SOME NOTES ON EXCEPTIONAL VALUES OF MEROMORPHIC FUNCTIONS

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1. Let E be a totally-disconnected compact set in the z-plane and let Ω be its complement with respect to the extended z-plane. Then Ω is a domain and we can consider a single-valued meromorphic function $w = f(z)$ on Ω which has a transcendental singularity at each point of E. Suppose that E is a null-set of the class W in the sense of Kametani [4] (= the class $N_{\mathfrak{B}}$ in the sense of Ahlfors and Beurling [1]). Then the cluster set of $f(z)$ at each transcendental singularity is the whole w-plane, and hence $f(z)$ has an essential singularity at each point of E. We shall say that a value w is exceptional for $f(z)$ at an essential singularity $\zeta \in E$ if there exists a neighborhood of ζ where the function $f(z)$ does not take this value w.

In our previous paper [6], we showed that, even if E is of capacity[1] zero, the set of all exceptional values of $f(z)$ at a point ζ of E may be non-countable. From this fact, there arises the following problem: Is there a perfect set E such that each $f(z)$ has at most a countable number of exceptional values at each essential singularity $\zeta \in E$?

Recently, Carleson [3] and the author [7] have given positive answers to this problem, that is, they have given sufficient conditions for sets E to satisty that every $f(z)$ has at most a finite number of exceptional values. Particularly, Carleson's paper is very interesting. Carleson has shown that there exist sets E of positive capacity for which every $f(z)$ has at most three exceptional values. His arguments are based on the fact that a set of linear measure zero is a null-set of the class W.

In this note, we shall give a sufficient condition, much better than Carleson's and the author's, using Carleson's arguments essentially.

2. Let $\{\Omega_n\}_{n=0,1,2,\ldots}$ be an exhaustion of Ω with the following conditions:

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[1] In this note, capacity is always logarithmic.

1) $\Omega_{n+1} \supset \bar{\Omega}_n$ for every n,

2) for each n, the boundary $\partial\Omega_n$ of Ω_n consists of a finite number of closed analytic curves,

3) each component of the open set $\mathscr{C}\bar{\Omega}_n$[2] contains points of E,

4) the open set $\Omega_n - \bar{\Omega}_{n-1}$ $(n \geqq 1)$ consists of a finite number of doubly-connected domains $R_{n,k}$ $(k = 1, 2, \ldots, N(n))$.

We shall use in the sequel the graph associated with $\{\Omega_n\}$ in the sense of Noshiro [9]. Let $u(z) + iv(z)$ be the mapping function of $\Omega - \Omega_0$ onto it and let R be its length.[3]

Let γ_r be the niveau curve $u(z) = r$ $(0 < r < R)$ on Ω. The niveau curve γ_r consists of a finite number of simple closed curves $\gamma_{r,k}$ $(k = 1, 2, \ldots, n(r))$. We shall call each component of the open set $\Omega_n - \bar{\Omega}_m$ $(n > m \geqq 0)$ an R-chain, consider for every $\gamma_{r,k}$ $(0 < r < R,\ 1 \leqq k \leqq n(r))$ the longest doubly-connected R-chain $R(\gamma_{r,k})$ such that $\gamma_{r,k}$ is contained in $R(\gamma_{r,k})$ or is the one of the two boundary components of $R(\gamma_{r,k})$, and denote by $\mu(\gamma_{r,k})$ the harmonic modulus of this R-chain. We set

$$\mu(r) = \min_{1 \leqq k \leqq n(r)} \mu(\gamma_{r,k}).$$

Generally $R_{n,k}$ may branch off into a finite number of $R_{n+1,m}$. If every $R_{n,k}$ $(n = 1, 2, \ldots;\ k = 1, 2, \ldots, N(n))$ branches off into at most ρ domains $R_{n+1,m}$, we say that the exhaustion $\{\Omega_n\}$ branches off at most ρ-times everywhere.

Now we state our theorem.

