A CRITERION FOR A SET AND ITS IMAGE UNDER QUASICONFORMAL MAPPING TO BE OF α $(0<\alpha\leqq 2)$-DIMENSIONAL MEASURE ZERO

KAZUO IKOMA

Let $w=w(z)$ be any K-quasiconformal mapping (in the sense of Pfluger-Ahlfors) of a domain D in the z-plane into the w-plane. Since $w=w(z)$ is a measurable mapping (vid. Bers [1]), it transforms any set of Hausdorff's 2-dimensional measure zero in D into such another one. However, A. Mori [5] showed that for $0<\alpha\leqq 2$, any set of $\frac{\alpha}{K}$-dimensional measure zero in a Jordan domain D is transformed by $w=w(z)$ into a set of α-dimensional measure zero. Further, Beurling and Ahlfors [2] proved that even the set of 1-dimensional measure zero on a segment S in D is not always transformed into such another one under $w=w(z)$ transforming S into another segment.

In this paper motivated by the above results, by extending our argument in the previous paper [3] where the following lemma due to Teichmüller is very useful, we shall give a criterion for both some closed set E in a Jordan domain D and its image set by any K-quasiconformal mapping $w=w(z)$ of D to be of α-dimensional measure zero, where $0<\alpha\leqq 2$.

Lemma (Teichmüller [6]). If one of the complementary continua of a doubly connected domain R contains $z=0$ and $z=re^{i\theta}$ and the other contains $z=\infty$ and $z=\rho e^{i\varphi}$, then it holds

$$\operatorname{mod} R\leqq \log \Psi\left(\frac{\rho}{r}\right),$$

where $\log\Psi(P)$ means the modulus of Teichmüller's extremal domain.

1. Let E be a compact set in the complex plane and let its complementary set be a connected domain G.

A set $\{R_n^{(j)}\}$ $(j=1,2,\ldots,\nu(n)<\infty;\ n=1,2,\ldots)$ of doubly connected domains $R_n^{(j)}$ will be simply referred a system inducing an exhaustion of G if it satisfies the following conditions:

Received May 22, 1962.

(i) the closure $\overline{R_n^{(j)}}$ of $R_n^{(j)}$ is contained in G,

(ii) the boundary of $R_n^{(j)}$ consists of the interior contour $C_n^{(j)}$ and the exterior $\Gamma_n^{(j)}$, which are rectifiable closed Jordan curves,

(iii) the complementary set of $\overline{R_n^{(j)}}$ consists of two domains, the one $F_n^{(j)}$ of which contains the point at infinity and the other $H_n^{(j)}$ has at least one point common with E,

(iv) any point of E is contained in a certain $H_n^{(j)}$,

(v) $R_n^{(k)}$ lies in $F_n^{(j)}$ if $k \neq j$,

(vi) each $R_{n+1}^{(k)}$ is contained in a certain $H_n^{(j)}$, and

(vii) $\{G_n\}_{n=1}^{\infty}$ is an exhaustion of G, where $G_n = \bigcap_{j=1}^{\nu(n)} (F_n^{(j)} \cup R_n^{(j)})$.

Put mod $R_n^{(j)} = \log \mu_n^{(j)}$ and $\min_{1 \leqq j \leqq \nu(n)} \log \mu_n^{(j)} = \log \mu_n$, then we can establish the following

THEOREM 1. *Let E, G and* $\log \Psi(P)$ *be defined as above and $w = w(z)$ be any K-quasiconformal mapping of a Jordan domain D containing E. If there exists a system $\{R_n^{(j)}\}$ inducing an exhaustion of G which satisfies*

$$\limsup_{n \to \infty} \left\{ \alpha \sum_{l=1}^{n} \log \Psi^{-1}(\mu_l^{1/K}) - \left(1 - \frac{\alpha}{2}\right) \log \nu(n) \right\} = +\infty$$

for some α such that $0 < \alpha \leqq 2$, then not only E but its image set by $w = w(z)$ is of α-dimensional measure zero, where $\Psi^{-1}(Q)$ is the inverse function of $Q = \Psi(P)$.

