ON THE NON-COMMUTATIVITY OF PONTRJAGIN RINGS MOD 3 OF SOME COMPACT EXCEPTIONAL GROUPS

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Introduction. Pontrjagin rings over the field of rational numbers of compact Lie groups are commutative in the sense of graded algebras (or anti-commutative in the classical terminology) [14]. Pontrjagin rings over the field Z_p $(p \neq 0)$ of several compact simple Lie groups were studied by Borel [5]. The most examples are commutative. However, this is generally not true.

The first example of non-commutative Pontrjagin rings of compact Lie groups is $H_*(\text{Spin}\ (10)\,;\ Z_2)$, [5], [1]. Then it was shown that the Pontrjagin rings $H_*(\text{Spin}\ (n)\,;\ Z_2)$, $n \geqq 10$ and $n \neq 2^s+1$, are non-commutative, [11], [18]. These known examples are all those over Z_2.

In this work we prove that the Pontrjagin rings mod 3 of all compact exceptional groups except G_2 are not commutative.

We denote by F_4, E_6, E_7 and E_8 the compact, connected and simply-connected groups among the local structures usually expressed by these notations.

In Chapter I we determine the ring structure of the Pontrjagin ring $H_*(F_4\,;\ Z_3)$. This is shown to be non-commutative. The proof is based on a theorem of Kudo [12] on one hand, and on the scheme used by Borel [5] to determine the Pontrjagin ring $H_*(\text{Spin}\ (10)\,;\ Z_2)$ on the other hand.

In Chapter II we discuss the homology maps $H_*(\Omega F_4\,;\ Z) \to H_*(\Omega E_6\,;\ Z)$ in degrees $\leqq 10$, $H_*(\Omega E_6\,;\ Z) \to H_*(\Omega E_7\,;\ Z)$ in degrees $\leqq 10$, and $H_*(\Omega E_7\,;\ Z) \to H_*(\Omega E_8\,;\ Z)$ in degrees $\leqq 14$, induced by the inclusions $F_4 \subset E_6 \subset E_7 \subset E_8$. In the description of these homology maps we use K-cycles due to Bott-Samelson [10].

In Chapter III we prove that the Pontrjagin rings mod 3 of E_6, $\text{Ad}\,E_6$, E_7 and E_8 are non-commutative using the results of Chapters I and II.

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Chapter I. The Pontrjagin ring $H_*(F_4;\ Z_3)$

§1. Kudo theorem and the comparison theorem of spectral sequences

1. Let $\{E_r,\ r\geqq 0\}$ be the cohomology spectral sequence of Serre [15] over Z_p (p is an odd prime) associated with a fibration. About terminologies and notations of cohomology spectral sequences we refer to [4], [15]. Kudo theorem on transgressions [12] is stated as follows:

THEOREM K. *Let $\alpha\in E_2^{o,2k}$ $(k>0)$ be transgressive. Choose a suitable representative $\beta\in E_2^{2k+1,o}$ of $\tau(\alpha)$, then*

i) *α^p and $\beta\cdot\alpha^{p-1}$ are transgressive,*

ii) *$\mathscr{P}_p^k\beta$ and $-\delta_p^*\mathscr{P}_p^k\beta\in E_2^{*o}$, represents $\tau(\alpha^p)$ and $\tau(\beta\cdot\alpha^{p-1})$ respectively, where τ denotes the transgression*

$$d_{b+1}\kappa_{b+1}^2:\ (\kappa_{b+1}^2)^{-1}(E_{b+1}^{a,b})\rightarrow E_{b+1}^{a+b+1,o}$$

for each (a,b), $\mathscr{P}_p^k$ a cyclic p-th reduced power and δ_p^* the Bockstein operation mod p.

2. Let $\{E_r;\ r\geqq 2\}$ and $\{'E_r;\ r\geqq 2\}$ be canonical cohomology spectral sequences ([4], p. 122) of modules (or of algebras) such that $E_2^{a,b}\approx E_2^{a,o}\otimes E_2^{o,b}$ and $'E_2^{a,b}\approx {}'E_2^{a,o}\otimes{}'E_2^{o,b}$ for all $(a,b)\geqq 0$. Let

$$\{h_r,\ r\geqq 2\}:\{E_r;\ r\geqq 2\}\rightarrow\{'E_r;\ r\geqq 2\}$$

be a homomorphism of the spectral sequences such that

$$h_2^{a,b}\approx h_2^{a,o}\otimes h_2^{o,b}\qquad\text{for all }(a,b)\geqq 0.$$

The comparison theorem of spectral sequences ([13], p. 110) has three statements. Among them the one we use later is the following:

THEOREM CSS. *Let $h_2^{o,b}$ be isomorphic for all $b\geqq 0$ and $h_\infty^{a,b}$ be isomorphic for all $(a,b)\geqq 0$. Then $h_2^{a,O}$ is isomorphic for all $a\geqq 0$.*

The conditions and the conclusion of this theorem can be weakened as in the following:

THEOREM WCSS. *Let $h_2^{o,b}$ be isomorphic for $0\leqq b\leqq n$ and $h_\infty^{a,b}$ be isomorphic for $0\leqq a+b\leqq n+1$. Then $h_2^{a,o}$ is isomorphic for $0\leqq a\leqq n+1$.*

This theorem is proved in the same way with the proof of the Theorem CSS so that the proof is omitted.

§ 2. Universal cohomology spectral sequence of F_4 over Z_3

1. By [5] we know that

$$H^*(F_4;\ Z_3) = \Lambda_3(x_3,\ x_7,\ x_{11},\ x_{15}) \otimes Z_3[x_8]/(x_8^3)$$

with the following relations of reduced powers and Bockstein operations

$$(1) \qquad x_7 = \mathscr{P}_3^1 x_3, \qquad x_{15} = \mathscr{P}_3^1 x_{11}, \qquad x_8 = \delta_3^* x_7,$$

where suffixes denote degrees.

PROPOSITION 1. *$H^*(F_4;\ Z_3)$ has no system of universally transgressive generators.*

Proof. Assume that $H^*(F_4;\ Z_3)$ has a system of universally transgressive generators: $x_3,\ x_7,\ x_8,\ x_{11},\ x_{15}$. We may regard the relation (1) as holding for these generators.

Denoting by τ the transgression in the universal spectral sequence of F_4 over Z_3, by the Theorem K we can choose representatives

$$H^{i+1}(B_{F_4};\ Z_3) \ni y_{i+1} \quad \text{of} \quad \tau(x_i) \qquad (i = 3,\ 7,\ 8,\ 11,\ 15)$$

and

$$H^{26}(B_{F_4};\ Z_2) \ni y_{26} \quad \text{of} \quad \tau(y_9 \otimes x_8^2)$$

satisfying

$$(2) \qquad y_9 = -\delta_3^* y_8 \quad \text{and} \quad y_{26} = -\delta_3^* \mathscr{P}_3^4 y_9.$$

Construct a graded algebra

$$B = Z_3[y'_4,\ y'_8,\ y'_{12},\ y'_{16},\ y'_{26}] \otimes \Lambda_3(y'_9)$$

and a spectral sequence over Z_3 which starts from

$${}'E_2 = {}'E_2^{*,0} \otimes {}'E_2^{0,*}$$

with

$${}'E_2^{*,0} = B \quad \text{and} \quad {}'E_2^{0,*} = H^*(F_4;\ Z_3).$$

Successive terms of this spectral sequence is defined as follows:

$$d_2 = d_3 = 0,\quad {}'E_2 = {}'E_3 = {}'E_4;$$
$$d_4 \kappa_4^2 x_3 = \kappa_4^2 y'_4,\quad d_4 \kappa_4^2 x_i = 0 \qquad \text{for } i > 3,$$

whence ${}'E_5 = H({}'E_4) = B_1 \otimes C_1$ where $B_1 = B/Z_3[y'_4]$ and $C_1 = H^*(F_4;\ Z_3)/\!/\Lambda_3(x_3)$ ($/\!/$ denotes a symbol to *take away* the tensor factor indicated thereafter. For the strict definition of this symbol as a kind of quotient, cf., J. Milnor and J.

Moore, "On the structure of Hopf algebras", to appear);

$$d_5 = d_6 = d_7 = 0,\ 'E_5 = 'E_6 = 'E_7 = 'E_8\,;$$
$$d_8 \kappa_8^2 x_7 = \kappa_8^2 y_8',\quad d_8 \kappa_8^2 x_i = 0 \qquad \text{for } i > 7,$$

whence $'E_9 = B_2 \otimes C_2$ where $B_2 = B_1 /\!\!/ Z_3[y_8']$ and $C_3 = C_1 /\!\!/ \Lambda_3(x_7)$;

$$d_9 \kappa_9^2 x_8 = \kappa_9^2 y_9' \text{ and } d_9 \kappa_9^2 x_i = 0 \qquad \text{for } i > 8,$$

whence $'E_{10} = B_3 \otimes C_3 \otimes \Lambda_3(x_{25}')$ where $B_3 = B_2 /\!\!/ \Lambda_3(y_9')$, $C_3 = C_2 /\!\!/ Z_3[x_8]/(x_8^3)$ and $x_{25}' = \kappa_{10}^2(y_9' \otimes x_8^2)$;

$$d_{10} = d_{11} = 0,\ 'E_{10} = 'E_{11} = 'E_{12}\,;$$
$$d_{12} \kappa_{12}^2 x_{11} = \kappa_{12}^2 y_{12}',\quad d_{12} \kappa_{12}^2 x_{15} = 0 \quad \text{and} \quad d_{12} \kappa_{12}^{10} x_{25}' = 0,$$

whence $'E_{13} = B_4 \otimes C_4 \otimes \Lambda_3(\kappa_{13}^{10} x_{25}')$ where $B_4 = B_3 /\!\!/ Z_3[y_{12}']$ and $C_4 = C_3 /\!\!/ \Lambda_3(x_{11})$;

$$d_{13} = d_{14} = d_{15} = 0,\ 'E_{13} = 'E_{14} = 'E_{15} = 'E_{16}\,;$$
$$d_{16} \kappa_{16}^2 x_{15} = \kappa_{16}^2 y_{16}' \quad \text{and} \quad d_{16} \kappa_{16}^{10} x_{25}' = 0,$$

whence $'E_{17} = Z_3[\kappa_{17}^2 y_{26}] \otimes \Lambda_3(\kappa_{17}^{10} x_{25}')$;

$$d_{17} \kappa_{17}^{10} x_{25}' = \kappa_{17}^2 y_{26},$$

whence $'E_{18} \approx Z_3$ (trivial); finally

$$d_r = 0 \text{ for } r > 17,$$
$$'E_{18} = 'E_{19} = \cdots = 'E_\infty \approx Z_3 \text{ (trivial)}.$$

Denoting by $\{E_r\}$ the universal spectral sequence of F_4 over Z_3, next we define a homomorphism of spectral sequences

$$\{h_r,\ r \geqq 2\} : \{'E_r,\ r \geqq 2\} \to \{E_r,\ r \geqq 2\}$$

in the following way:

$$E_2 = H^*(B_{F_4};\ Z_3) \otimes H^*(F_4;\ Z_3)$$

canonically. Then, h_2 is defined by

$$h_2 y_i' = y_i \quad \text{and} \quad h_2 x_i = x_i.$$

The choices of x_i, y_i and the construction of $'E_r$ allows us the successive definition of h_r; namely h_2 commutes with d_2 and induces the homomorphism h_3, h_3 commutes with d_3 and induces the homomorphism h_1, and so on.