THEOREM 1. *Let E be a totally-disconnected compact set in the z-plane and let Ω be its complementary domain. If there exists an exhaustion $\{\Omega_n\}$ of Ω which satisfies the conditions* 1), 2), 3) *and* 4) *stated above, branches off at most ρ-times everywhere and has the graph satisfying the condition that*

$$\lim_{r \to R} \mu(r) = +\infty,$$

then every function which is single-valued and meromorphic in Ω and has an

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[2] We denote the complement of a set A with respect to the extended complex plane by $\mathscr{C}A$.

[3] Cf. K. Matsumoto [7], § 2.

essential singularity at each point of E,[4] *has at most $\rho+1$ exceptional values at each singularity.*

This is an amelioration of the theorem given in [7]. In fact, we proved the same assertion under the additional conditions that the length R is infinite and further

$$\overline{\lim_{r\to\infty}} \frac{n(r)}{r} < +\infty.$$

3. We shall prove a little stronger theorem than Theorem 1. Let $f(z)$ be a single-valued meromorphic function in Ω possessing at least one essential singularity in E, not necessarily at each point of E. We shall say that $f(z)$ omits a value w in Ω at an essential singularity $\zeta \in E$ if there is a neighborhood $U(\zeta)$ of ζ such that $f(z)$ does not take this value w in $\Omega \cap U(\zeta)$. Such a value may be taken by $f(z)$ only at points of E near ζ.

We shall prove the following

THEOREM 2. *Under the same conditions as Theorem 1, every function, which is single-valued and meromorphic in Ω and has at least one essential singularity in E, omits at most $\rho+1$ values in Ω at each singularity.*

In the case where $\rho=1$, E consists of just one point and hence our assertion is true from Picard's theorem. We shall give a proof only in the case where $\rho=2$. In the same way, we can prove in general cases.

4. Before proving the theorem, we give two lemmas. We shall consider the Riemann sphere Σ with radius $1/2$ touching the w-plane at the origin. For w and w' in the w-plane we denote by $[w, w']$ the chordal distance between them, that is,

$$[w, w'] = \frac{|w-w'|}{\sqrt{(1+|w|^2)(1+|w'|^2)}}.$$

Further we denote by $C(w;\ \delta)$ $(\delta>0)$ the spherical open disc with center at w and with chordal radius δ.

Let $w=f(z)$ be a single-valued meromorphic function in an annulus

[4] We shall see in the proof of the theorem that E is a null set of the class W and hence every transcendental singularity of single-valued meromorphic functions in Ω is always an essential singularity.

$1<|z|<e^{\sigma}$ $(\sigma>0)$ omitting four values w_1, w_2, w_3 and w_4, and let $\delta>0$ be so small that spherical discs $C(w_i\ ;\ \delta)$ $(i=1,2,3,4)$ are mutually disjoint. Now we prove the following lemma which is a consequence of Bohr-Landau's theorem [2]:

If $g(z)$ is regular in $|z|<1$ and $g(z)\neq 0, 1$ there, then

$$\max_{|z|=r}|g(z)|\leqq \exp\left(\frac{K\log(|g(0)|+2)}{1-r}\right)\qquad \text{for all } r<1,$$

where K is a positive constant (a precise form of Schottky's theorem).

Lemma 1. *There is a positive constant δ' such that, if $f(z)$ takes a value outside $C(w_i\ ;\ \delta)$ for some i $(1\leqq i\leqq 4)$ at a point on $|z|=e^{\sigma/2}$, then the image of $|z|=e^{\sigma/2}$ by $f(z)$ lies completely outside the concentric spherical disc $C(w_i\ ;\ \delta')$. Here δ' depends only on σ, w_i $(1\leqq i\leqq 4)$ and δ, and does not depend on $f(z)$.*