Proof. First, take a point $z_n^{(j)}$ inside $C_n^{(j)}$ and put

$$r_n^{(j)} = \max_{z \in C_n^{(j)}} |z - z_n^{(j)}|, \ \rho_n^{(j)} = \min_{z \in \Gamma_n^{(j)}} |z - z_n^{(j)}|.$$

Since there exists a number N_D such that the sub-system of $\{R_n^{(j)}\}$ for $n \geqq N_D$ is contained in D, for all such n we have $\widetilde{R_n^{(j)}}$, $\widetilde{C_n^{(j)}}$ and $\widetilde{\Gamma_n^{(j)}}$ denoting images of $R_n^{(j)}$, $C_n^{(j)}$ and $\Gamma_n^{(j)}$ under $w = w(z)$ and we can define $\widetilde{\mu_n^{(j)}}$, $\widetilde{r_n^{(j)}}$ and $\widetilde{\rho_n^{(j)}}$ similarly just as $\mu_n^{(j)}$, $r_n^{(j)}$ and $\rho_n^{(j)}$.

Then, from a fundamental property of a K-quasiconformal mapping and the above Teichmüller's lemma, it follows for $n \geqq N_D$

$$\frac{1}{K} \log \mu_n \leqq \log \widetilde{\mu_n^{(j)}} \leqq \log \Psi(\widetilde{\rho_n^{(j)}} / \widetilde{r_n^{(j)}}),$$

so that

$$\widetilde{r_n^{(j)}} \leqq \frac{1}{\Psi^{-1}(\mu_n^{1/K})} \widetilde{\rho_n^{(j)}}. \tag{1.1}$$

Starting from this and applying Hölder's inequality, we have for $n \geqq N_D$ and $0 < \alpha \leqq 2$

$$\begin{aligned} \sum_{j=1}^{\nu(n)} (\widetilde{r_n^{(j)}})^\alpha &\leqq \frac{1}{\{\Psi^{-1}(\mu_n^{1/K})\}^\alpha} \sum_{j=1}^{\nu(n)} (\widetilde{\rho_n^{(j)}})^\alpha \\ &\leqq \frac{\{\nu(n)\}^{1-(\alpha/2)}}{\{\Psi^{-1}(\mu_n^{1/K})\}^\alpha} \{\sum_{j=1}^{\nu(n)} (\widetilde{\rho_n^{(j)}})^2\}^{\alpha/2}. \end{aligned} \tag{1.2}$$

Now, it is obvious that $\pi \sum_{j=1}^{\nu(n)} (\widetilde{\rho_n^{(j)}})^2$ is not greater than the sum of areas bounded by $\bigcup_{j=1}^{\nu(n)} \widetilde{\Gamma_n^{(j)}}$ and that $\pi \sum_{k=1}^{\nu(n-1)} (\widetilde{r_{n-1}^{(k)}})^2$ is not less than the sum of areas bounded by $\bigcup_{k=1}^{\nu(n-1)} \widetilde{C_{n-1}^{(k)}}$, and so that

$$\sum_{j=1}^{\nu(n)} (\widetilde{\rho_n^{(j)}})^2 \leqq \sum_{k=1}^{\nu(n-1)} (\widetilde{r_{n-1}^{(k)}})^2.$$

Further, from (1.1), it holds for $n-1 \geqq N_D$

$$\sum_{k=1}^{\nu(n-1)} (\widetilde{r_{n-1}^{(k)}})^2 \leqq \frac{1}{\{\Psi^{-1}(\mu_{n-1}^{1/K})\}^2} \sum_{k=1}^{\nu(n-1)} (\widetilde{\rho_{n-1}^{(k)}})^2.$$

Substituting these in (1.2), we have for $n-1 \geqq N_D$

$$\sum_{j=1}^{\nu(n)} (\widetilde{r_n^{(j)}})^\alpha \leqq \frac{\{\nu(n)\}^{1-(\alpha/2)}}{\{\Psi^{-1}(\mu_n^{1/K})\}^\alpha \{\Psi^{-1}(\mu_{n-1}^{1/K})\}^\alpha} \{\sum_{k=1}^{\nu(n-1)} (\widetilde{\rho_{n-1}^{(k)}})^2\}^{\alpha/2}. \tag{1.3}$$

This process can be continued up to $R_{N_D}^{(m)}$ and finally we obtain

$$\sum_{j=1}^{\nu(n)} (\widetilde{r_n^{(j)}})^\alpha \leqq \frac{\{r(n)\}^{1-(\alpha/2)}}{\prod_{l=N_D}^{n} \{\Psi^{-1}(\mu_l^{1/K})\}^\alpha} \{\sum_{m=1}^{\nu(N_D)} (\widetilde{\rho_{N_D}^{(m)}})^2\}^{\alpha/2} \tag{1.4}$$

From our assumption,

$$\limsup_{n\to\infty} \frac{\prod_{l=1}^{n} \{\Psi^{-1}(\mu_l^{1/K})\}^\alpha}{\{\nu(n)\}^{1-(\alpha/2)}} = +\infty,$$

and hence it follows that

$$\liminf_{n\to\infty} \sum_{j=1}^{\nu(n)} (\widetilde{r_n^{(j)}})^\alpha = 0,$$

which shows that α-dimensional measure of E is equal to zero.