Now, since $h_\infty : 'E_\infty \approx E_\infty$ and $h_2^{0,*} : 'E_2^{0,*} = E_2^{0,*}$ an identity map, by Theorem CSS we can conclude that

(3) $$h_2^{*,o} : H^*(B_{F_4};\ Z_3) \approx B.$$

In particular $y_{26} \neq 0$ and this is the δ_3^*-image of $-\mathscr{P}^4 y_9$ by (2). Since B contains only one odd degree generator y_9 and deg $\mathscr{P}^4 y_9 = 25$ is odd, we have

(4) $$-\mathscr{P}^4 y_9 = y_9 a_{16}$$

with $a_{16} \in H^{16}(BF_4;\ Z_3)$. Consider $\delta_3^* a_{16} = b$. b has odd degree so that $b = y_9 c$ with $c \in H^8(B_{F_4};\ Z_3)$ by the same reason as above. Then, applying δ_3^* on both sides of (4) we have

$$y_{26} = \delta_3^*(y_9 a_{16}) = -y_9 \delta_3^* a_{16} = -y_9 y_9 c = 0.$$

This contradicts to (3). Therefore $H^*(F_4;\ Z_3)$ has no system of universally transgressive generators. (q.e.d.)

2. *Remark.* The same discussion can be applied to many other compact Lie groups G for which p-torsion (p odd prime) exist and the cohomology ring $H^*(G;\ Z_p)$ is known. For example,

$$H^*(PU(p);\ Z_p)\ (p \neq 2),\quad H^*(E_6;\ Z_3),$$
$$H^*(E_7;\ Z_3),\quad H^*(E_8;\ Z_3),\quad H^*(E_8;\ Z_5)$$

have no system of universally transgressive generators. These cohomology rings are determined in [3; 5; 6]. Here we mention only that these cohomology rings contain only one (or two in case $H^*(E_8;\ Z_3)$) generator of even degree with height p.

But this property does not imply immediately that the "Pontrjagin ring $H_*(G;\ Z_p)$ is not commutative", as examples of Kojima [11] $H_*(\mathrm{Spin}(2^s+1);\ Z_2)$ $(s \geqq 4)$ show it.

3. By (1) and the Prop. 1 we can determine the behaviors of generators of $H^*(F_4;\ Z_3)$ in the universal spectral sequence.

LEMMA 1. *$H^*(B_{F_4};\ Z_3) = Z_3[y_4, y_8] \otimes \Lambda_3(y_9)$ in deg. $\leqq 11$ with relations $y_4 = \tau(x_3)$, $y_8 = \mathscr{P}^1 y_4$ and $y_9 = \delta_3^* y_8$.*

Proof. By (1) we see immediately that x_3, x_7, x_8 are universally transgressive because x_3 is a generator with the lowest degree and $\mathscr{P}^1$ and δ_3^* commutes with the transgression. Then a similar construction with that in the proof of the Prop. 1 and Theorem WCSS prove this lemma. (q.e.d.)

PROPOSITION 2. *In the universal spectral sequence* $\{E_r\}$ *of* F_4 *over* Z_3 *the behaviors of generators of* $H^*(F_4;\ Z_3)$ *are as follows:* x_3, x_7, x_8 *are universally transgressive;* x_{11} *and* x_{15} *cannot be chosen to be universally transgressive and they can be chosen such that*

$$d_i\kappa_i^2 x_{11} = d_i\kappa_i^2 x_{15} = 0 \qquad \textit{for } 2 \leqq i < 9,$$
$$d_9\kappa_9^2 x_{11} = \kappa_9^2(y_9 \otimes x_3) \neq 0,$$
$$d_9\kappa_9^2 x_{15} = \kappa_9^2(y_9 \otimes x_7) \neq 0.$$

Proof. By adding some decomposable elements we can change the generator x_{11} to satisfy

$$d_i\kappa_i^2 x_{11} = 0 \qquad \text{for } 2 \leqq i \leqq 8.$$

If $d_9\kappa_9^2 x_{11} = 0$, then the generators x_{11} and $x_{15} = \mathscr{P}^1 x_{11}$ must be universally transgressive. This contradicts to the Prop. 1. Therefore

$$E_9^{9,3} \ni d_9\kappa_9^2 x_{11} \neq 0.$$

On the other hand $E_2^{9,3}$ is 1-dimensional and generated by $y_9 \otimes x_3$. Hence $y_9 \otimes x_3$ is d_r-cocycles for all $r \geqq 2$, $y_4 y_9 = 0$ and

$$d_9\kappa_9^2 x_{11} = \kappa_9^2(y_9 \otimes x_3)$$

after changing the coefficient of x_{11} suitably. Then $x_{15} = \mathscr{P}^1 x_{11}$ (the changed generator of deg. 15) is d_r-cocycles for $2 \leqq r \leqq 8$ and

$$d_9\kappa_9^2 x_{15} = \kappa_9^2(y_9 \otimes \mathscr{P}^1 x_3) = \kappa_9^2(y_9 \otimes x_7)$$

by some properties of reduced powers in spectral sequences [2] (i e., relations with d_r and the Cartan formula). This implies that $y_8 y_9 = 0$. Then $d_9\kappa_9^2 x_{15} \neq 0$.

(q.e.d.)

By this proposition we can discuss the universal spectral sequence of F_4 over Z_3 in low degrees immediately, and we have

COROLLARY. $H^*(B_{F_4};\ Z_3) = Z_3[y_4,\ y_8] \otimes \Lambda_3(y_9)/(y_4y_9,\ y_8y_9)$ *in deg.* $\leqq 19$.

§3. The coproduct in $H^*(F_4;\ Z_3)$ and the Pontrjagin ring $H_*(F_4;\ Z_3)$

1. For any connected compact Lie group G the group multiplication $h: G \times G \to G$ induces the coproduct

$$h^*: H^*(G;\ Z_p) \to H^*(G;\ Z_p) \otimes H^*(G;\ Z_p)$$

for any prime p. For a principal G-bundle $(E,\ B,\ \pi,\ G)$ the right translation

$h : E \times G \to E$ induces a homomorphism

$$h^* : E_r \to E_r \otimes H^*(G; Z_p) \qquad (r \geqq 2)$$

of associated spectral sequences over Z_p [4], p. 174. The operation of $u \in H_s(G; Z_p)$ on $H^*(G; Z_p)$ of deg $-s$ or on E_r of deg $(0, -s)$,

$$\vartheta_u x = \sum < y_i, \ u > x_i$$

where $x \in H^*(G; Z_p)$ or $\in E_r$ and $h^* x = \sum x_i \otimes y_i$, was first defined by Leray and then used by Borel [5] for the study of the coproduct in $H^*(\mathrm{Spin}(10); Z_2)$.

ϑ_u in the spectral sequence of a principal G-bundle has the following properties [5]:

i) ϑ_u commutes with d_r and κ_t^r,

ii) in $E_2 = H^*(B; Z_p) \otimes H^*(G; Z_p)$ we have $\vartheta_u(b \otimes x) = b \otimes \vartheta_u x$ for $b \in H^*(B; Z_p)$ and $x \in H^*(G; Z_p)$.

2. THEOREM 1. *For generators of $H^*(F_4; Z_3)$ which behave in the universal spectral sequence as stated in the Prop. 2, their coproducts are as follows:*

a) x_3, x_7, x_8 *are primitive,*

b) $h^* x_{11} = 1 \otimes x_{11} + x_{11} \otimes 1 + x_3 \otimes x_3$,

c) $h^* x_{15} = 1 \otimes x_{15} + x_{15} \otimes 1 + x_8 \otimes x_7$.

Proof. a) is immediate by the Prop. 20.1 of [4].

Let u_i be the dual class of x_i and ϑ_i be the operation associated with u_i $(i = 3, 8)$. Then

$$h^* x_{11} = 1 \otimes x_{11} + x_{11} \otimes 1 + \vartheta_3 x_{11} \otimes x_3 + \vartheta_8 x_{11} \otimes x_8.$$

By the Prop. 2 and the property i) of ϑ_u we see that $\vartheta_8 x_{11}$ is a permanent element. Therefore

$$\vartheta_8 x_{11} = 0$$

since universal spectral sequences have no permanent elements except zero. By the Prop. 2 and the properties i) and ii) of ϑ_u we see that $\vartheta_3 x_{11}$ is d_r-cocycles for $2 \leqq r \leqq 8$, and

$$d_9 \kappa_9^2 \vartheta_3 x_{11} = \kappa_9^2 y_9 = d_9 \kappa_9^2 x_8.$$

Therefore $\vartheta_3 x_{11} - x_8$ is permanent. Hence

$$\vartheta_3 x_{11} = x_8.$$

And b) is proved.

c) is obtained by applying $\mathscr{P}^1$ on both sides of b) since $\mathscr{P}^1 x_{11} = x_{15}$, $\mathscr{P}^1 x_3 = x_7$ and $\mathscr{P}^1 x_8 = 0$. (q.e.d.)

3. Since the generators x_i, $i = 3, 7, 8, 11, 15$, form a 3-simple system of generators of $H^*(F_4; Z_3)$ by a terminology of [5],

$$x_{i_1} \cdots x_{i_r}, \quad x_{i_1} \cdots x_{i_r} x_8, \quad x_{i_1} \cdots x_{i_r} x_8^2$$

form an additive basis of $H^*(F_4; Z_3)$ where $\{i_1, \ldots, i_r\}$ are subsequences of $\{3, 7, 11, 15\}$. We denote by $v_{i_1,\ldots,i_r}$, $v_{i_1,\ldots,i_r;1}$, $v_{i_1,\ldots,i_r;2}$ the dual basis in homology. We use the symbol $[v_i, v_j]$ to denote $v_i \vee v_j - (-1)^{ij} v_j \vee v_i$.