Proof. From Bohr-Landau's theorem, we can see easily that if $w=g(z)$ is a regular function in $1<|z|<e^{\sigma}$ such that

$$g(z)\neq 0, 1 \text{ and } \min_{|z|=e^{\sigma/2}}|g(z)|<M \text{ for a positive } M,$$

then there is a positive constant M' depending only on M and σ and satisfying

$$\max_{|z|=e^{\sigma/2}}|g(z)|\leqq M'.$$

We denote by $\zeta=T^i_{j,m}(w)$ $(1\leqq i, j, m\leqq 4,\ i\neq j,\ m$ and $j\neq m)$ the linear transformation which transforms w_i, w_j and w_m to the point at infinity, the origin and the point $\zeta=1$ respectively. Since the number of such $T^i_{j,m}$ is finite, we can find a positive M so large that, for each $T^i_{j,m}$, its image of the outside of $C(w_i\ ;\ \delta)$ is contained completely in $|\zeta|<M$. For this M, $\zeta=T^i_{j,m}(f(z))$ has the same properties as $g(z)$ stated above, and hence it holds that

$$|T^i_{j,m}(f(z))|\leqq M' \qquad \text{on} \qquad |z|=e^{\sigma/2},$$

where $M'>0$ depends only on M and σ. The image of the outside V of $|\zeta|\leqq M'$ by $(T^i_{j,m})^{-1}$ is an open disc containing w_i. If we denote by $d^i_{j,m}$ the chordal distance between w_i and the boundary of $(T^i_{j,m})^{-1}(V)$ and set

$$\delta'=\min_{\substack{1\leqq i,j,m\leqq 4\\ i\neq j,m\\ j\neq m}} d^i_{j,m},$$

then $\delta'>0$ and obviously satisfies all conditions of the lemma.

Next lemma is a revised form of Carleson's [3].

LEMMA 2. *Let $w=f(z)$ be a single-valued meromorphic function on an annulus $1\leqq|z|\leqq e^{\mu}$ $(\mu>0)$. If $f(z)$ takes there no value in a spherical disc $C(w_0\,;\,\delta)$, then there exists a positive constant A depending only on δ such that the diameter of the image of $|z|=e^{\mu/2}$ by $f(z)$ with respect to the chordal distance is dominated by $Ae^{-\mu/2}$ for sufficiently large μ.*

In particular, if δ is sufficiently close to 1, that is, the complementary sherical disc $C(-1/\overline{w}_0\,;\,d)$ of $C(w_0\,;\,\delta)$ has a radius d sufficiently small, we have

$$A<Bd,$$

where B is a positive constant.

Proof. We may assume without loss of generality that the center w_0 of $C(w_0\,;\,\delta)$ is the point at infinity, for otherwise we can transform w_0 to the point at infinity by the linear transformation $(1+\overline{w}_0w)/(w-w_0)$, under which the chordal distance remains invariant. Let $|w|>M$ be the domain in the w-plane corresponding to $C(w_0\,;\,\delta)$. Then

$$|f(z)|\leqq M \quad \text{on} \quad 1\leqq|z|\leqq e^{\mu}.$$

By Cauchy's integral theorem, we have

$$f'(z)=\frac{1}{2\pi i}\left\{\int_{|\zeta|=e^{\mu}}\frac{f(\zeta)}{(\zeta-z)^2}d\zeta-\int_{|\zeta|=1}\frac{f(\zeta)}{(\zeta-z)^2}d\zeta\right\}$$

for every z on $|z|=e^{\mu/2}$ and hence, if $\mu\geqq 2$,

$$|f'(z)|\leqq\frac{M}{2\pi}\left\{\frac{2\pi e^{\mu}}{(e^{\mu}-e^{\mu/2})^2}+\frac{2\pi}{(e^{\mu/2}-1)^2}\right\}\leqq\frac{2e^2}{(e-1)^2}Me^{-\mu}.$$

Therefore we have

$$\int_{|z|=e^{\mu/2}}|f'(z)|\,|dz|\leqq\frac{2e^2}{(e-1)^2}Me^{-\mu}\cdot 2\pi e^{\mu/2}=\frac{4\pi e^2}{(e-1)^2}M\cdot e^{-\mu/2}.$$

The left side shows the length of the image curve $f(|z|=e^{\mu/2})$, and hence the diameter of the image of $|z|=e^{\mu/2}$ by $f(z)$ with respect to the metric $|dw|$, consequently with respect to the chordal distance, is dominated by $(2\pi e^2/(e-1)^2)Me^{-\mu/2}$. We can take $(2\pi e^2/(e-1)^2)M$ as A, for M depends only on δ.

