EXAMPLES OF WEAK BOUNDARY COMPONENTS

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1. Let D be a plane domain and Γ be a component of the boundary of D *consisting of a single point.* According to Sario [5] we shall call Γ a ***weak*** boundary component of D if its image under any conformal mapping of D consists of a single point. A weak boundary component has been introduced by Grötzsch [2], who called it "vollkommen punktförmig". If Γ is not weak we shall say that it is *unstable* (Sario [5]). We know that the weakness depends merely on the configuration of D in a neighborhood of Γ (see [4], p. 274).

Let E be a compact set on the non-negative real axis such that $0 \in E$, $E \subset [0, 1]$, and that the component of 0 contains no other point. Let $h(\xi)$ be a real (finite) valued function which is defined on E, upper semi-continuous, non-negative, and such that $h(0) = 0$. For any $\xi \in E$, let

$$S_{\xi, h} = \{z\,;\ \mathrm{Re}\, z = \xi,\ |\mathrm{Im}\, z| \leqq h(\xi)\}.$$

Then $D_{E,h} = \{z\,;\ |z| \leqq \infty\} - \bigcup_{\xi \in E} S_{\xi, h}$ is a domain and $\Gamma_{E,h} = \{0\}$ is its boundary component consisting of a single point.

It would be useful to give convenient condition on E and $h(\xi)$ to determine when $\Gamma_{E,h}$ is weak or unstable.

2. We remark first that the following "comparison theorem" would enlarge the range of applicability of criteria given in the sequel: *If Γ_{E,h_1} is* ***weak and***

$$\overline{\lim_{\xi \in E,\ \xi \to 0}} \frac{h_2(\xi)}{h_1(\xi)} < \infty$$

then *Γ_{E,h_2} is also* ***weak.*** The proof is immediate from the local property and the quasi-conformal invariance of weakness (see [4], p. 274).

3. The former author has shown that, if $E = \{a_n\}_{n=1}^{\infty}(a_n > a_{n+1} > 0, \lim_{n\to\infty} a_n = 0)$ and $h(\xi) \leqq c\xi$ $(c > 0)$, then $\Gamma_{E,h}$ is weak (see [1]). It is generalized as follows (cf. the comparison theorem):

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THEOREM 1. *If $D_{F,h}$ and $\Gamma_{F,h}$ are given by the restriction onto E of a function $h(\xi)$ defined on $0 \leqq \xi \leqq 1$ such that*

(i) $\sqrt{\xi^2+h(\xi)^2}$ *is a non-decreasing function of ξ with the derivative* (existing almost every where) *bounded away from zero,*

(ii) $$\int_{[0,1]-E} \frac{d\xi}{\sqrt{\xi^2+h(\xi)^2}} = \infty,$$

then $\Gamma_{F,h}$ is a weak boundary component.

Proof. Take $b>0$ such that $\{z;\ b \leqq |z| \leqq \infty\} \subset D_{E,h}$. Let $\{\gamma\}$ be the family of all the closed rectifiable curves in $D_{F,h} \cap \{z;\ |z|<b\}$ separating $\Gamma_{E,h}$ from $|z|=b$. It has been shown by Jurchescu [3] that $\Gamma_{E,h}$ is weak if and only if the extremal length $\lambda\{\gamma\}$ of the family $\{\gamma\}$ vanishes (see also [4], Theorems 2, 3).

For a $\xi \in [0, 1]-E$, let γ_ξ be the union of $\gamma'_\xi = \{z;\ \operatorname{Re} z = \xi, |\operatorname{Im} z| \leqq h(\xi)\}$ and $\gamma''_\xi = \{z;\ |z|^2 = \xi^2 + h(\xi)^2,\ \arctan(h(\xi)/\xi) \leqq |\arg z| \leqq \pi\}$. Evidently $\{\gamma_\xi\} = \{\gamma_\xi;\ \xi \in [0, 1]-E\}$ is contained in $\{\gamma\}$ and, therefore, it is sufficient to show that $\lambda\{\gamma_\xi\}=0$. On making use of usual notations, we have

$$L_\rho\{\gamma_\xi\}^2 \leqq \left(\int_{\gamma_\xi} \rho ds\right)^2 \leqq \left(\int_{\gamma_\xi} ds\right)\left(\int_{\gamma_\xi} \rho^2 ds\right)^2 \leqq 2\pi\sqrt{\xi^2+h(\xi)^2}\left(\int_{\gamma_\xi'} \rho^2 dy + \int_{\gamma_\xi''} \rho^2 r d\theta\right).$$