THEOREM 2. *The Pontrjagin ring $H_*(F_4; Z_3)$ is non-commutative and has a 3-simple system of generators v_3, v_7, v_8, v_{11}, v_{15} satisfying the following relations:*

$$v_i \vee v_i = 0 \text{ for } i \neq 8, \quad v_8 \vee v_8 \vee v_8 = 0$$
$$[v_i, v_j] = 0 \text{ for } i < j \text{ and } (i, j) \neq (3, 8), (7, 8),$$
$$[v_8, v_3] = v_{11}, \quad [v_8, v_7] = v_{15}.$$

Proof. Every $y \in H^*(F_4; Z_3)$ can be written as

$$y = x + a_1 x_{11} + a_2 x_{15} + a_3 x_{11} x_{15}$$

with x, $a_i (1 \leqq i \leqq 3) \in \Lambda_3(x_3, x_7) \otimes Z_3[x_8]/(x_8^3)$. If we put

$$h^*(a_1 x_{11}) = h^*(a_1)(x_{11} \otimes 1 + 1 \otimes x_{11}) + b_1,$$
$$h^*(a_2 x_{15}) = h^*(a_2)(x_{15} \otimes 1 + 1 \otimes x_{15}) + b_2,$$
$$h^*(a_3 x_{11} x_{15}) = h^*(a_3)(x_{11} \otimes 1 + 1 \otimes x_{11})(x_{15} \otimes 1 + 1 \otimes x_{15}) + b_3,$$

then $h^*(y) - \sum b_j$ is symmetric in the sense of [5], p. 283. In case y is monomial, put $y = x_{i_1} \cdots x_{i_r} x_8^{\varepsilon}$ $(0 \leqq \varepsilon \leqq 2)$, then

$$h^*(y) - \sum b_j = \prod (x_{i_s} \otimes 1 + 1 \otimes x_{i_s}) \cdot (x_8 \otimes 1 + 1 \otimes x_8)^{\varepsilon}. \tag{5}$$

If we write b_j explicitly like $b_1 = h^*(a_1) \cdot (x_8 \otimes x_3)$, etc., then we see easily that

$$\begin{aligned} &< b_j, v_{i_1,\ldots,i_{r-1}} \otimes v_{i_r} > = 0 \\ &< b_j, v_{i_1,\ldots,i_r} \otimes v_8 > = 0 \\ &< b_j, v_{i_1,\ldots,i_r;1} \otimes v_8 > = 0 \end{aligned} \tag{6}$$

for any subsequence $\{i_1, \ldots, i_r\} \subset \{3, 7, 11, 15\}$ and $j = 1, 2, 3$, and that

$$
(7) \qquad
\begin{aligned}
&< b_1,\ v_i \otimes v_j > \ = 0 && \text{if } (i,\ j) \neq (8,\ 3), \\
&< b_2,\ v_i \otimes v_j > \ = 0 && \text{if } (i,\ j) \neq (8,\ 7), \\
&< b_3,\ v_i \otimes v_j > \ = 0 && \text{for all } (i,\ j).
\end{aligned}
$$

Using (5) and (6) we can see that

$$
\begin{aligned}
&v_{i_1, \dots, i_r} = v_{i_1, \dots, i_{r-1}} \vee v_{i_r} \\
&v_{i_1, \dots, i_r; 1} = v_{i_1, \dots, i_r} \vee v_8 \\
&v_{i_1, \dots, i_r; 2} = 2\, v_{i_1, \dots, i_r; 1} \vee v_8,
\end{aligned}
$$

whence by an induction on r we have

$$
\begin{aligned}
&v_{i_1, \dots, i_r} = v_{i_1} \vee \cdots \vee v_{i_r} \\
&v_{i_1, \dots, i_r; 1} = v_{i_1} \vee \cdots \vee v_{i_r} \vee v_8 \\
&v_{i_1, \dots, i_r; 2} = 2 v_{i_1} \vee \cdots \vee v_{i_r} \vee v_8 \vee v_8
\end{aligned}
$$

for any $\{i_1, \dots, i_r\} \subset \{3, 7, 11, 15\}$. Hence v_i, $i = 3, 7, 11, 15, 8$, form a 3-simple system of generators.

(7) and the fact that $h^*(y) - \sum b_j$ is symmetric show that

$$[v_i, v_j] = 0 \text{ for } (i,\ j) \neq (8,\ 3),\ (8,\ 7) \text{ and } i \geqq j.$$

In particular we have $v_i \vee v_i = 0$ for i odd.

The proofs of the last two relations are entirely the same with the corresponding part of the proof of [5], Théorème 16.4, and are omitted. (q.e.d.)

Chapter II. On some homology relations of loop spaces of compact exceptional groups

§ 1. Preliminaries

1. Let K be a compact connected and simply connected Lie group. Bott-Samelson [10] described a homology basis of $H_*(\Omega K;\ Z)$ (ΩK is the loop space of K) by making use of K-cycles.

Let $T \subset K$ be a fixed maximal torus in K, R be the universal covering group of T and $\eta : R \to T$ be the covering map. Further we denote by D the diagram defined on R.

By a singular plane of codimension 1 in D we mean a pair $(\theta,\ n)$ of a root form θ and an integer n. It is oriented and $(\theta,\ n)$ and $(-\theta,\ -n)$ are distinguished. Let $P = \{p_1, \dots, p_k\}$ be a finite ordered set of singular planes in D of codimension 1. We use the following notations due to [10]: $\bar{p}_i = \eta(p_i)$,

$K(p_i)$ = the connected centralizer of $\cdot \bar{p}_i$, $W(P) = \prod_1^k K(p_i)$. Then the K-cycle $\Gamma(P)$ associated with P is defined by $\Gamma(P) = W(P)/T^k$ where the right translations of T^k on $W(P)$ are defined by

$$(x_1, \ldots, x_k) \cdot (t_1, \ldots, t_k) = (x_1 t_1, t_1^{-1} x_1 t_2, \ldots, t_{k-1}^{-1} x_k t_k)$$

for $(x_1, \ldots, x_k) \in W(P)$ and $(t_1, \ldots, t_k) \in T^k$.

Later we need K-cycles $\Gamma(P)$ without the restriction of codim $p_i = 1$ to obtain some relations between homology classes in $H_*(\Omega K; Z)$ expressed by K-cycles. They are called here the general K-cycles to distinguish from the original K-cycles.

The K-cycle $\Gamma(P)$ admits an iterated fibration with cross-sections. The base space is $K(p_1)/T$ and the successive fibres are $K(p_2)/T, \ldots, K(p_k)/T$. $K(p_i)/T$ are orientable, whence $\Gamma(P)$ is orientable. If codim $p_i = 1$, then $K(p_i)/T$ is a 2-sphere and has a canonical orientation determined by the root vector of p_i as in [10], Chap. II, § 3. Hence the original K-cycle has a natural orientation. The orientation of a general K-cycle $\Gamma(P)$ is determined depending on the choice of the orientations of $K(p_i)/T$ with codim $p_i \geqq 2$.

2. Let $P' = \{p_{i_1}, \ldots, p_{i_r}\}$ be a subsequence of P. We imbed $\Gamma(P')$ as a submanifold of $\Gamma(P)$. Let $i : W(P') \to W(P)$ be an inclusion defined by

$$\begin{aligned} \pi_t \circ i(x_1, \ldots, x_r) &= x_t \qquad \text{if } t \in \{i_1, \ldots, i_r\} \\ &= e \qquad \text{if } t \notin \{i_1, \ldots, i_r\} \end{aligned}$$

for $(x_1, \ldots, x_r) \in W(P')$, where $\pi_t : W(P) \to K(p_t)$ is the natural projection onto the t-th factor and e is the neutral element of the group. Let $h : T^r \to T^k$ be a homomorphism defined by

$$\begin{aligned} \pi_t \circ h(t_1, \ldots, t_r) &= e \qquad \text{if } t < i_1 \\ &= t_1 \qquad \text{if } i_1 \leqq t < i_2 \\ &\cdots\cdots\cdots\cdots \\ &= t_s \qquad \text{if } i_s \leqq t < i_{s+1} \text{ for } s < r \\ &= t_r \qquad \text{if } i_r \leqq t \end{aligned}$$

for $(t_1, \ldots, t_r) \in T^r$ where $\pi_t : T^k \to T$ is the natural projection onto the t-th factor. Then i is clearly a boundle map relative to the homomorphism h and induces an injection

$$\bar{i} : \Gamma(P') \subset \Gamma(P).$$

This imbedding may be considered as a canonical one, and $\Gamma(P')$, oriented naturally or in a suitable way, gives an integral cycle in $\Gamma(P)$. The submanifold $\Gamma(P')$ is called a sub K-cycle in $\Gamma(P)$.

3. $H_*(\Gamma(P)\,;\,Z)$ has no torsion, and in case $\Gamma(P)$ being an original one Bott-Samelson [10], Chap. II, Prop. 4.2, determined the cohomology ring $H^*(\Gamma(P)\,;\,Z)$. Sub K-cycles $\Gamma(p_i)$ in $\Gamma(P)$ associated with a single plane p_i $(1 \leqq i \leqq k)$ form a basis of $H_2(\Gamma(P)\,;\,Z)$. Let $\{x_i\}$ be the dual basis of $H^2(\Gamma(P)\,;\,Z)$. Then by [10], Chap. II, Prop. 4.2, we see easily that $x_{j_1}\cdots x_{j_s}$, $j_1<j_2<\cdots<j_s$, form an additive basis of $H^*(\Gamma(P)\,;\,Z)$. On the other hand $\bar{i}^*\ (x_{j_1}\cdots x_{j_s})=$ the top dimensional generator of $H^*(\Gamma(P')\,;\,Z)$ or zero according as $\{j_1, \ldots, j_s\}=\{i_1, \ldots, i_r\}$ or not. From these we see easily that sub K-cycles in $\Gamma(P)$ form an additive basis of $H_*(\Gamma(P)\,;\,Z)$ when $\Gamma(P)$ is an original one. In case of a general K-cycle the description of the homology basis is more involved.

4. For any K-cycle $\Gamma(P)$ there are associated a homology class P_* in $H_*(\Omega K;\,Z)$ [9, 10]. For the definition of a chain $c=\{c_0, c_1, \ldots, c_k\}$ subject to P we refer to [9], §3. For any chanin c subject to P we have a map

$$f_P^c : \Gamma(P) \to \Omega K.$$

P_* is defined by $P_*=f^c_{P*}$(the fundamental class of $\Gamma(P)$). P_* is determined independently of c. These constructions are valid also for general K-cycles. The only thing we should take care of is the orientation of $\Gamma(P)$ as we mentioned already.

5. Let $P=\{p_2, \ldots, p_k\}$ be an ordered set of $k-1$ singular planes of condimension 1 in D. Let θ_1 and θ_2 be two roots of K such that they have the same length and $(\theta_1, \theta_2)<0$. Then $\theta_3=\theta_1+\theta_2$ is a root of K. Let n be an integer and we put $q_1=(\theta_1, n)$, $q_2=(\theta_2, 0)$, $q_3=(\theta_3, n)$ and $p_1=q_1\cap q_2$. Then q_1, q_2 and q_3 contain p_1. Put $P'=\{p_1, P\}$. $\Gamma(P')$ is a general K-cycle containing $\Gamma(P)$ as a sub K-cycle. Put $P_i=\{q_i, P\}$ $(i=1, 2, 3)$. The inclusions $K(q_i)\subset K(p_1)$ induce inclusions $W(P_i)\subset W(P')$, which are bundle maps and induce in turn imbeddings

$$\Gamma(P_i)\subset \Gamma(P') \qquad \text{for each } i=1,\ 2,\ 3.$$

The semi-simple part of $K(p_1)$ is of type A_2 and is denoted by A_2, which have $\pm\theta_i$ $(i=1, 2, 3)$ as roots. $A_2/T' \approx K(p_1)/T$ canonically where $T'=A_2\cap T$ is a maximal torus in A_2. Let τ_i $(i=1, 2, 3)$ be root vectors associated with θ_i in the sense of Stiefel [17], i.e., τ_i is perpendicular to θ_i and $\theta_i(\tau_i)=2$. τ_i defines an integral cycle in $H_1(T')$ with a relation $\tau_3=\tau_1+\tau_2$. 2-spheres $K(q_i)/T$ in A_2/T' represent 2-cycles y_i of $H_2(A_2/T')$ such that their homology transgression images in the fibration $(A_2, A_2/T', T')$ are τ_i. Since this transgression defined on $H_2(A_2/T')$ is injective, we have the relation

$$(1)\qquad y_3=y_1+y_1.$$

$\Gamma(P')$ is fibred with $\Gamma(P)$ as its fibre and with A_3/T' as its base space. The associated homology spectral sequence over Z is collapsed because the fibre and the base space have trivial homology groups in odd degrees. The integral cycle $\Gamma(P_i)$ represents $y_i\otimes P_*$ in the E_∞ term for each $i=1, 2, 3$, where P_* denotes the fundamental class of the homology group of the fibre. (1) implies that

$$(1')\qquad y_3\otimes P_*=y_1\otimes P_*+y_2\otimes P_* \qquad \text{in } E_\infty.$$

Since P_* is a homology class of the fibre with the highest degree, (1′) implies in turn that

$$(2)\qquad \Gamma(P_3)_*=\Gamma(P_1)_*+\Gamma(P_2)_*$$

where $\Gamma(P_i)_*$ denote the classes in $H_*(\Gamma(P'); Z)$ represented by the integral cycles $\Gamma(P_i)$ in $\Gamma(P')$.