If $d<1/2$, then $M<2d$. Hence

$$B = 4\pi e^2/(e-1)^2$$

is one of the wanted. Thus our lemma is established.

5. *Proof of the theorem.* Contrary to our assertion, let us suppose that there exists a function $f(z)$ which is single-valued and meromorphic in Ω, has at least one essential singularity in E and omits more than three values in Ω at an essential singularity $\zeta_0 \in E$. Then there is a neighborhood $U(\zeta_0)$ of ζ_0 such that $f(z)$ omits four values w_1, w_2, w_3 and w_4 in $U(\zeta_0) \cap \Omega$. We take a positive δ so small that spherical discs $C(w_i\ ;\ \delta)$ $(1 \leqq i \leqq 4)$ are mutually disjoint. For this δ and a $\sigma>0$, Lemma 1 determines $\delta'>0$.

Next we consider this δ' as δ in Lemma 2 and take μ_0 so large that

$$Ae^{-\mu_0/2} < \min\ \{1/24,\ \delta'/3\} \text{ and } Be^{-\mu_0/2} < 1/12,$$

where A and B are constants in Lemma 2. From the assumption

$$\lim_{r \to R} \mu(r) = +\infty,$$

there is an $r_0>0$ such that

$$\mu(r) > \mu_0 + 2\sigma \qquad \text{for all } r : r_0 < r < R.$$

The niveau curve $\gamma_r : u(z) = r$ $(r_0<r<R)$ consists of a finite number of simple closed curves $\gamma_{r,k}$ $(k=1, 2, \ldots, n(r))$, and one of them, say $\gamma_{r,1}$, encloses ζ_0. For r sufficiently near R, the longest doubly-connected R-chain $R(\gamma_{r,1}) = S_{1,1}$ for $\gamma_{r,1}$, which we defined in §2, is contained in $U(\zeta_0)$. The harmonic modulus of $S_{1,1}$ is of course greater than $\mu_0 + 2\sigma$ but is not infinite, for, if so, ζ_0 must be isolated and $f(z)$ cannot omit four values at ζ_0. Therefore $S_{1,1}$ must branch off. Now suppose that $S_{1,1}$ is a component of the open set $\Omega_n - \bar{\Omega}_{n'}$ $(n>n')$, and branches off into two domains $R_{n+1,k}$ and $R_{n+1,k'}$, and consider the longest doubly-connected R-chains $S_{2,1}$ and $S_{2,2}$ containing $R_{n+1,k}$ and $R_{n+1\ k'}$ respectively. Then they have harmonic moduli greater than $\mu_0 + 2\sigma$, one of them, say $S_{2,1}$, separates ζ_0 from $S_{1,1}$ and its harmonic modulus is finite by the same reason as above. Hence $S_{2,1}$ is a component of the open set $\Omega_m - \Omega_n$ for some m and branches off into two domains $R_{m+1,k}$ and $R_{n+1,k'}$. We shall denote by $S_{3,1}$ and $S_{3,2}$ the longest doubly-connected R-chains containing them. On the other hand, the harmonic modulus of $S_{2,2}$ may be infinite. If it is infinite, one of the boundary components of $S_{2,2}$ is a point $\eta \in E$ and $f(z)$ is meromorphic at η. If it is finite, we obtain two R-chains $S_{3,3}$ and $S_{3,4}$ in the same manner as above. Thus we have at most 2^2 R-chains $S_{3,q}$ such that their harmonic moduli are greater than $\mu_0 + 2\sigma$, one of them encloses ζ_0 and each of them branches off into two domains, if its harmonic modulus is finite, or has a point $\eta \in E$ as one of its boundary components at which $f(z)$ is meromorphic, if its harmonic modulus is infinite. Continuing inductively, we obtain a set of R-chains $S_{p,q}$ ($p = 1, 2, 3, \ldots$; $q = 1, 2, \ldots, Q(p) \leqq 2^{p-1}$) with the following properties:

1) $\bigcup_{p=1}^{\infty} \bigcup_{q=1}^{Q(p)} \bar{S}_{p,q} \supset \Delta$, where Δ denotes the intersection of the inside of the simple closed curve $\gamma_{r,1}$ and Ω,

2) their harmonic moduli are greater than $\mu_0 + 2\sigma$,

3) each $S_{p,q}$ branches off into two $S_{p+1,q}$ if its harmonic modulus is finite, or

4) it has a point $\eta \in E$ as one of its boundary components and $f(z)$ is meromorphic at η if its harmonic modulus is infinite. In this case we shall denote the point η and the value $f(\eta)$ by $\eta_{p,q}$ and $w_{p,q}$ respectively.