Corresponding to (1.1), evidently it holds that

$$r_n^{(j)} \leqq \frac{1}{\Psi^{-1}(\mu_n)} \rho_n^{(j)}.$$

Since $\Psi^{-1}(\mu_n) \geqq \Psi^{-1}(\mu_n^{1/K})$ because of $\mu_n > 1$, we have

$$(1.1') \qquad r_n^{(j)} \leqq \frac{1}{\Psi^{-1}(\mu_n^{1/K})} \rho_n^{(j)}.$$

Starting from this instead of (1.1) and proceeding similarly just as stated above, we arrive at the corresponding relation (1.4′) to (1.4):

$$(1.4') \qquad \sum_{j=1}^{\nu(n)} (r_n^{(j)})^\alpha \leqq \frac{\{\nu(n)\}^{1-(\alpha/2)}}{\prod_{l=1}^{n} \{\Psi^{-1}(\mu_l^{1/K})\}^\alpha} \{\sum_{m=1}^{\nu(1)} (\rho_1^{(m)})^2\}^{\alpha/2},$$

from which our assertion is completed.

Corollary 1. *Let E, $\{R_n^{(j)}\}$, $\Psi(P)$ and $w = w(z)$ be same ones as in Theorem 1. If there exist a positive number δ and a system $\{R_n^{(j)}\}$ which satisfy*

$$\liminf_{n\to\infty} \mu_n > \{\Psi(1+\delta)\}^K \quad and \quad \liminf_{n\to\infty} \frac{\{\nu(n)\}^{2-\alpha}}{(1+\delta)^{n\alpha}} = 0$$

for some α such that $0<\alpha\leqq 2$, then not only E but its image set by $w = w(z)$ is of α-dimensional measure zero.

2. Next, we consider the particular case where the set E lies on a segment S and $w = w(z)$ transforms S into a segment. Obviously the above relations (1.1) and (1.1′) hold also in such a case.

Starting from (1.1) and applying Hölder's inequality, we have for $n \geqq N_D$ and $0<\alpha\leqq 1$

$$(2.2) \qquad \sum_{j=1}^{\nu(n)} (\widetilde{r_n^{(j)}})^\alpha \leqq \frac{\{\nu(n)\}^{1-\alpha}}{\{\Psi^{-1}(\mu_n^{1/K})\}^\alpha} \{\sum_{j=1}^{\nu(n)} \widetilde{\rho_n^{(j)}}\}^\alpha.$$

Now, take a point $w_n^{(j)}$ on a part, included in $\widetilde{C_n^{(j)}}$, of the image segment of S by $w = w(z)$, and define $\widetilde{r_n^{(j)}}$, $\widetilde{\rho_n^{(j)}}$ as before, then it is evident that $2\sum_{j=1}^{\nu(n)} \widetilde{\rho_n^{(j)}}$ is not greater than the sum of lengths of parts, included in $\bigcup_{j=1}^{\nu(n)} \widetilde{\Gamma_n^{(j)}}$, of S or its stretching line and that $2\sum_{k=1}^{\nu(n-1)} \widetilde{r_{n-1}^{(k)}}$ is not less than the sum of lengths of parts, included in $\bigcup_{k=1}^{\nu(n-1)} \widetilde{C_{n-1}^{(k)}}$, of S or its stretching line, and so it holds

$$\{\sum_{j=1}^{\nu(n)} \widetilde{\rho_n^{(j)}}\}^\alpha \leqq \{\sum_{k=1}^{\nu(n-1)} \widetilde{r_{n-1}^{(k)}}\}^\alpha.$$

Further, from (1.1) follows

$$\{\sum_{k=1}^{\nu(n-1)} \widetilde{r_{n-1}^{(k)}}\}^\alpha \leqq \frac{1}{\{\Psi^{-1}(\mu_{n-1}^{1/K})\}^\alpha}\{\sum_{k=1}^{\nu(n-1)} \widetilde{\rho_{n-1}^{(k)}}\}^\alpha.$$

Substitute these in (2.2), then we have for $n-1 \geqq N_D$

$$\sum_{j=1}^{\nu(n)} (\widetilde{r_n^{(j)}})^\alpha \leqq \frac{\{\nu(n)\}^{1-\alpha}}{\{\Psi^{-1}(\mu_n^{1/K})\}^\alpha \{\Psi^{-1}(\mu_{n-1}^{1/K})\}^\alpha}\{\sum_{k=1}^{\nu(n-1)} \widetilde{\rho_{n-1}^{(k)}}\}^\alpha. \tag{2.3}$$