PROPOSITION 1. *P_1, P_2 and P_3 are defined as above, then*

$$P_{3*}=P_{1*}+P_{2*} \qquad \textit{in } H_{2k}(\Omega K; Z).$$

Proof. Let c be a chain subject to P'. Then c is also subject to P_i, $i=1$, 2, 3. And

$$f^c_{P_i}=f^c_{P'}\mid P_i \qquad \text{for each } i=1, 2, 3.$$

Then (2) and the definition of P_{i*} imply the conclusion of this proposition.

(q.e.d.)

6. Let W be the Weyl group of K operated in R or in T. An element $w\in W$ transforms an ordered set $P=\{p_1, \ldots, p_k\}$ of singular planes to $wP=\{wp_1, \ldots, wp_k\}$.

Let a be a representative of w in the normalizer $N(T)$ of T. The inner automorphism $\varphi_a : K \to K$ defined by a, maps $K(p_i)$ onto $K(wp_i)$ as is easily seen, and induces $\overline{\varphi}_a$: $W(P) \to W(wP)$ defined by $\overline{\varphi}_a = \varphi_a \times \cdots \times \varphi_a$. This is a bundle map relative to a homomorphism $\overline{w} : T^k \to T^k$ defined by the diagonal action of w, whence we have an induced map

$$\varphi_{a\#} : \Gamma(P) \to \Gamma(wP).$$

$\varphi_{a*} : H_*(\Gamma(P)) \to H_*(\Gamma(wP))$ depends only on w and does not depend on the choice of a. The inner automorphism φ_a induces a map $\psi_a : \Omega K \to \Omega K$. Since K is connected, ψ_a is homotopic to the identity map and $\psi_{a*} : H(\Omega K) \to H(\Omega K)$ is equal to the identity.

Let c be a chain subject to P, then wc is a chain subject to wP. Now the following diagram

$$\begin{array}{ccc} \Gamma(P) & \xrightarrow{f_P^c} & \Omega K \\ {\scriptstyle\varphi_{a\#}}\downarrow & & \downarrow{\scriptstyle\psi_a} \\ \Gamma(wP) & \xrightarrow{f_{wP}^{wc}} & \Omega K \end{array}$$

is clearly commutative, whence we have the

PROPOSITION 2. $P_* = (wP)_*$ *for any* $w \in W$.

In case $\Gamma(P)$ is a general K-cycle we must take care of the orientations of $\Gamma(P)$ and $\Gamma(wP)$ in the above proposition.

7. Let $P = \{p_1, \ldots, p_k\}$ be an ordered set of singular planes with codim $p_j = 1$ except a plane p_i. Assume that p_i has codimension 2 and $p_i = (\theta_1, n) \cap (\theta_2, o)$ such that θ_1 and θ_2 have the same length and $(\theta_1, \theta_2) < 0$. Then the semisimple part of $K(p_i)$ is of type A_2 and is denoted by A_2 as in No. 1.5. Put $\theta_3 = \theta_1 + \theta_2$, then the plane (θ_3, n) contains p_i. The Weyl group $W(A_2)$ of A_2 is a subgroup of the Weyl group $W(K)$ of K. Let $R_2 \in W(A_2)$ be the reflection across $(\theta_2, 0)$. R_2 maps (θ_1, n) onto (θ_3, n) and vice versa. Hence R_2 keeps p_i invariant.

If we fix an orientation of $K(p_i)/T$, then the orientation of $\Gamma(P)$ as well as the orientation of $\Gamma(R_2 P)$ is determined. Now R_2 reverses the orientation of T, whereas φ_a preserves the orientation of $K(p_i)$ where φ_a is an inner automorphism of $K(p_i)$ defined by a representative a of R_2 in $N(T \cap A_2)$. Hence the induced map $\varphi_{a\#} : K(p_i)/T \to K(p_i)/T$ reverses the orientation. Hence by the Prop. 2 and the remark at the end of No. 2.6 we see that $P_* = -(R_2 P)_*$.

In particular, if R_2 keeps planes p_j invariant of $j \neq i$, i.e., $P = R_2 P$, then we see that $P_* = 0$.

PROPOSITION 3. *Let P be as above and assume that θ_2 is orthogonal to all roots of p_j for $j \neq i$. Then $P_* = 0$.*

8. Here we give a modified description of an original K-cycle $\Gamma(P)$ which is convenient in the next No.

Let $P = \{p_1, \ldots, p_k\}$ be an ordered set of k singular planes of codimension 1. The semisimple part of $K(p_i)$ is a 3-sphere denoted by $S^3(p_i)$ whose maximal torus $S^1(p_i) = S^3(p_i) \cap T$ is a circle perpendicular to $\bar{p}_i$. We remark that for any $t \in T$ the inner automorphism defined by t maps $S^3(p_i)$ into itself.

Put $W'(P) = \prod_1^k S^3(p_i)$ and $T'(P) = \prod_1^k S^1(p_i)$. Let $T'(P)$ operate on $W'(P)$ by

$$(x_1, \ldots, x_k) \cdot (t_1, \ldots, t_k) = (x_1 t_1, t_1^{-1} x_2 t_1 t_2, \ldots, \\ t_1^{-1} \cdots t_{i-1}^{-1} x_i t_1 \cdots t_i, \ldots, t_1^{-1} \cdots t_{k-1}^{-1} x_k t_1 \cdots t_k)$$

for $(x_1, \ldots, x_k) \in W'(P)$ and $(t_1, \ldots, t_k) \in T'(P)$. Then $W'(P)$ is a principal $T'(P)$ bundle. Let $f : T'(P) \to T^k$ be a homomorphism defined by

$$f(t_1, \ldots, t_k) = (t_1, t_1 t_2, \ldots, t_1 \cdots t_i, \ldots, t_1 \cdots t_k)$$

for $(t_1, \ldots, t_k) \in T'(P)$. Then the natural inclusion $W'(P) \to W(P)$ is clearly a bundle map relative to f and induces a homeomorphism of base spaces, i.e., $W'(P)/T'(P) \approx \Gamma(P)$.

9. Here we assume that the group K is simple and simply laced, i.e., all roots of K have the same length. In an original K-cycle $\Gamma(P)$ with $P = \{p_1, \ldots, p_k\}$ let us assume that two successive planes p_i and p_{i+1} are orthogonal to each other and put $P' = \{p_1, \ldots, p_{i-1}, p_{i+1}, p_i, p_{i+2}, \ldots, p_k\}$.

PROPOSITION 4. $$P_* = P'_*.$$

Proof. The roots of p_i and p_{i+1} are denoted by θ_i and θ_{i+1}. Since K is simply laced, $\theta_i \pm \theta_{i+1}$ are not roots. This implies that the semisimple part of $K(q)$, $q = p_i \cap p_{i+1}$, is of type $A_1 \times A_1$, and the direct factors are $S^3(p_i)$ and $S^3(p_{i+1})$ respectively. Hence $S^3(p_i)$ and $S^3(p_{i+1})$ are elementwise commutative in K.

Let $\chi : W'(P) \to W'(P')$ be a map defined by $\chi(x_1, \ldots, x_k) = (x_1, \ldots, x_{i-1}, x_{i+1}, x_i, x_{i+2}, \ldots, x_k)$ for $(x_1, \ldots, x_k) \in W'(P)$. $\bar{\chi} = \chi \mid T'(P) : T'(P)$

$\to T'(P')$ is a homomorphism and χ is a bundle map relative to $\bar{\chi}$ inducing a homeomorphism of base spaces

$$\tilde{\chi} : \Gamma(P) \approx \Gamma(P').$$

$\tilde{\chi}$ is orientation preserving since $\tilde{\chi}^*$ transforms the cohomology fundamental class $x_1 \cdots x_k$ to the cohomology fundamental class as is easily seen.

Let $c = \{c_0, \ldots, c_{k-1}\}$ be a chain subject to $P'' = \{p_1, \ldots, p_{i-1}, q, p_{i+2}, \ldots, p_k\}$. c may be considered as a chain subject to P or to P' with $(i+1)$-th polygon collapsed to a point. Since $S^3(p_i)$ and $S^3(p_{i+1})$ are elementwise commutative in K we see easily that the diagram

$$\begin{tikzcd} \Gamma(P) \arrow[dr, "f_P^c"] \arrow[dd, "\tilde{\chi}"'] & \\ & \Omega K \\ \Gamma(P') \arrow[ur, "f_{P'}^c"'] & \end{tikzcd}$$

is commutative. Hence $P_* = P'_*$. (q.e.d.)

10. Let $\Gamma(P)$ be an original K-cycle with $P = \{p_1, \ldots, p_k\}$. Further, assume that $p_i = p_{i+1}$. Then

PROPOSITION 5. $P_* = 0$.

Proof. Put $P' = \{p_1 \cdots \overset{i}{\vee} \cdots p_k\}$ by deleting p_i from P. Let

$$\alpha : K(p_i) \times K(p_{i+1}) \to K(p_{i+1})$$

be the map defined by the multiplication in the group.

$$\bar{\alpha} = \iota \times \cdots \times \iota \times \alpha \times \iota \times \cdots \times \iota : W(P) \to W(P')$$

is a bundle map relative to $\bar{\alpha} \mid T^k$, where ι is identity map, and induces a may

$$\tilde{\alpha} : \Gamma(P) \to \Gamma(P').$$

Let c be a chain subject to P'. c is also subject to P with $(i+1)$-th polygon collapsed to a point. The diagram

$$\begin{tikzcd} \Gamma(P) \arrow[dr, "f_P^c"] \arrow[dd, "\tilde{\alpha}"] & \\ & \Omega K \\ \Gamma(P') \arrow[ur, "f_{P'}^c"'] & \end{tikzcd}$$

is clearly commutative and

$$\tilde{\alpha}_* \text{ (the fundamental class of } \Gamma(P)) = 0$$

since $\dim \Gamma(P) > \dim \Gamma(P')$. Hence $P_* = 0$. (q.e.d.)

11. Let $\mathfrak{F}$ be a fundamental chamber in D, i.e.,

$$\mathfrak{F} = \{X \in R;\ \varphi_i(X) > 0 \quad \text{for all } 1 \leqq i \leqq l\}$$

where φ_i, $1 \leqq i \leqq l$, are simple roots of a fundamental system of roots of K.