Each $S_{p,q}$ is conformally equivalent to the annulus $1 < |\zeta| < e^{\mu}$, where μ is the harmonic modulus of $S_{p,q}$. In the case where $\mu < +\infty$, we denote by $S^1_{p,q}$, $S^2_{p,q}$ and $S^3_{p,q}$ subdomains of $S_{p,q}$ corresponding to the annuli $1 < |\zeta| < e^{\sigma}$, $e^{\sigma} < |\zeta| < e^{\mu-\sigma}$ and $e^{\mu-\sigma} < |\zeta| < e^{\mu}$ respectively and by $\Gamma^1_{p,q}$, $\Gamma^2_{p,q}$ and $\Gamma^3_{p,q}$ closed curves corresponding to $|\zeta| = e^{\sigma/2}$, $|\zeta| = e^{\mu/2}$ and $|\zeta| = e^{\mu-\sigma/2}$ respectively. We observe that for each $\Gamma^2_{p,q}$ the diameter of its image by $f(z)$ with respect to the chordal distance is dominated by $K = \min\ \{1/24, \delta'/3\}$. In fact, for $z' \in \Gamma^1_{p,q}$ and $z'' \in \Gamma^3_{p,q}$, the images $f(z')$ and $f(z'')$ lie in the outside of at least one $C(w_i\ ;\ \delta)$, say $C(w_1\ ;\ \delta)$, and hence, applying Lemma 1 in $S^1_{p,q}$ and $S^3_{p,q}$, we see that the images of $\Gamma^1_{p,q}$ and $\Gamma^3_{p,q}$, consequently the image of the ring domain bounded by them by the maximum principle, lie completely outside $C(w_1\ ;\ \delta')$. Thus our assertion follows from Lemma 2, because the harmonic modulus of $S^2_{p,q}$ is greater than μ_0.

Every $S_{p+1,q'}$ ($p \geqq 1$) has in common with another $S_{p+1,q''}$ an $S_{p,q}$ branching off into them, and we shall denote by $\Delta_{p,q}$ the triply-connected domain bounded by $\Gamma^2_{p,q}$, $\Gamma^2_{p+1,q'}$ and $\Gamma^2_{p+1,q''}$, where we consider $\Gamma^2_{p+1,q'} = \eta_{p+1,q'}$ or $\Gamma^2_{p+1,q''} = \eta_{p+1,q''}$ if $S_{p+1,q'}$ or $S_{p+1,q''}$ has infinite harmonic modulus. For $w \in f(\Gamma^2_{p,q})$, $w' \in f(\Gamma^2_{p+1,q'})$ and $w'' \in f(\Gamma^2_{p+1,q''})$ we consider spherical discs $C(w\ ;\ K)$, $C(w'\ ;\ K)$ and

$C(w''\ ;\ K)$, which of course contain $f(\Gamma^2_{p,q})$, $f(\Gamma^2_{p+1,q'})$ and $f(\Gamma^2_{p+1,q''})$ respectively. Since $K<\delta'/3$, they cannot contain at least one of w_i $(1\leqq i\leqq 4)$, say w_1, and hence each one of them cannot be disjoint from the union of the others, for, if so for some one, there is $z_0\in\Delta_{p,q}$ such that $f(z_0)$ is contained and can be joined to w_1 with a curve Λ in the outside of the union of these three discs, and we are led to a contradiction that the element of the inverse function f^{-1} corresponding to z_0 can be continued analytically along Λ up to a point arbitrarily near w_1 so that $f(z)$ takes the value w_1 in $\Delta_{p,q}$. Therefore we can conclude that

(1°) for every $\Delta_{p,q}$, there is a spherical disc with the chordal radius $3K$ containing its image $f(\Delta_{p,q})$.

