e^{i\theta} | f^*(e^{i\theta}) = \frac{\alpha}{|\alpha|}\}$ and $M_2 = M - M_1$. The set M_1 is by hypothesis of logarithmic capacity zero and at each point $e^{i\theta} \in M_2 \cap E$, the radial cluster set $C_\rho(f, e^{i\theta})$ is a single point $f^*(e^{i\theta}) \neq \frac{\alpha}{|\alpha|}$. Each point $e^{i\theta} \in M_2$ is arcwise accessible from $G(\alpha, \rho, k)$ and the curvilinear cluster set $C_{\lambda\theta}(f, e^{i\theta})$ along any path $\lambda\theta$ lying in $G(\alpha, \rho, k)$ and terminating at $e^{i\theta}$ is contained in $\{|w-\alpha| \leq \rho\}$. The intersection $C_\rho(f, e^{i\theta}) \cap C_{\lambda\theta}(f, e^{i\theta}) = \phi$ for every point $e^{i\theta} \in E \cap M_2$. By a result of Bagemihl [1], $E \cap M_2$ is at most a denumerable set and so $E \cap M$ is of logarithmic capacity zero.

Finally, to prove that $\Gamma = Fr(G(\alpha, \rho, k))$ is a Jordan curve, we must show that the complement of Γ consists of two components G_1 and G_2 and that every point of Γ is arcwise accessible from each of G_1 and G_2. Since we may identify $G(\alpha, \rho, k)$ with G_1, it is sufficient to show that the complement G of the closure of $G(\alpha, \rho, k)$ is connected and that each point of Γ is arcwise accessible from G. Now E is a closed set of logarithmic capacity zero, so that between any two points of E exists at least one arc of $|z|=1$ belonging to G. Thus if there exists a point of G interior to $|z|<1$ which cannot be joined to a point of $|z|>1$ by an arc lying in G, there must exist a simple closed curve γ which lies, except for one point q of E, entirely inside $G(\alpha, \rho, k)$ and which encloses points of G.

Since $f(z)$ is a bounded analytic function in the domain Ω bounded by γ and since except for the one point $q \in \gamma$, $\limsup_{z \to \xi} |f(z) - \alpha| < \rho$ for all points $\xi \in \gamma = Fr(\Omega)$, we see by the extended maximum principle [8; p. 14] that $|f(z) - \alpha| < \rho$ for all points in Ω which contradicts the statement that there exists a point of G interior to $|z|<1$ which cannot be joined to a point of $|z|>1$ by an arc lying in G. Hence all points of G which lie in $|z|<1$ can be joined by some arc of G to $|z|>1$, so that the complement of Γ consists of two components G_1 and G_2. The accessibility of each point of Γ from G_2 is trivial however since E lies on $|z|=1$ and that part of Γ inside $|z|<1$ consists of smooth level curves. Hence Lemma 2 is proved.

THEOREM.

Let $w = f(z)$ be a non-constant function of class (U^*) and let (c) be any circular disk $|w - a| < \rho$ lying inside $|w|<1$ whose periphery may be tangent to the circumference $|w|=1$. Denote by $G(\alpha, \rho, k)$ any component of the open set $G(\alpha, \rho) = \{z \mid |f(z) - a| < \rho\}$. Let $z = z(\xi)$ be a function which maps $|\xi|<1$ in a one-to-one conformal way onto the simply connected domain $G(\alpha,\rho,k)$. Then, if for every $e^{i\beta}$ on $|w|=1$, cap $\{e^{i\theta} \mid f^*(e^{i\theta}) = e^{i\beta}\} = 0$, the function

$$W = F(\xi) = \frac{1}{\rho}[f(z(\xi)) - \alpha]$$

is also of class (U^*) and for every $e^{i\gamma}$ on $|W|=1$, cap $\{e^{i\theta} \mid F^*(e^{i\theta}) = e^{i\gamma}\} = 0$.

Proof. If the closure of $G(\alpha, \rho, k)$ lies in D: $|z|<1$, the theorem is clearly valid since by a well-known theorem of Carathéodory on the conformal mapping of Jordan domains $W=F(\xi)$ is continuous on the closed disk $|\xi|\leqq 1$.

We shall now consider the case where $G(\alpha, \rho, k)$ has at least one boundary point on $|z|=1$. We define $E(\alpha, \rho, k)=Fr(G(\alpha, \rho, k)\cap\{|z|=1\})$ and observe that Lemma 2 states that cap $(E(\alpha, \rho, k))=0$.