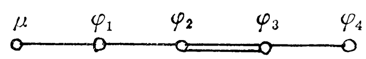
$\mathfrak{F}$ is subdivided in cells by singular planes in it. Let ϕ be the 1-skeleton of the dual subdivision of $\mathfrak{F}$, called graph. To each vertex of ϕ there corresponds a cell of $\mathfrak{F}$ and to each edge of $\mathfrak{F}$ there corresponds a cell in a singular plane (θ, n) in $\mathfrak{F}$. Let μ denote the dominant root of the fundamental system of roots $\{\varphi_i\}$. $(\mu, 1)$ is the nearest to the origin among the singular planes in $\mathfrak{F}$ so that $\{X \in \mathfrak{F};\ \mu(X) < 1\}$ is a cell in $\mathfrak{F}$, i.e., the fundamental cell in $\mathfrak{F}$ denoted by $\Delta_{\mathfrak{F}}$. $v_{\mathfrak{F}}$ is the vertex dual to $\Delta_{\mathfrak{F}}$. The dual vertex of a cell Δ is denoted by v_Δ.

Bott-Samelson [10], Chap. II, Prop. 9.1, described a homology basis of $H_*(\Omega K;\ Z)$ using K-cycles. To each vertex $v \in \phi$ there corresponds an element P_* of this basis such that P consists of the planes in a suitable path connecting v_Δ to $v_{\mathfrak{F}}$ in that order. Actually it would yield some difficulties to determine the path carrying the generator. Nevertheless, in cases discussed in the next section this point is solved by the Prop. 4.

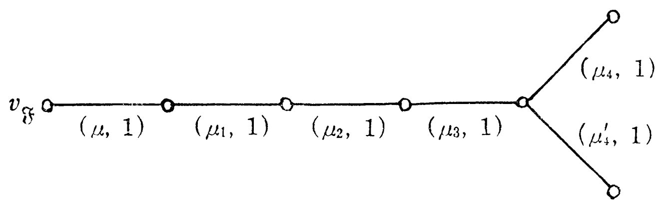
By ϕ_i we denote the subgraph of ϕ containing all vertices and edges which are connected to $v_{\mathfrak{F}}$ through, at most, i successive edges. ϕ_i is sufficient to determine the homology basis of ΩK in deg. $\leqq 2i$.

§2. Homology basis in low degrees of loop spaces of F_4, E_6, E_7, E_8

1. The Schläfli figure of F_4 is as follows:



where φ_i, $1 \leqq i \leqq 4$, is a fundamental system of roots of F_4 and μ is the dominant root of the fundamental system. $\phi_5(F_4)$ is described as follows:



where $\mu_i=\mu_{i-1}-\varphi_i$ for $1\leqq i\leqq 4$ $(\mu_0=\mu)$ and $\mu'_4=\mu_3-\varphi_3$. This is obtained without difficulties by a succession of reflections of cells on some incident singular planes starting from $\Delta_{\mathfrak{F}}$.

Then the homology basis of $H_*(\Omega F_4;\ Z)$ for deg. $\leqq 10$ is given by

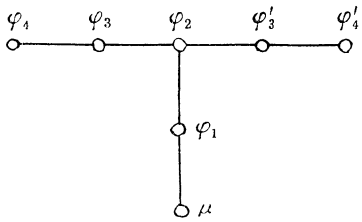
$$P_{1*},\ P_{2*},\ P_{3*},\ P_{4*},\ P^1_{5*} \text{ and } P^1_{5*},$$

where

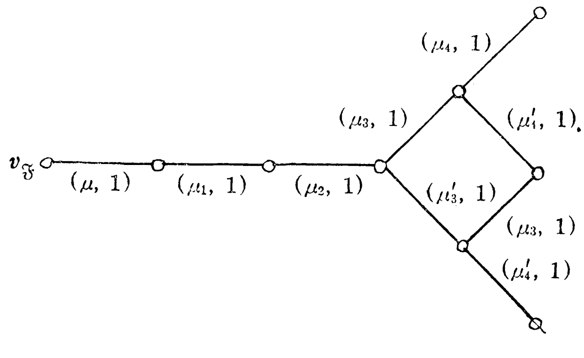
$$P_i=\{(\mu_{i-1},1),\ (\mu_{i-2},1),\ \ldots,\ (\mu,1)\} \qquad \text{for } 1\leqq i\leqq 4,$$
$$P^1_5=\{(\mu_4,1),\ (\mu_3,1),\ \ldots,\ (\mu,1)\},$$
$$P^2_5=\{(\mu'_4,1),\ (\mu_3,1),\ \ldots,\ (\mu,1)\}$$

and the double of the lower suffixes denote the dimensions of the corresponding F_4-cycles.

2. The Schläfli figure of E_6 is as follows:



where φ_i and φ'_i are simple roots of a fundamental system of roots of E_6 and μ is the dominant root. $\phi_5(E_6)$ is described as follows:



where $\mu_i=\mu_{i-1}-\varphi_i$ for $1\leqq i\leqq 4$ $(\mu_0=\mu)$ and $\mu'_3=\mu_2-\varphi'_3$, $\mu'_1=\mu'_3-\varphi'_4$.

Notations of ordered sets of singular planes:

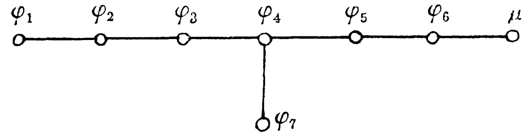
$$P_i=\{(\mu_{i-1},1),\ (\mu_{i-2},1),\ \ldots,\ (\mu,1)\} \qquad \text{for } 1\leqq i\leqq 3,$$
$$P^1_4=\{(\mu_3,1),\ P_3\},\quad P^2_4=\{(\mu'_3,1),\ P_3\},$$
$$P^1_5=\{(\mu_4,1),\ (\mu_3,1),\ P_3\},\quad P^2_5=\{(\mu'_4,1),\ (\mu'_3,1),\ P_3\},$$
$$P^3_5=\{(\mu'_3,1),\ (\mu_3,1),\ P_3\},\quad P^4_5=\{(\mu_3,1),\ (\mu'_3,1),\ P_3\}.$$

Here we remark that μ_3 and μ_3' are orthogonal to each other, then by the Prop. 4 we see that $P_{5*}^3 = P_{5*}^4$. Hence the homology basis of $H_*(\Omega E_6;\, Z)$ in deg. $\leqq 10$ is given by

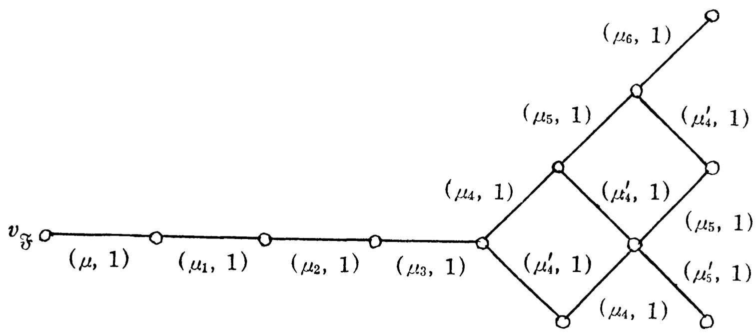
$$P_{1*},\ P_{2*},\ P_{3*},\ P_{4*}^1,\ P_{4*}^2,\ P_{5*}^1,\ P_{5*}^2 \text{ and } P_{5*}^3$$

where the double of lower suffixes denote degrees of these generators.

3. The Schläfli figure of E_7 is as follows:



where μ is the dominant root. $\phi_7(E_7)$ is described as follows:



where $\mu_i = \mu_{i-1} - \varphi_{7-i}$ for $1 \leqq i \leqq 6$ ($\mu_0 = \mu$) and $\mu_4' = \mu_3 - \varphi_7$, $\mu_5' = \mu_4 - \varphi_7$.

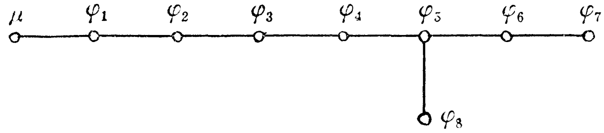
Notations of ordered sets of singular planes:

$$\begin{aligned}
&P_i = \{(\mu_{i-1}, 1), (\mu_{i-2}, 1), \ldots, (\mu, 1)\} \qquad \text{for } 1 \leqq i \leqq 4,\\
&P_5^1 = \{(\mu_4, 1), P_4\}, \quad P_5^2 = \{(\mu_4', 1), P_4\},\\
&P_6^1 = \{(\mu_5, 1), P_5^1\}, \quad P_6^2 = \{(\mu_4', 1), P_5^1\}, \quad P_6^2 = \{(\mu_4, 1), P_5^2\},\\
&P_7^1 = \{(\mu_6, 1), P_6^1\}, \quad P_7^2 = \{(\mu_4', 1), P_6^1\}, \quad P_7^3 = \{(\mu_5', 1), P_6^2\},\\
&P_7^4 = \{(\mu_5, 1), P_6^2\}, \quad P_7^5 = \{(\mu_5, 1), P_6^3\}, \quad P_7^6 = \{(\mu_5', 1), P_6^3\}.
\end{aligned}$$

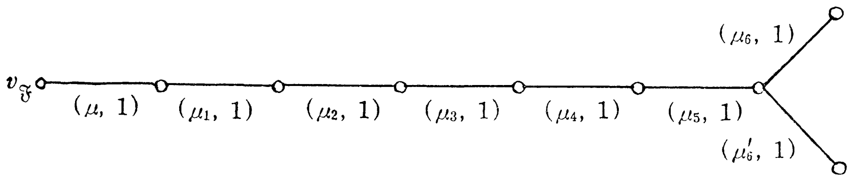
We remark that μ_4' is orthogonal to μ_4 and μ_5 respectively. Then by the Prop. 4 we see that $P_{6*}^2 = P_{6*}^3$, $P_{7*}^2 = P_{7*}^4 = P_{7*}^5$ and $P_{7*}^3 = P_{7*}^6$. Hence the homology basis of $H_*(\Omega E_7;\, Z)$ in deg. $\leqq 14$ is given by

$$P_{i*}\ (1 \leqq i \leqq 4),\ P_{5*}^1,\ P_{5*}^2,\ P_{6*}^1,\ P_{6*}^2,\ P_{7*}^1,\ P_{7*}^2 \text{ and } P_{7*}^3.$$

4. The Schläfli figure of E_8 is as follows:



where μ is the dominant root. $\phi_7(E_8)$ is described as follows:



where $\mu_i = \mu_{i-1} - \varphi_i$ for $1 \leqq i \leqq 6$ $(\mu_0 = \mu)$ and $\mu_6' = \mu_5 - \varphi_8$.