Next we shall consider $\Gamma^2_{p,q}$ for $p\geqq 2$. Then $\Delta_{p,q}$ and some $\Delta_{p-1,q'}$ have $\Gamma^2_{p,q}$ as the common boundary and $\Delta_{p,q}\cup\Gamma^2_{p,q}\cup\Delta_{p-1,q'}\supset S_{p,q}$. From (1°) the images of $\Delta_{p,q}\cup\Gamma^2_{p,q}\cup\Delta_{p-1,q'}$, consequently of $S^2_{p,q}\subset S_{p,q}$, are contained in a spherical disc with the chordal radius $6K<1/2$, so that, applying Lemma 2 in $S^2_{p,q}$ for $d=6K$, we see that the diameter of $f(\Gamma^2_{p,q})<6KBe^{-\mu_0/2}<K/2$. For $p\geqq 2$, each boundary component of $\Delta_{p,q}$ thus has the image with diameter less than $K/2$. By the same reasoning as above we now conclude that

(2°) for $p\geqq 2$, the image of any $\Delta_{p,q}$ is contained in a spherical disc with chordal radius $3K/2$.

By induction we also see for every n that

(n°) for $p\geqq n$, the image of any $\Delta_{p,q}$ is contained in a spherical disc with chordal radius $3K/2^{n-1}$.

Let Δ' be the intersection of the inside of the simple closed curve $\Gamma^2_{1,1}$ and Ω and let z_0 be a point of $\Gamma^2_{1,1}$. Then it follows from the property 1) of $\{S_{p,q}\}$ that

$$\bigcup_{p=1}^{\infty}\bigcup_{q=1}^{Q(p)}\bar{\Delta}_{p,q}\supset\Delta',$$

and hence, for any $z'\in\Delta'$, there is $\Delta_{p',q'}$ whose closure contains z'. From (n°) obtained above, we have for a chain of $\Delta_{p,q}$ joining $\Delta_{1,1}$ and $\Delta_{p',q'}$

$$\begin{aligned}[f(z'),\ f(z_0)]&\leqq\sum_{p=1}^{p'}\text{diam. of } f(\Delta_{p,q}) \text{ w.r.t. the chordal distance}\\&<2\sum_{p=1}^{\infty}3K/2^{p-1}=12K<1/2.\end{aligned}$$

By means of a linear transformation we can consider from the above that

$f(z)$ is bounded in Δ'. On the other hand, only applying Pfluger-Mori's criterion ([10], [8]) to the ring domains $\{S_{p,q}\}$, we can see easily that the part E' of E contained in the inside of $\Gamma^2_{1,1}$ is a null-set of the class W. Hence each point of E' must be a removable singularity of a bounded function $f(z)$. This contradicts our assumption that $\zeta_0 \in E'$ is an essential singularity of $f(z)$, and hence $f(z)$ cannot omit four values in Ω at ζ_0. Thus our theorem is proved completely.

6. For Cantor sets, we have the following which is an immediate consequence of our theorem.

THEOREM 3. *Let E be a Cantor set on the closed interval* [0, 1] *with the successive ratios ξ_n satisfying the condition*

$$\lim_{n\to\infty} \xi_n = 0.$$

Then every function, which is single-valued and meromorphic in $\Omega = \mathscr{C}E$ and has at least one essential singularity in E, omits at most three values in Ω at each essential singularity.

Carleson proved in his paper [3] the same assertion under a stronger condition

$$\lim_{n\to\infty} \frac{\log \xi_n^{-1}}{\log n} = +\infty.$$

As he remarked there, there is a set E of positive capacity because E is of capacity zero if and only if

$$\sum_{n=1}^{\infty} \frac{\log \xi_n^{-1}}{2^n} = +\infty.$$

7. In our paper [7], we showed, by using its Theorem 1, that there is a set E such that all $f(z)$ possessing E as the set of essential singularities have at most three exceptional values and some one of them indeed has just three exceptional values at each point of E. But the condition

$$\varlimsup_{r\to\infty} \frac{n(r)}{r} < +\infty,$$

which gave at that time some difficulties in constructing E, is unnecessary as we saw above and so for each $\rho \geqq 2$ we can give easily in the similar manner a set E such that all $f(z)$ have at most ρ exceptianal values and some one of them indeed has just ρ exceptional values.