where $r=\sqrt{\xi^2+h(\xi)^2}$. Divide it by r and integrate it with respect to ξ over $[0, 1]-E$. Since it is assumed that $dr/d\xi \geqq \alpha > 0$, we have, on putting $\Delta' = \cup_\xi \gamma'_\xi$ and $\Delta'' = \cup_\xi \gamma''_\xi$, that

$$L_\rho\{\gamma_\xi\}^2 \int_{[0,1]-E} \frac{d\xi}{\sqrt{\xi^2+h(\xi)^2}} \leqq 2\pi \iint_{\Delta'} \rho^2 dxdy + \frac{2\pi}{\alpha}\iint_{\Delta''} \rho^2 r dr d\theta \leqq \text{const} \iint_D \rho^2 dxdy.$$

Therefore, $L_\rho\{\gamma_\xi\}^2=0$ for any square integrable ρ, i.e., $\lambda\{\gamma_\xi\}=0$.

4. A result of the former author [1] saying that $\Gamma_{F,h}$ is unstable when $E=\{1/n\}_{n=1}^{\infty}$ and $h(\xi)=\xi^p$ $(0<p<1)$ will be contained in the following:

THEOREM 2. *If $D_{E,h}$ and $\Gamma_{E,h}$ are given by the restriction onto E of a function $h(\xi)$ defined on $0 \leqq \xi \leqq 1$ such that*

(i) *monotone non-decreasing*

(ii) *there exists a constant K such that, for any $\xi \in E-\{1\}$, it is possible to find a $\xi' \in E$ with $\xi < \xi'$ and $h(\xi') \leqq Kh(\xi)$.*

(iii) $$\int_{[0,1]-E} \frac{d\xi}{h(\xi)} < \infty,$$

then $\Gamma_{E,h}$ is an unstable boundary component.

That the condition (ii) cannot be omitted for the case $\int_0^1 d\xi/h < \infty$ is seen from an easily constructed example. Whether or not we can omit it for the case $\int_0^1 d\xi/h = \infty$ is not clear, however, we can do so for $h(\xi) = \xi^p (p \geqq 1)$ as follows:

COROLLARY. *If $h(\xi) = \xi^p$ $(p \geqq 1)$, then $\Gamma_{E,h}$ is an unstable boundary component of $D_{E,h}$ provided*

$$\int_{[0,1]-E} \frac{d\xi}{\xi^p} < \infty .$$

Proof of Theorem 2. We shall apply the following criterion due to Grötzsch ([2]; see also [4], Theorem 3): Take b such that $\{z; b \leqq |z| \leqq \infty\} \subset D_{E,h}$; $\Gamma_{E,h}$ is unstable if and only if there exists a finite number M such that $\sum_{\nu=1}^{k} \mathrm{mod}\ A_\nu \leqq M$ holds for any finite set $\{A_1, A_2, \ldots, A_k\}$ of doubly connected domains A_ν with the following conditions:

(1) $A_\nu \subset D_{E,h} \cap \{z; |z| < b\}$,

(2) A_ν separates $\Gamma_{E,h}$ from $|z| = b$,

(3) A_ν separates $A_{\nu-1}$ from $A_{\nu+1}$.

On looking over the argument in [4] we understand that the result is true if every A_ν is so restricted that the boundary consists of closed analytic curves.

Let $(0, b) - E = \cup_{n=1}^{\infty} I_n$, where $I_n = (\xi_n, \xi_n')$ are mutually disjoint open intervals. Consider the quadrilaterals $Q_n = \{z; \mathrm{Re}\, z \in I_n, |\mathrm{Im}\, z| \leqq h(\xi_n)\}$ $(n = 1, 2, \ldots)$. By the condition (2) every A_ν passes through a Q_n vertically, i.e., every closed arc in A_ν separating its boundary components contains a subarc connecting in $Q_n \cap A_\nu$ the top and the bottom sides of Q_n. There may be more than one Q_n's; we then take the Q_n corresponding to the left most I_n (remember that the boundary of A_ν consists of analytic curves). For a Q_n, consider all the A_ν's with the above property. Then, by the condition (3), the sum of their moduli does not exceed $2\pi/\lambda\{\gamma\}_n$, where $\{\gamma\}_n$ is the family of all the closed curves in $D_{E,h} \cap \{z; |z| < b\}$ which separate $\Gamma_{E,h}$ from $|z| = b$ and pass through Q_n vertically.