Notations of ordered sets of singular planes:

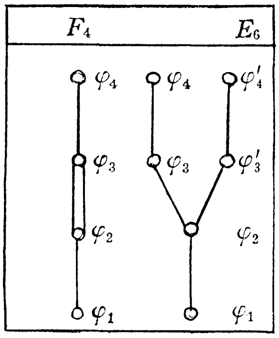
$$P_i = \{(\mu_{i-1}, 1), (\mu_{i-2}, 1), \ldots, (\mu, 1)\} \qquad \text{for } 1 \leqq i \leqq 6,$$
$$P_7^1 = \{(\mu_6, 1), P_6\}, \quad P_7^2 = \{(\mu_6', 1), P_6\}.$$

The homology basis of $H_*(\Omega E_8; Z)$ in deg. $\leqq 14$ is given by

$$P_{i*} \ (1 \leqq i \leqq 6), \ P_{7*}^1 \text{ and } P_{7*}^2.$$

§ 3. The homology map $H_*(\Omega F_4) \to H_*(\Omega E_6)$

1. The inclusion $f : F_4 \subset E_6$ is described by the following table, [16]. Let T be a fixed maximal torus of E_6 and $T' = F_4 \cap T$. T' is a maximal torus of F_4. Let R and R' be the universal covering groups of E_6 and F_4. R' is identified with $\eta^{-1}T' \subset R$ where $\eta : R \to T$ is the covering map. The map defined by this identification is denoted by $\bar{f}$. Under the inclusion f every root form of E_6 determines a root form of F_4 if restricted to R'. This correspondence of root forms is given in the above table, i.e.,



$$\varphi_4(E_6) \mid R' = \varphi_4'(E_6) \mid R' = \varphi_4(F_4),$$
$$\varphi_3(E_6) \mid R' = \varphi_3'(E_6) \mid R' = \varphi_3(F_4),$$
$$\varphi_2(E_6) \mid R' = \varphi_2(F_4), \ \varphi_1(E_6) \mid R' = \varphi_1(F_4).$$

For every long root of F_4 there corresponds one root of E_6 and for every short root of F_4 there correspond two mutually orthogonal roots of E_6 [16]. These roots of E_6 are called the associated roots with the given roots of F_4. The dominant root of E_6 is associated with the dominant root of F_4, since

$$\mu(E_6) = 2\varphi_1 + 3\varphi_2 + 2(\varphi_3 + \varphi_3') + (\varphi_4 + \varphi_4')$$

and

$$\begin{aligned}\mu(E_6) \mid R' &= 2\varphi_1(F_4) + 3\varphi_2(F_4) + 4\varphi_3(F_4) + 2\varphi_4(F_4)\\ &= \mu(F_4).\end{aligned}$$

By τ_ρ (or τ_θ) we denote the root vector corresponding to the root form ρ (or θ) of F_4 (or of E_6) in the sense of Stiefel [17]. If ρ is a long root of F_4 and θ is the root of E_6 associated with ρ, then

$$\bar{f}(\tau_\rho) = \tau_\theta \tag{3}$$

since τ_ρ is orthogonal to the plane $(\theta, 0)$ in R. In particular

$$\bar{f}(\tau_\nu(F_4)) = \tau_\mu(E_6)$$

where μ denotes the dominant root of F_4 or of E_6. If ρ is a short root of F_4 and θ_1, θ_2 are roots of E_6 associated with ρ, then

$$\bar{f}(\tau_\rho) = \tau_{\theta_1} + \tau_{\theta_2}. \tag{4}$$

This is proved as follows: since τ_ρ is orthogonal to $(\theta_1, 0) \cap (\theta_2, 0)$ in R, $\bar{f}(\tau_\rho)$ is a linear combination of τ_{θ_1} and τ_{θ_2}, say $a\tau_{\theta_1} + b\tau_{\theta_2}$. Then, since θ_1 and θ_2 are orthogonal we have $\theta_1(\bar{f}(\tau_\rho)) = 2a$ and $\theta_2(\bar{f}(\tau_\rho)) = 2b$. On the other hand $\theta_i(\bar{f}(\tau_\rho)) = \rho(\tau_\rho) = 2$. Hence $a = b = 1$.

2. The inclusion $f : F_4 \to E_6$ induces the map of Lie algebras

$$df : L(F_4) \to L(E_6).$$

Let

$$\begin{aligned} L(F_4) &= R' + \textstyle\sum \mathfrak{e}_\alpha \qquad (\dim \mathfrak{e}_\alpha = 2),\\ L(E_6) &= R + \textstyle\sum \mathfrak{e}_\beta \qquad (\dim \mathfrak{e}_\beta = 2)\end{aligned}$$

be respectively canonical direct sum decompositions of $L(F_4)$ and of $L(E_6)$ into the invariant subspaces under $\operatorname{Ad} T'$ and $\operatorname{Ad} T$, where α (or β) runs the positive roots subject to the simple system of roots $\{\varphi_i\}$ (or $\{\varphi_i, \varphi_i'\}$).

Let p be a singular plane of codimension 1 in R' and ρ be the root of p. For the sake of simplicity we assume that ρ is positive. By a standard argument (3) and (4) imply that i) if ρ is a long root of F_4 and θ is the associated root of E_6, then

(5) $$df(\mathfrak{e}_\rho) \subset \mathfrak{e}_\theta ;$$

ii) if ρ is a short root of F_4 and θ_1, θ_2 are the associated roots of E_6, then

(6) $$df(\mathfrak{e}_\rho) \subset \mathfrak{e}_{\theta_1} + \mathfrak{e}_{\theta_2}.$$

Then we see that

(7) $$f(K(p)) \subset K(q) \qquad \text{for each singular plane } p = (\rho, n)$$

where q is a singular plane of codimension 1 or 2 according as ρ is long or short such that in case ρ is long $q = (\theta, n)$ with $\theta \mid R' = \rho$ and in case ρ is short $q = (\theta_1, n) \cap (\theta_2, n)$ with $\theta_i \mid R' = \rho$. The singular plane q is called the singular plane in R associated with the plane p in R'.

3. When $p = (\rho, n)$ is a singular plane in R' such that ρ is a long root of F_4 and $q = (\theta, n)$ is the associated plane in R, then

$$f'_p = f \mid K(p) : K(p) \to K(q)$$

induces a homeomorphism

(8) $$f_p : K(p)/T' \approx K(q)/T$$

which is orientation preserving by the reason of (3).

When $p = (\rho, n)$ is a singular plane in R' such that ρ is a short root of F_4 and $q = (\theta_1, n) \cap (\theta_2, n)$ is the associated plane, then the semisimple part of $K(q)$ is of type $A_1 \times A_1$. Let $S^3(\theta_1)$ and $S^3(\theta_2)$ be the semisimple part of $K(\theta_1)$ and $K(\theta_2)$ respectively. Then the semisimple part of $K(q)$ is $S^3(\theta_1) \cdot S^3(\theta_2)$ and is isomorphic to $S^3(\theta_1) \times S^3(\theta_2)$ or to $S^3(\theta_1) \times_{Z_2} S^3(\theta_2)$ where Z_2 is identified with the centers of $S^3(\theta_i)$, $i = 1, 2$. Since f is an injection $f \mid S^3(\rho)$ is injective, where $S^3(\rho)$ is the semisimple part of $K(p)$, which shows that the semisimple part of $K(q)$ is isomorphic to $S^3(\theta_1) \times S^3(\theta_2)$. Then the map

$$f \mid S^3(\rho) : S^3(\rho) \to S^3(\theta_1) \times S^3(\theta_2)$$

is the diagonal injection by the reason of (4). And the induced map

(9) $$f_p : K(p)/T' \approx S^2(\rho) \to S^2(\theta_1) \times S^2(\theta_2) \approx K(q)/T$$

is the diagonal injection of 2-sphere, where $S^2(\rho) = S^3(\rho)/S^1(\rho)$ and $S^2(\theta_i) = S^3(\theta_i)/S^1(\theta_i)$.

Now $S^2(\theta_i)$ is oriented by τ_{θ_i} and defines an integral 2-class in $K(q)/T$ for each $i=1, 2$. They are homology basis of $H_*(K(q)/T; Z)$. Let y_i, $i=1, 2$, be the dual basis of $H^2(K(q)/T)$, then

$$f_q^*(y_i) = x \tag{10}$$

where x is the cohomology fundamental class of $K(p)/T'$.

4. Let $\Gamma(P)$ be an original F_4-cycle associated with $P=\{p_1, \ldots, p_k\}$. Put $fP=\{q_1, \ldots, q_k\}$ where q_i are singular planes associated with p_i. Then the map

$$f'_{p_1} \times \cdots \times f'_{p_k} : W(P) \to W(fP)$$

induces a map of K-cycles

$$f_p : \Gamma(P) \to \Gamma(fP)$$

as is easily seen.

Put $p_i = (\rho_i, n_i)$ and let $\rho_{i_1}, \ldots, \rho_{i_r}$ be short roots and the rest be long roots. Let θ'_{i_s}, θ''_{i_s} be the roots of E_6 associated with ρ_{i_s}, $1 \leqq s \leqq r$. Further we put $q'_{i_s} = (\theta'_{i_s}, n_{i_s})$ and $q''_{i_s} = (\theta''_{i_s}, n_{i_s})$ for $1 \leqq s \leqq r$. Now consider an ordered set of singular planes

$$Q = \{q_1, \ldots, q_{i_1-1}, q'_{i_1}, q''_{i_1}, \ldots, q'_{i_s}, q''_{i_s}, \ldots, q_k\}$$

consisting of $k+r$ planes of codimension 1 in R. Then the product of maps

$$\begin{aligned} &a_j : K(q_j) \to K(q_j) && \text{for } j \notin I, \\ &b_j : K(q'_j) \times K(q''_j) \to K(q'_j) \cdot K(q''_j) \subset K(q_j) && \text{for } j \in I, \end{aligned}$$

where $I=\{i_1, \ldots, i_r\}$, a_j are identity maps and b_j are defined by the group multiplication in E_6, induces a homeomorphism

$$\chi_P : \Gamma(Q) \approx \Gamma(fP).$$

If we identify $\Gamma(Q)$ and $\Gamma(fP)$ by χ_P, then $H_2(\Gamma(fP); Z)$ has a homology basis consisting of sub E_6-cycles $K(q_j)/T$, $j \notin I$, and $K(q'_j)/T$, $K(q''_j)/T$, $j \in I$. Its dual cohomology basis is written as y_j, $j \notin I$, y'_j, y''_j, $j \in I$. Then

$$\begin{aligned} &f_P^*(y_j) = x_j && \text{for } j \notin I, \\ &f_P^*(y'_j) = f_P^*(y''_j) = x_j && \text{for } j \in I \end{aligned} \tag{11}$$

by (8) and (10), where $\{x_j\}$, $1 \leqq j \leqq k$, is the cohomology basis of $H^2(\Gamma(P); Z)$ which is dual to the F_4-cycles $K(p_i)/T'$, $1 \leqq j \leqq k$.

Let

$$Q(i_1,\ \varepsilon_1;\ i_2,\ \varepsilon_2;\ \ldots;\ i_r,\ \varepsilon_r) \qquad (\varepsilon_s = 1 \text{ or } 2)$$

be a subsequence of Q which is obtained from fP replacing q_{i_s} by q'_{i_s} or by q''_{i_s} according as $\varepsilon_s = 1$ or 2 for $1 \leqq s \leqq r$. This defines a $2k$-dimensional sub E_6-cycle of $\Gamma(fP)$ for each $\varepsilon_1, \ldots, \varepsilon_r$. Since $Q(i_1, \varepsilon_1; i_2, \varepsilon_2; \ldots; i_r, \varepsilon_r)$ is the dual homology class of $y_1 \cdots y_{i_1}^{(\varepsilon_1)} \cdots y_{i_s}^{(\varepsilon_s)} \cdots y_k$ by No. 1.3, (11) shows the

PROPOSITION 6. *The $2k$-cycles $f_P\Gamma(P)$ and $\sum\Gamma(Q(i_1, \varepsilon_1; \ldots; i_r, \varepsilon_r))$ represent the same class in $H_{2k}(\Gamma(fP); Z)$.*

Let $\Omega f: \Omega F_4 \to \Omega E_6$ be the map of loop spaces induced by f. Then we have the

PROPOSITION 7. *$\Omega f_*(P_*) = \sum Q(i_1, \varepsilon_1; \ldots; i_r, \varepsilon_r)_*$ where Ωf_* denotes the homology map induced by Ωf.*

Let c be a chain subject to P in R'. Then $\bar{f}c$ is a chain subject to fP and to $Q(i_1, \varepsilon_1; \ldots; i_r, \varepsilon_r)$ in R, and we have a commutative diagram

$$\begin{array}{ccc} \Gamma(P) & \xrightarrow{f_P^c} & \Omega F_4 \\ \downarrow{\scriptstyle f_P} & & \downarrow{\scriptstyle \Omega f} \\ \Gamma(fP) & \xrightarrow{f_{fP}^{\bar{f}c}} & \Omega E_6 \end{array} .$$

Now by the Prop. 6 we have immediately a proof of the Prop. 7.