8. In the last section we shall be concerned with single-valued meromorphic functions which have as the set of essential singularities a set E satisfying the conditions of Theorem 1 for $\rho=2$ and have three exceptional values at a point $\zeta\in E$. We begin with

LEMMA 3. *Let $f(z)$ be a single-valued meromorphic function on the closure of a triply-connected domain $\varDelta$ omitting three values w_1, w_2 and w_3 there, let Γ_1, Γ_2 and Γ_3 be the boundary components of $\varDelta$ and let $\delta>0$ be so small that the discs $C(w_i;\ \delta)$ $(1\leqq i\leqq 3)$ are mutually disjoint. If $f(\Gamma_i)\subset C(w_i\ ;\ \delta)$ for all i, $f(z)$ takes in $\varDelta$ each value outside $\bigcup_{i=1}^{3} C(w_i\ ;\ \delta)$ once and only once.*

Proof. Contrary, suppose that $f(z)$ takes a value w_0 outside $\bigcup_{i=1}^{3} C(w_i\ ;\ \delta)$ at two points $z'\in\varDelta$ and $z''\in\varDelta$. We can join w_0 to $C(w_1\ ;\ \delta)$ and $C(w_2\ ;\ \delta)$ with curves Λ_1 and Λ_2, respectively, which lie outside $\bigcup_{i=1}^{3} C(w_i\ ;\ \delta)$, do not intersect each other except at w_0 and do not pass through any projection of branch points of the Riemannian image of $\varDelta$ by $f(z)$. The elements of the inverse function f^{-1} corresponding to z' and z'' can be continued analytically along these curves to their end points and further from them along radii of $C(w_1\ ;\ \delta)$ and $C(w_2\ ;\ \delta)$ so that the curves in $\varDelta$ corresponding to these continuations join each of z' and z'' to Γ_1 and Γ_2 and with parts of Γ_1 and Γ_2 bound a domain not containing Γ_3. Since $\varDelta$ has no boundary other than Γ_1, Γ_2 and Γ_3, this domain must be a subdomain of $\varDelta$ and $f(z)$ must take the value w_3 in it; this contradiction proves the lemma.

Let E be a compact set in the z-plane satisfying the conditions of Theorem 1 for $\rho=2$ and let $f(z)$ be a single-valued meromorphic function which has E as the set of essential singularities. Of course, $f(z)$ has at most three exceptional values at any point of E. From Kuroda's theorem [5] we can easily see that the complementary domain Ω of E belongs to the class O°_{AB} as a Riemann surface and hence the covering surface, which is the Riemannian image of Ω by $w=f(z)$, has Inversen's property. Consequently, any exceptional value α of $f(z)$ at an essential singularity $\zeta\in E$ is an asymptotic value at ζ or there is a sequence $\{\zeta_n\}$ of points of E such that $\lim_{n\to\infty}\zeta_n=\zeta$ and α is an asymptotic value at every ζ_n. Now, suppose that $f(z)$ has indeed three exceptional values w_1, w_2 and w_3 at $\zeta \in E$. We prove

THEOREM 4. *There exists a sequence* $\{\zeta_n\}$ *of points of* E *such that* $\lim_{n\to\infty} \zeta_n = \zeta$ *and for any curve terminating at* ζ_n *the cluster set of* $f(z)$ *along it contains always all exceptional values* w_1, w_2 *and* w_3 *at* ζ.