We thus have a grouping of the set $\{A_1, A_2, \ldots, A_k\}$ in terms Q_n. Since an A_ν does not appear in different groups, $\sum_{\nu=1}^{k} \mathrm{mod}\ A_\nu \leqq 2\pi \sum_{n=1}^{\infty} 1/\lambda\{\gamma\}_n$.

Evidently $\lambda\{\gamma\}_n$ is not less than mod Q_n, the "vertical" modulus of the quadrilateral Q_n, which is equal to $h(\xi_n)/(\xi_n' - \xi_n)$. We conclude, on using the condition (ii) that

$$\sum_{\nu=1}^{k} \text{mod } A_\nu \leqq 2\pi \sum_{n=1}^{\infty} \frac{\xi_n' - \xi_n}{h(\xi_n)} \leqq 2\pi K \sum_{n=1}^{\infty} \frac{\xi_n' - \xi_n}{h(\xi_n')} \leqq 2\pi K \int_{[0,b]-F} \frac{d\xi}{h(\xi)} < \infty$$

and that $\Gamma_{E,h}$ is unstable.

Proof of Corollary. Since

$$\sum_{n=1}^{\infty}\left(1 - \frac{\xi_n}{\xi_n'}\right) \leqq \sum_{n=1}^{\infty} \int_{\xi_n}^{\xi_n'} \frac{d\xi}{\xi^p} = \int_{[0,b]-E} \frac{d\xi}{\xi^p} < \infty,$$

$(\xi_n')^p/(\xi_n)^p$ is bounded. This fact plays the role of (ii) in the above proof.

5. In a paper of the latter author, we proved the following ([4], Theorem 8):

Consider in particular $E = \{0\} \cup \bigcup_{n=1}^{\infty} [u_n, u_n']$, *where* $0 < u_n < u_n' < u_{n-1} < 1$ $(n = 2, 3, \ldots)$ *and* $\lim_{n\to\infty} u_n = 0$. *Then, under the assumption that* $\lim_{n\to\infty} (u_n/u_{n+1}') = 1$ *and* $u_n/u_{n+1} \geqq 1 + \delta > 1$, *the* $\Gamma_{E,h}$ *for* $h \equiv 0$ *is weak if and only if*

$$\sum_{n=1}^{\infty} \frac{1}{\log \dfrac{u_{n+1}'}{u_n - u_{n+1}'}} = \infty.$$

Concerning such an E, Theorem 1 is merely saying that $\Gamma_{E,h}$ for $h \equiv 0$ is weak if $\sum_{n=1}^{\infty}((u_n/u_{n+1}') - 1) = \infty$. Theorem 2 is not applicable to the case where $h \equiv 0$. We see that there is a wide room into which our Theorems 1 and 2 should be extended.

6. Our theorems, however, may be extended into a different direction. To show this, we first introduce the notations

$$S(\xi, \theta, c) = \{z;\ |z + c| = \xi + c,\ |\arg z| \leqq \theta\} \qquad (0 \leqq c < \infty,\ 0 \leqq \theta < \pi,\ 0 < \xi)$$

and

$$S(\xi, \theta, \infty) = \{z;\ \text{Re } z = \xi,\ |\text{Im } z| \leqq \xi \tan \theta\} \qquad (0 \leqq \theta < \pi/2,\ 0 < \xi).$$

Let E be, as before, a compact set on the non-negative real axis such that $0 \in E$, $E \subset [0, 1]$, and that the component of 0 contains no other point. Let $\theta(\xi)$ and $c(\xi)$ be functions which are defined on E and satisfy the following conditions: $\theta(\xi)$ is upper semi-continuous and is such that $0 \leqq \theta(\xi) < \pi$; $c(\xi)$

is continuous, non-decreasing, and is such that $0 \leqq c(\xi) \leqq \infty$. Suppose further that $\theta(\xi) < \pi/2$ whenever $c(\xi) = \infty$, and that

$$\overline{\lim_{\xi \in E,\, \xi \to 0}}\, \theta(\xi) \leqq \frac{\pi}{2} \qquad \text{if } 0 < \lim_{\xi \in E,\, \xi \to 0} c(\xi) < \infty,$$

$$\overline{\lim_{\xi \in E,\, \xi \to 0}}\, \xi \tan \theta(\xi) = 0 \qquad \text{if } c(\xi) \equiv \infty.$$

Then

$$D(E, \theta, c) = \{z;\ 0 < |z| \leqq \infty\} - \bigcup_{\xi \in E - \{0\}} S(\xi, \theta(\xi), c(\xi))$$

is a domain and $\Gamma(E, \theta, c) = \{0\}$ is its boundary component. The domain $D_{E,h}$ discussed in the previous sections is the $D(E, \theta, c)$ for $\theta(\xi) = \arctan(h(\xi)/\xi)$ and $c(\xi) \equiv \infty$.