5. Now we discuss the homology map

$$\Omega f_*: H_*(\Omega F_4;\ Z) \to H_*(\Omega E_6;\ Z)$$

in deg. $\leqq 10$. Notations of Nos. 2.1 and 2.2 are used. If necessary to distinguish things associated with F_4 or with E_6 we denote them by attaching (F_4) or (E_6).

The homology bases of $H_*(\Omega F_4; Z)$ and of $H_*(\Omega E_6; Z)$ in deg. $\leqq 10$ are given in Nos. 2.1 and 2.2.

The planes $(\mu_i(E_6), 1)$ are associated with $(\mu_i(F_4), 1)$ for $0 \leqq i \leqq 2$ $(\mu_0 = \mu)$. The planes $(\mu_i(E_6), 1) \cap (\mu'_i(E_6), 1)$ are associated with $(\mu_i(F_4), 1)$ for $i = 3, 4$. The plane $(\mu_3 - \varphi'_3(E_6), 1)$ is associated with $(\mu'_4(F_4), 1)$. Then by the Prop. 7 we see that

$$(12)\qquad \begin{aligned} \Omega f_*(P_{i*}(F_4)) &= P_{i*}(E_6) \qquad\qquad \text{for } 1 \leqq i \leqq 3, \\ \Omega f_*(P_{4*}(F_4)) &= P^1_{4*}(E_6) + P^2_{4*}(E_6), \end{aligned}$$

$$(12') \qquad \Omega f_*(P^1_{5*}(F_4)) = P^1_{5*}(E_6) + P^2_{5*}(E_6) + \{(\mu'_4, 1), P^1_4\}_* + \{(\mu_4, 1), P^2_4\}_*,$$

$$(12'') \qquad \Omega f_*(P^2_{5*}(F_4)) = \{(\mu_3 - \varphi'_3, 1), P^1_4\}_* + \{(\mu_3 - \varphi'_3, 1), P^2_4\}_*.$$

Now

$$R_{\varphi'_4}\{(\mu'_4, 1), P^1_4\} = \{(\mu'_3, 1), P^1_4\} = P^3_5(E_6).$$

Hence by the Prop. 2

$$\{(\mu'_4, 1), P^1_4\}_* = P^3_{5*}(E_6).$$

Similarly

$$\{(\mu_4, 1), P^2_4\}_* = P^4_{5*} = P^3_{5*}(E_6).$$

Therefore

$$(13) \qquad \Omega f_*(P^1_{5*}(F_4)) = P^1_{5*}(E_6) + P^2_{5*}(E_6) + 2P^3_{5*}(E_6).$$

Next, $\mu_3 - \varphi'_3 = \mu'_3 - \varphi_3$. Then by the Prop. 1

$$\{(\mu_3 - \varphi'_3, 1), P^1_4\}_* = P^3_{5*} - \{(\varphi_3, 0), P^1_4\}_*.$$

Put $q = (\varphi_3, 0) \cap (\mu_3, 1)$. Since $\varphi_3 + \mu_3 = \mu_2$, $(\mu_2, 1)$ contains q and the semisimple part of $K(q)$ is of type A_2. Put $P' = \{(\varphi_3, 0), P^1_4\}$ and $P'' = \{q, (\mu_1, 1), (\mu, 1)\}$, and let

$$\alpha : K(\varphi_3) \times K(\mu_3) \times K(\mu_2) \to K(\varphi_3) \cdot K(\mu_3) \cdot K(\mu_2) \subset K(q)$$

be defined by the group multiplication.

$$\alpha \times \iota \times \iota : W(P') \to W(P'')$$

induces

$$\bar{\alpha} : \Gamma(P') \to \Gamma(P'')$$

where ι is identity map.

Let c be a chain in R subject to P''. c is considered also as a chain subject to P' with the second and third polygons collapsed to a point. The diagram

$$\begin{tikzcd} \Gamma(P') \arrow[d, "\bar{\alpha}"'] \arrow[dr, "f^c_{P'}"] & \\ \Gamma(P'') \arrow[r, "f^c_{P''}"'] & \Omega E_6 \end{tikzcd}$$

is clearly commutative. Then, since $\Gamma(P')$ and $\Gamma(P'')$ have the same dimension we see that

$$P'_* = \beta P''_* \qquad \text{with } \beta \in Z.$$

On the other hand, by the Prop. 3 $P''_* = 0$, whence $P'_* = 0$. Consequently

$$\{(\mu_3 - \varphi'_3,\ 1),\ P^1_4\}_* = P^3_{5*}(E_6).$$

Similarly

$$\{(\mu_3 - \varphi'_3,\ 1),\ P^2_4\}_* = P^4_{5*} = P^3_{5*}(E_6).$$

Therefore

$$\Omega f_*(P^2_{5*}(F_4)) = 2P^3_{5*}(E_6). \tag{14}$$

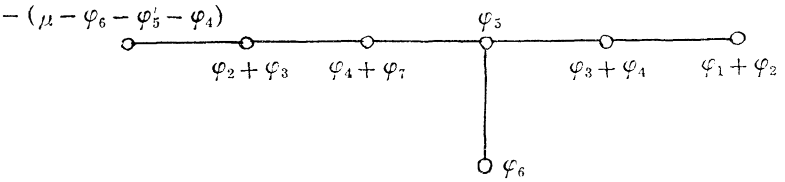
(12), (13) and (14) describe the integral homology map Ωf_* in deg. $\leqq 10$. From these we see easily that

PROPOSITION 8. *The homology map Ωf_* mod p is injective in deg.* $\leqq 10$ *for any odd prime p.*

PROPOSITION 8′. *The homology map Ωf_* mod 2 is injective in deg.* $\leqq 8$. *$\Omega f_* \mid H_{10}(\Omega F_4;\ Z_2)$ has the kernel of dimension* 1 *(generated by $P^2_{5*}(F_4)$).*

§ 4. The homology map $H_*(\Omega E_6) \to H_*(\Omega E_7)$

1. In the Schläfli figure of No. 2.3, the centralizer of the straight line $\bigcap_{i=2}^{7} (\varphi_i,\ 0)$ is of type $E_6 \times T_1$, [7]. As the semisimple part of this subgroup we obtain the canonical inclusion $E_6 \subset E_7$. In this inclusion the dominant root of E_6 is not obtained as the restriction of the dominant root of E_7 to $E_6 \cap T$ (T is the maximal torus of E_7). By this reason we use another fundamental system of roots described by the following figure.



Then the inclusion $g : E_6 \subset E_7$ is defined as the subgroup generated by the closed system of roots spanned by $(\varphi_2 + \varphi_3,\ \varphi_4 + \varphi_7,\ \varphi_5,\ \varphi_3 + \varphi_4,\ \varphi_1 + \varphi_2,\ \varphi_6)$. This is equivalent to the canonical inclusion. Hence its homological effects are the same with the effects of the canonical inclusion.

2. Let T be the maximal torus of E_7, $T' = E_6 \cap T$ be the maximal torus of E_6 and $\bar{g} : R' \subset R$ be the induced inclusion of the universal covering groups as in § 3.

For each singular plane $p = (\rho,\ n)$ in R', let $q = (\theta,\ n)$ be the associated plane in R in the sense that $p = R' \cap q$. Then clearly $g(K(p)) \subset K(q)$ and the induced homeomorphism $g_p : K(p)/T' \approx K(q)/T$ is orientation preserving.

Let $\Gamma(P)$ be an original E_6-cycle with $P=\{p_1, \ldots, p_k\}$. Put $gP=\{q_1, \ldots, q_k\}$ where q_i singular planes in R associated with p_i. $\Gamma(gP)$ is an original E_7-cycle. The map

$$g'_{p_1}\times\cdots\times g'_{p_k} : W(P)\to W(gP)$$

induces a homeomorphism of K-cycles

$$g_P : \Gamma(P)\approx\Gamma(gP)$$

preserving orientations. Let

$$\Omega g : \Omega E_6\to\Omega E_7$$

be the map of loop spaces induced by g. Then it is immediate to see that

$$\Omega g_*(P_*)=(gP)_* \tag{15}$$

where Ωg_* is the map of integral homology groups induced by Ωg.

3. We discuss the homology map

$$\Omega g_* : H_*(\Omega E_6;\ Z)\to H_*(\Omega E_7;\ Z)$$

in deg. $\leqq 10$. Notations of Nos. 2.2 and 2.3 are used.

The planes $(\mu_i(E_7),\ 1)$ are associated with $(\mu_i(E_6),\ 1)$ for $0\leqq i\leqq 2$ $(\mu_0=\mu)$. The planes $(\mu'_4(E_7),\ 1)$, $(\mu_4(E_7),\ 1)$, $(\mu_5-\varphi_7(E_7),\ 1)$ and $(\mu_6(E_7),\ 1)$ are associated with $(\mu_3(E_6),\ 1)$, $(\mu'_3(E_6),\ 1)$, $(\mu_4(E_6),\ 1)$ and $(\mu'_4(E_6),\ 1)$ respectively.