This procedure can be continued up to $R_{N_D}^{(m)}$, and finally we obtain

$$\sum_{j=1}^{\nu(n)} (\widetilde{r_n^{(j)}})^\alpha \leqq \frac{\{\nu(n)\}^{1-\alpha}}{\prod_{l=N_D}^{n} \{\Psi^{-1}(\mu_l^{1/K})\}^\alpha}\{\sum_{m=1}^{\nu(N_D)} \widetilde{\rho_{N_D}^{(m)}}\}^\alpha. \tag{2.4}$$

Starting from (1.1′) instead of (1.1) and proceeding as stated above, we arrive at the relation resulting except the wave mark $\sim$ from both sides of (2.4). Hence we have

THEOREM 2. *Let E be a closed set on a segment S in the z-plane and $w = w(z)$ be any K-quasiconformal mapping, transforming S into a segment, of a Jordan domain D containing S. If there exists a system $\{R_n^{(j)}\}$ inducing an exhaustion of the complementary domain of E which satisfies*

$$\limsup_{n\to\infty} \{\alpha \sum_{l=1}^{n} \log \Psi^{-1}(\mu_l^{1/K}) - (1-\alpha)\log \nu(n)\} = +\infty$$

for some α such that $0<\alpha\leqq 1$, then not only E but its image set by $w=w(z)$ is of α-dimensional measure zero.

COROLLARY 2. *Let E, $\{R_n^{(j)}\}$ and $w=w(z)$ be same ones as in Theorem 2. If there exist a positive number δ and a system $\{R_n^{(j)}\}$ which satisfy*

$$\liminf_{n\to\infty} \mu_n > \{\Psi(1+\delta)\}^K \quad \textit{and} \quad \liminf_{n\to\infty} \frac{\{\nu(n)\}^{1-\alpha}}{(1+\delta)^{n\alpha}} = 0$$

for some α such that $0<\alpha\leqq 1$, then not only E but its image set by $w=w(z)$ is of α-dimensional measure zero.

Considering that $\Psi(P)$ is a strictly increasing and continuous function of P and $\log \Psi(1) = \pi$, we have

COROLLARY 3. *Let E, $\{R_n^{(j)}\}$ and $w = w(z)$ be same ones as in Theorem 2.*

If there exists a system $\{R_n^{(j)}\}$ *satisfying*

$$\liminf_{n\to\infty} \mu_n > e^{K\pi},$$

then not only E but its image set of E by $w = w(z)$ *is of* 1-*dimensional measure zero.*

3. Finally, we shall give examples of two sets, of positive logarithmic capacity, to each of which a system satisfying the condition at Theorem 1 or 2 corresponds.

Take a closed segment S with length l_1 on the real axis and delete from S_1 an open segment T_1 with length $\frac{l_1}{p_1}$ $(p_1 > 1)$ such that the set $S_2 = S_1 - T_1$ consists of two closed segments $S_2^{(j)}$ $(j = 1, 2)$ with equal length l_2. In general, we delete from the set S_{m-1} open segments $T_{m-1}^{(j)}$ $(j = 1, 2, \ldots, 2^{m-2})$ such that each $T_{m-1}^{(j)}$ has length $\frac{l_{m-1}}{p_{m-1}}$ $(p_{m-1} > 1)$ and the set $S_m = S_{m-1} - \bigcup_{j=1}^{2^{m-2}} T_{m-1}^{(j)}$ consists of closed segments $S_m^{(j)}$ $(j = 1, 2, \ldots, 2^{m-1})$ with equal length l_m. Then $\bigcap_{m=1}^{\infty} S_m$ is a non-empty perfect closed set which is called the ordinary Cantor set and is denoted by $E(p_1, p_2, \ldots, p_n, \ldots)$.