Proof. It is sufficient to prove that in any neighborhood $U(\zeta)$ of ζ there is a $\zeta' \in E$, different from ζ, having the property stated in the theorem. We shall take $\delta > 0$ so small that discs $C(w_i;\ 2\delta)$ $(1 \leqq i \leqq 3)$ are mutually disjoint, use the notations in §5 considering 2δ as δ there and note that in the present case all $S_{p,q}$ branch off into two $S_{p+1,q}$, because every point of E is an essential singularity of $f(z)$. Further, we can prove by the same reasoning as used in §5 that if $f(\Delta_{p,q})$ lies completely outside some $C(w_i;\ \delta)$, then there is a spherical disc $C_{p,q}$ with the chordal radius $3K$ containing $f(\Delta_{p,q})$. In fact, if one of the spherical discs $C(w;\ K)$, $C(w';\ K)$ and $C(w'';\ K)$ $(w \in f(\Gamma^2_{p,q}),\ w' \in f(\Gamma^2_{p+1,q'})$ and $w'' \in f(\Gamma^2_{p+1,q''}))$ is disjoint from the union of the others, there is a point $z_0 \in \Delta_{p,q}$ with the image $f(z_0)$ outside these three discs, $f(z_0)$ can be joined to the center w_i of $C(w_i;\ \delta)$ with a curve Λ outside them and hence $f(z)$ takes the value w_i in $\Delta_{p,q}$; this is a contradiction. Thus if, for every $\Delta_{p,q}$ there is one in $C(w_i;\ \delta)$ $(1 \leqq i \leqq 3)$ which is disjoint from $f(\Delta_{p,q})$, then the assertion (1°) in §5 holds and we can repeat the arguments there to reach the contradiction that all points of E near ζ are removable singularities of $f(z)$. Thus we can conclude that there is an infinite number of $\Delta_{p,q}$ such that three discs $C(w_i;\ 2\delta)(1 \leqq i \leqq 3)$ contain the images of its three boundary components one by one.

Let δ_ν be a sequence of positive numbers decreasing to zero. Then from the above there is a $\Delta_{p,q}$ such that three discs $C(w_i;\ 2\delta_1)$ contain the images of its three boundary components one by one. We denote by Γ_0 the boundary component of $\Delta_{p,q}$ with the image $f(\Gamma_0)$ contained in $C(w_1;\ 2\delta_1)$ and consider all $\Delta_{p,q}$ contained in the inside of Γ_0. Among these $\Delta_{p,q}$ there is one such that three discs $C(w_i;\ 2\delta_2)$ contain the images of its three boundary components one by one and we denote by Γ_1 its boundary component with the image $f(\Gamma_1)$ contained in $C(w_2;\ 2\delta_2)$. Next we consider all $\Delta_{p,q}$ contained in the inside of Γ_1 and obtain Γ_2 with the image $f(\Gamma_2)$ contained in $C(w_3;\ 2\delta_3)$. We proceed inductively and obtain a sequence of closed curves $\{\Gamma_\nu\}$ with the following conditions: 1) the inside of Γ_ν contains $\Gamma_{\nu+1}$, 2) the image of $\Gamma_{3\kappa+\tau}$ ($\kappa=0, 1, 2, \ldots$; $\tau=0, 1, 2$) is contained in $C(w_{\tau+1}; 2\delta_{3\kappa+\tau})$. We set

$$\zeta' = \bigcap_\nu \overline{(\Gamma_\nu)},$$

where we denote by (Γ_ν) the inside of Γ_ν and by $\overline{(\Gamma_\nu)}$ the closure of (Γ_ν). Obviously $\zeta' \in E$ and has the property stated in the theorem. Our proof is now complete.

The fact in the above proof that for any $\delta>0$, there is $\Delta_{p,q}$ such that three discs $C(w_i ; \delta)$ contain the images of its three boundary components one by one implies by Lemma 3 the following theorem under the same conditions as Theorem 4.

THEOREM 5. *Let Φ be the covering surface of the w-plane which is the Riemannian image of Ω by $w=f(z)$. Then, for arbitrary four discs C_i $(1 \leqq i \leqq 4)$ in the w-plane with the closures being mutually disjoint, there is at least one over which Φ has an infinite number of univalent discs $\{\tilde{C}_k\}$ such that $f^{-1}(\tilde{C}_k)$ are compact relative to Ω and are contained in any neighborhood of ζ except for a finite number of them.*

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