THEOREM 1′. *Suppose that $D(E, \theta, c)$ and $\Gamma(E, \theta, c)$ described above are given by restrictions onto E of $\theta(\xi)$ and $c(\xi)$ defined on $0 \leqq \xi \leqq 1$, where $c(\xi)$ is non-decreasing on $0 \leqq \xi \leqq 1$. If either*

(I) $\lim_{\xi \to 0} c(\xi) = 0$ *and*

$$\int_{[0,1]-E} \frac{d\xi}{\xi + c(\xi)} = \infty$$

or

(II) *the distance $r(\xi)$ between 0 and the endpoints of $S(\xi, \theta(\xi), c(\xi))$ is a non-decreasing function of ξ with the derivative bounded away from zero and*

$$\int_{[0,1]-E} \frac{d\xi}{r(\xi)} = \infty,$$

then $\Gamma(E, \theta, c)$ is weak.

Proof. If (I) is assumed, we may suppose without loss of generality that $c(\xi)$ is finite. The weakness of $\Gamma(E, \xi, c)$ follows from the vanishing of the extremal length of the family $\{\gamma_\xi;\ \xi \in [0, 1] - E\}$, where $\gamma_\xi = \{z;\ |z + c(\xi)| = \xi + c(\xi)\}$. Under the supposition of (II), we similarly consider $\{\gamma'_\xi \cup \gamma''_\xi;\ \xi \in [0, 1] - E\}$, where $\gamma'_\xi = S(\xi, \theta(\xi), c(\xi))$ and $\gamma''_\xi = \{z;\ |z| = r(\xi),\ \theta(\xi) \leqq |\arg z| \leqq \pi\}$. The proof in detail will be omitted since it is completely analogous to that of Theorem 1.

THEOREM 2′. *Suppose that $D(E, \theta, c)$ and $\Gamma(E, \theta, c)$ described above are given by restrictions onto E of $\theta(\xi)$ and $c(\xi)$ defined on $0 \leqq \xi \leqq 1$, where $c(\xi)$ is non-decreasing on $0 \leqq \xi \leqq 1$. If*

(i) *the length* $l(\xi)$ *of* $S(\xi, \theta(\xi), c(\xi))$ *is a non-decreasing function of* ξ *provided that* $c(\xi) \equiv \infty$, *and* $l(\xi)/(\xi + c(\xi))$ *is non-decreasing otherwise,*

(ii) *there exists a constant* K *such that, for any* $\xi \in E - \{1\}$, *it is possible to find a* $\xi' \in E$ *with* $\xi < \xi'$ *and* $l(\xi') \leqq Kl(\xi)$,

(iii)

$$\int_{[0,1]-E} \frac{d(\xi + c(\xi))}{l(\xi)} < \infty,$$

where it is regarded that $dc(\xi) \equiv 0$ *on the interval on which* $c(\xi) \equiv \infty$, *then* $\Gamma(E, \theta, c)$ *is unstable.*

Proof is completely similar to that of Theorem 2. We shall just indicate the estimation of the modulus of the quadrilateral Q defined by the domain bounded by $C_\xi = \{z;\ |z + c(\xi)| = \xi + c(\xi)\}$, $C_{\xi'} = \{z;\ |z + c(\xi')| = \xi' + c(\xi')\}$, $\{z;\ \arg z = l(\xi)/(\xi + c(\xi))\}$, and $\{z;\ \arg z = -l(\xi)/(\xi + c(\xi))\}$ where $\xi < \xi'$ and $c(\xi') < \infty$. Map the interior of $C_{\xi'}$ onto $|\zeta| < 1$ by a linear transformation which maps C_ξ onto the circle $|\zeta| = a < 1$. An elementary estimation of non-euclidean quantities shows that $a > (\xi + c(\xi))/(\xi' + c(\xi'))$ and that the image of Q contains $\{\zeta;\ 1/a < |\zeta| < 1,\ |\arg \zeta| < l(\xi)/(\xi + c(\xi))\}$. We conclude that

$$\text{mod } Q \leqq \frac{\xi + c(\xi)}{l(\xi)} \log \frac{\xi' + c(\xi')}{\xi + c(\xi)} \leqq \frac{\xi' - \xi + c(\xi') - c(\xi)}{l(\xi)} \leqq K \int_\xi^{\xi'} \frac{d(\xi + c(\xi))}{l(\xi)}$$

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