The functions $z=z(\xi)$ and $W=F(\xi)$ are bounded analytic functions in $|\xi|<1$. Let us denote by E_ξ the set of points $e^{i\theta}$ on $|\xi|=1$ for which the radial limit $z(e^{i\theta})$ exists and the radial limit $\lim_{r\to 1}F(re^{i\theta})$ either fails to exist or if it does exist is of modulus less than one. Let E_z denote the image of E_ξ under $z=z(\xi)$ i.e. $E_z=\{z(e^{i\theta})|e^{i\theta}\in E_\xi\}$. The set E_z lies on Γ: $|z|=1$. Because $E_z\subset E(\alpha, \rho, k)$ we can conclude that cap $E_z=0$. We shall now prove that the logarithmic capacity of E_ξ is zero.

Let Γ_ξ denote the circle $|\xi|=\frac{1}{4}$ and Γ_z its image under $z=z(\xi)$. Since cap $(E_z)=0$, by Pfluger's theorem [9; p. 122] it follows that the extremal length of the totality of paths $\hat{J}$ joining E_z to Γ_z is infinite, $\lambda(\hat{J})=\infty$. Now as we observed above, the frontier of $G(\alpha, \rho, k)$ is locally connected and if we consider the subfamily J_z of paths joining E_z to Γ_z and lying in $G(\alpha, \rho, k)$ then, since $J_z\subset\hat{J}$, it follows that $\lambda(J_z)\geq\lambda(\hat{J})=\infty$. Because the extremal length is a conformal invariant, we obtain $\lambda(J_\xi)=\infty$ where J_ξ is the family of preimages of J_z in $|\xi|<1$ under the transformation $z=z(\xi)$. It now follows from another application of Pfluger's theorem cited above that cap $(E_\xi)=0$. Since E_ξ the set of points $e^{i\theta}$ on $|\xi|=1$ such that the radial limits $\lim_{r\to 1}F(re^{i\theta})$ either fails to exist or is of modulus less than one, is of logarithmic capacity zero we conclude that $W=F(\xi)\in(U^*)$ and hence $w=f(z)$ is locally of class (U^*) in $|z|<1$.

We now prove that for every $e^{i\tau}$ on $|W|=1$, if $S_\tau\{e^{i\theta}|F^*(e^{i\theta})=e^{i\tau}\}$ then cap $(S_\tau)=0$. Let Z_τ denote the image of S_τ under $z=z(\xi)$. The set Z_τ must lie on $|z|=1$ except for at most a denumerable subset in $|z|<1$. Because for every $e^{i\theta}\in Z_\tau$ the radial limit $f^*(e^{i\theta})$ exists and equals $\rho e^{i\tau}+\alpha$, we observe that cap $(Z_\tau)=0$ and thus by the previous argument cap $(S_\tau)=0$ and the proof of the theorem is complete.

For conformal mappings of Riemann surfaces, in addition to proving that one may localize the notion type-*Bl*, Maurice Heins also proved that maps which are locally of type-*Bl* and have as range a Riemann surface with positive ideal boundary are also globally of type-*Bl*. This fact leads one to conjecture that if $w=f(z)$ is locally of class (U^*) in $|w|<1$, then it is of class (U^*).

The following example, which Kiyoshi Noshiro has kindly shown to the author answers the conjecture in the negative. The example is based on a result of P. J. Myrberg (see also Noshiro [8, p. 26]). Consider a domain Φ obtained by excluding two points α_1, α_2 from the disk $|w|<1$. Let $\tilde{\Phi}$ be the universal covering surface of Φ. Let $w=f(z)$ be a function which maps the unit disk $|z|<1$ conformally onto $\tilde{\Phi}$ in a one-to-one manner. Then, the perfect set E, on $|z|=1$, of essential singularities of $w=f(z)$ must be of linear measure zero but the capacity of E must be positive. The radial cluster set $C_\rho(f, e^{i\theta})$ does not lie on the circumference $|w|=1$ for every $e^{i\theta}\in E$. Therefore, E is considered as the exceptional set in the definition of class (U). The function $w=f(z)$ is locally of class (U^*) in $|w|<1$ because $\tilde{\Phi}$ has only logarithmic singularities at $w=\alpha_1$ and $w=\alpha_2$.

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