If we take $p_n = 3\{\Psi(2^{1/\alpha})\}^K / [3\{\Psi(2^{1/\alpha})\}^K - 1]$ $(n = 1, 2, \ldots)$, then we can construct, as was showed in Kuroda [4], a system $\{R_n^{(j)}\}$ $(j = 1, 2, \ldots, 2^n;$ $n = 1, 2, \ldots)$ inducing an exhaustion of the complementary domain of $E(p_1, p_2, \ldots, p_n, \ldots)$, where $R_n^{(j)}$ is bounded by concentric circles $C_n^{(j)}$, $\Gamma_n^{(j)}$ having the center at the middle point of $S_{n+1}^{(j)}$ and having respectively the radius $r_n = \frac{l_n}{4}\left(1 + \frac{1}{p_{n+1}}\right)\left(1 - \frac{1}{p_n}\right)$, $\rho_n = \frac{l_n}{4}\left(1 + \frac{1}{p_n}\right)$. Then, we can see easily $\mu_n \geqq \frac{1}{2\left(1 - \frac{1}{p_n}\right)}$, and hence, taking $\delta = 2^{1/\alpha} - 1$ $(0 < \alpha \leqq 1)$ we have

$$\begin{aligned}\liminf_{n\to\infty} \mu_n &\geqq 2^{-1}\left\{1 - \frac{1}{\limsup_{n\to\infty} p_n}\right\}^{-1} \\ &= \frac{3}{2}\{\Psi(2^{1/\alpha})\}^K > \{\Psi(1+\delta)\}^K\end{aligned}$$

and

$$\liminf_{n\to\infty} \frac{\{\nu(n)\}^{1-\alpha}}{(1+\delta)^{n\alpha}} = \liminf_{n\to\infty} \frac{(2^n)^{1-\alpha}}{(2^{1/\alpha})^{n\alpha}} = \lim_{n\to\infty} \frac{1}{2^{n\alpha}} = 0,$$

which shows that the condition in Corollary 2 or Theorem 2 is satisfied Furthermore, it is seen that

$$\sum_{n=1}^{\infty} \frac{\log \dfrac{1}{1-\dfrac{1}{p_n}}}{2^n} = \sum_{n=1}^{\infty} \frac{\log 3\{\Psi(2^{1/\alpha})\}^K}{2^n}$$
$$= \{\log 3 + K \log \Psi(2^{1/\alpha})\} \sum_{n=1}^{\infty} \frac{1}{2^n} < \infty,$$

which implies that $E(p_1, p_2, \ldots, p_n, \ldots)$ is of positive logarithmic capacity.

On the other hand, take the ordinary Cantor set $E(p_1, p_2, \ldots, p_n, \ldots)$, where $p_n = 3\{\Psi(16^{1/\alpha})\}^K/[3\{\Psi(16^{1/\alpha})\}^K - 1]$ $(n = 1, 2, \ldots)$, and consider the Cartesian product $E(p_1, p_2, \ldots, p_n, \ldots) \times E(p_1, p_2, \ldots, p_n, \ldots)$ which is referred the symmetric Cantor set. Then, we can construct a system $\{R_n^{(jk)}\}$ $(j = 1, 2, \ldots, 2^n; k = 1, 2, \ldots, 2^n;\ n = 1, 2, \ldots)$ inducing an exhaustion of the complementary domain of $E(p_1, p_2, \ldots, p_n, \ldots) \times E(p_1, p_2, \ldots, p_n, \ldots)$, where $R_n^{(jk)}$ denotes the circular annulus translating $R_n^{(j)}$ and having its center at the center of the square $S_{n+1}^{(j)} \times S_{n+1}^{(k)}$. Since moduli of $R_n^{(jk)}$ for all j, k $(j, k = 1, 2, \ldots, 2^n)$ are equal one another, we can put $\operatorname{mod} R_n^{(jk)} = \log \mu_n$. Then it is easily seen similarly as showed above that for $\delta = 16^{1/\alpha} - 1$, this system $\{R_n^{(jk)}\}$ satisfies the condition in Corollary 1 or Theorem 1 and the set $E(p_1, p_2, \ldots, p_n, \ldots) \times E(p_1, p_2, \ldots, p_n, \ldots)$ has the positive logarithmic capacity.

References

[1] Bers, L.: On a theorem of Mori and the definition of quasiconformality, Trans. Amer. Math. Soc., **84**, 78–84 (1957).

[2] Beurling, A. and Ahlfors, L.: The boundary correspondence under quasiconformal mapping, Acta Math., **96**, 125–142 (1956).

[3] Ikoma, K.: On a property of the boundary correspondence under quasiconformal mappings, Nagoya Math. Journ., **16**, 185–188 (1960).

[4] Kuroda, T.: A criterion for a set to be of 1-dimensional measure zero, Jap. Journ. Math., **29**, 48–51 (1959).

[5] Mori, A.: On quasi-conformality and pseudo-analyticity, Trans. Amer. Math. Soc., **84**, 56–77, (1957).

[6] Teichmüller, O.: Untersuchungen über konforme und quasikonforme Abbildung, Deutsche Math., **3**, 621–678 (1938).

Yamagata University