Then by (15) we see that

$$\Omega g_*(P_{i*}(E_6))=P_{i*}(E_7) \qquad \text{for } 1\leqq i\leqq 3, \tag{16}$$

$$\Omega g_*(P^1_{4*}(E_6))=\{(\mu'_4,\ 1),\ P_3(E_7)\}_*, \tag{16$'$.i}$$

$$\Omega g_*(P^2_{4*}(E_6))=\{(\mu_4,\ 1),\ P_3(E_7)\}_*, \tag{16$'$.ii}$$

$$\Omega g_*(P^1_{5*}(E_6))=\{(\mu_5-\varphi_7,\ 1),\ (\mu'_4,\ 1),\ P_3(E_7)\}_*, \tag{16$''$.i}$$

$$\Omega g_*(P^2_{5*}(E_6))=\{(\mu_6,\ 1),\ (\mu_4,\ 1),\ P_3(E_7)\}_*, \tag{16$''$.ii}$$

$$\Omega g_*(P^3_{5*}(E_6))=\{(\mu_4,\ 1),\ (\mu'_4,\ 1),\ P_3(E_7)\}_*. \tag{16$''$.iii}$$

Now

$$R_{\varphi_7}\{(\mu'_4,\ 1),\ P_3\}=P_4(E_7),\ R_{\varphi_3}\{(\mu_4,\ 1),\ P_3\}=P_4(E_7),$$

where R_{φ_i} denotes the reflection across the plane $(\varphi_i,\ 0)$. Hence by the Prop. 2 and (16'.i) and (16'.ii) we see that

(17) $$\Omega g_*(P^i_{4*}(E_6)) = P_{4*}(E_7) \qquad \text{for } i=1,\ 2.$$

Next

$$R_{\varphi_2}R_{\varphi_7}\{(\mu_5-\varphi_7,\ 1),\ (\mu'_4,\ 1),\ P_3\} = P^1_5(E_7),$$
$$R_{\varphi_2}R_{\varphi_1}R_{\varphi_3}\{(\mu_6,\ 1),\ (\mu_4,\ 1),\ P_3\} = P^1_5(E_7),$$
$$R_{\varphi}\{(\mu_4,\ 1),\ (\mu'_4,\ 1),\ P_3\} = \{(\mu'_5,\ 1),\ P_4(E_7)\}.$$

Since $\mu'_5+\varphi_3=\mu'_4$, by the Prop. 1 we have

$$\{(\mu'_5,\ 1),\ P_4(E_7)\}_* = P^2_{5*}(E_7) - \{(\varphi_3,\ 0),\ P_4(E_7)\}_*.$$

Then, applying the same argument with the proof of (14) (use the Prop. 3 for $q=(\varphi_3,\ 0)\cap(\mu_3,\ 1)\cap(\mu_2,\ 1))$, we see that

$$\{(\varphi_3,\ 0),\ P_4(E_7)\}_* = 0.$$

Hence by the Prop. 2 and (16$''$.i), ii) and iii) we have the

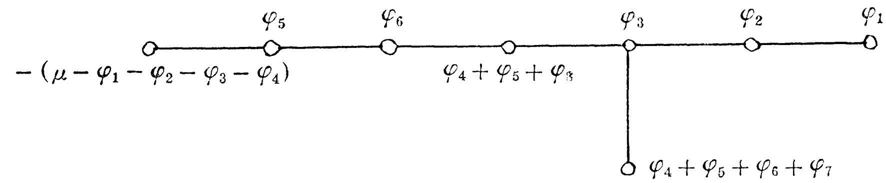
(18) $$\begin{aligned} &\Omega g_*(P^i_{5*}(E_6)) = P^1_{5*}(E_7) \qquad \text{for } i=1,\ 2,\\ &\Omega g_*(P^3_{5*}(E_6)) = P^2_{5*}(E_7). \end{aligned}$$

(16)-(18) describe the integral homology map Ωg_* in deg. $\leqq 10$. From these we see immediately the

Proposition 9. *The homology map Ωg_* mod p is surjective in deg. $\leqq 10$ for any prime p.*

§5. The homology map $H_*(\Omega E_7)\to H_*(\Omega E_8)$

1. By the same reason with §4 we use the different fundamental system of roots described by the following figure to define the inclusion $h:E_7\subset E_8$.



The inclusion h is defined as the subgroup generated by the closed system of roots spanned by $(\varphi_5,\ \varphi_6,\ \varphi_4+\varphi_5+\varphi_8,\ \varphi_3,\ \varphi_2,\ \varphi_1,\ \varphi_4+\varphi_5+\varphi_6+\varphi_7)$. This is equivalent to the canonical inclusion and its homological effects are the same with those of the canonical one.

Let T be the maximal torus of E_8, $T'=T\cap E_7$ be the maximal torus of E_7 and $\bar{h}:R'\subset R$ be the inclusion of the universal covering groups of T' and T.

With each singular plane $p=(\rho,\ n)$ in R' there is associated a singular plane $q=(\theta, n)$ in R in the sense that $p=R'\cap q$. For each ordered set $P=\{p_1, \dots, p_k\}$ of singular planes with codim $p_i=1$ we define $hP=\{q_1, \dots, q_k\}$ such that q_i are associated with p_i for $1\leqq i\leqq k$. Let

$$\Omega h : \Omega E_7 \to \Omega E_8$$

be the map of loop spaces induced by h and Ωh_* be the map of integral homology groups induced by Ωh. Then, in the same way with No. 4.2 we see that

$$\Omega h_*(P_*)=(hP)_*. \tag{19}$$

2. We discuss the homology map

$$\Omega h_* : H_*(\Omega E_7; Z) \to H_*(\Omega E_8; Z)$$

in deg. $\leqq 14$. Notations of Nos. 2.3 and 2.4 are used.

The planes $(\mu_i(E_8), 1)$ are associated with $(\mu_i(E_7), 1)$ for $0\leqq i\leqq 3$ $(\mu_0=\mu)$. The planes $(\mu_6'(E_8), 1)$, $(\mu_6-\varphi_8(E_8), 1)$ and $(\mu-\varphi_8-\varphi_5(E_8), 1)$ are associated with $(\mu_i(E_7), 1)$ for $4\leqq i\leqq 6$ in its order, and $(\mu_6-\varphi_7(E_8), 1)$ and $(\mu_6-\varphi_7-\varphi_8-\varphi_5-\varphi_4(E_8), 1)$ are associated with $(\mu_4'(E_7), 1)$ and $(\mu_5'(E_7), 1)$ respectively.

Then by (19) we see that

(20) $\Omega h_*(P_{i*}(E_7))=P_{i*}(E_8)$ for $1\leqq i\leqq 4$,

(20'.i) $\Omega h_*(P^1_{5*}(E_7))=\{(\mu_6', 1), P_4(E_8)\}_*$,

(20'.ii) $\Omega h_*(P^2_{5*}(E_7))=\{(\mu_6-\varphi_7, 1), P_4(E_8)\}_*$,

(20''.i) $\Omega h_*(P^1_{6*}(E_7))=\{(\mu_6-\varphi_8, 1), (\mu_6', 1), P_4\}_*$,

(20''.ii) $\Omega h_*(P^2_{6*}(E_7))=\{(\mu_6-\varphi_7, 1), (\mu_6', 1), P_4\}_*$,

(20'''.i) $\Omega h_*(P^1_{7*}(E_7))=\{(\mu_6-\varphi_8-\varphi_5, 1), (\mu_6-\varphi_8, 1), (\mu_6', 1), P_4\}_*$,

(20'''.ii) $\Omega h_*(P^2_{7*}(E_7))=\{(\mu_6-\varphi_7, 1), (\mu_6-\varphi_8, 1), (\mu_6', 1), P_4\}_*$.

Firstly

$$R_{\varphi_5}R_{\varphi_8}\{(\mu_6', 1), P_4\}=P_5(E_8),\ R_{\varphi_5}R_{\varphi_6}R_{\varphi_8}\{(\mu_6-\varphi_7, 1), P_4\}=P_5(E_8).$$

Hence by the Prop. 2 and (20'.i) and ii) we see that

$$\Omega h_*(P^i_{5*}(E_7))=P_{5*}(E_8) \qquad \text{for } i=1, 2. \tag{21}$$

Secondly

$$R_{\varphi_6}R_{\varphi_5}R_{\varphi_8}\{(\mu_6-\varphi_8, 1), (\mu_6', 1), P_4\}=P_6(E_8),$$
$$R_{\varphi_5}R_{\varphi_8}R_{\varphi_7}\{(\mu_6-\varphi_7, 1), (\mu_6', 1), P_4\}=\{(\mu_6-\varphi_8-\varphi_5, 1), P_5(E_8)\}.$$

Since $\mu_6-\varphi_8-\varphi_5+(\varphi_5+\varphi_6+\varphi_8)=\mu_5$, by the Prop. 1

$$\{(\mu_6-\varphi_8-\varphi_5,\ 1),\ P_5\}_*=P_{6*}-\{(\varphi_5+\varphi_6+\varphi_8,\ 0),\ P_5\}_*.$$

Further $R_{\varphi_8}R_{\varphi_6}\{(\varphi_5+\varphi_6+\varphi_8,\ 0),\ P_5\}=\{(\varphi_5,\ 0),\ P_5\}$,

$$\begin{aligned}\{(\varphi_5,\ 0),\ P_5\}_* &= -P_{6*}+\{(\mu_4,\ 1),\ (\mu_4,\ 1),\ P_4\}_* && \text{by the Prop. 1,}\\ &= -P_{6*} && \text{by the Prop. 5.}\end{aligned}$$

Therefore by the Prop. 2 and (20$''$.i) and ii) we see the

$$\begin{aligned}\Omega h_*(P^1_{6*}(E_7))&=P_{6*}(E_8),\\ \Omega h_*(P^2_{6*}(E_7))&=2P_{6*}(E_8).\end{aligned}\tag{22}$$

Thirdly

$$\begin{aligned}&R_{\varphi_6}R_{\varphi_5}R_{\varphi_8}\{(\mu_6-\varphi_8-\varphi_5,\ 1),\ (\mu_6-\varphi_8,\ 1),\ (\mu_6',\ 1),\ P_4\}=P^2_7,\\ &R_{\varphi_6}R_{\varphi_5}R_{\varphi_8}\{(\mu_6-\varphi_7,\ 1),\ (\mu_6-\varphi_8,\ 1),\ (\mu_6',\ 1),\ P_4\}\\ &\qquad=\{(\mu_6-\varphi_7-\varphi_8-\varphi_5-\varphi_6,\ 1),\ P_6\}.\end{aligned}$$

By the Prop. 1 we have

$$\{(\mu_6-\varphi_7-\varphi_8-\varphi_5-\varphi_6,\ 1),\ P_6\}_*=P^1_{7*}-\{\varphi_5+\varphi_6+\varphi_7+\varphi_8,\ 0),\ P_6\}_*.$$

Further

$$R_{\varphi_7}\{(\varphi_5+\varphi_6+\varphi_7+\varphi_8,\ 0),\ P_6\}=\{(\varphi_5+\varphi_6+\varphi_8,\ 0),\ P_6\}$$

and

$$\{(\varphi_5+\varphi_6+\varphi_8,\ 0),\ P_6\}_*=\{(\varphi_5,\ 0),\ P_6\}_*+\{(\varphi_6,\ 0),\ P_6\}_*+\{(\varphi_8,\ 0),\ P_6\}_*$$

by the Prop. 1. Here

$$\{(\varphi_5,\ 0),\ P_6\}_*=0$$

by an argument used in the proof of (14) (for $q=(\varphi_5,\ 0)\cap(\mu_5,\ 0)\cap(\mu_4,\ 0)$), and

$$\{(\varphi_6,\ 0),\ P_6\}_*=-P^1_{7*},\ \{(\varphi_8,\ 0),\ P_6\}_*=-P^2_{7*}$$

by an argument used in the proof of (21). Hence by the Prop. 2 and (20$'''$.i) and ii) we see that

$$\begin{aligned}\Omega h_*(P^1_{7*}(E_7))&=P^2_{7*}(E_8),\\ \Omega h_*(P^2_{7*}(E_7))&=2P^1_{7*}+P^2_{7*}(E_8).\end{aligned}\tag{23}$$

(20)-(22) describe the integral homology map Ωh_* in deg. $\leqq 12$ completely, whereas (23) describes Ωh_* in deg. 14 partly with the exception of $\Omega h_*(P^3_{7*}(E_7))$. Nevertheless by (20)-(23) we see immediately the following

PROPOSITION 10. *The homology map Ωg_* mod p is surjective in deg. $\leqq 14$ for any odd prime p.*

Chapter III. The cohomology maps mod 3 induced by the inclusions $F_4 \subset E_6 \subset E_7 \subset E_8$

§1. Preliminaries

1. By K we denote any one of groups F_4, E_6, E_7, E_8 when they are discussed at the same time. Similarly by k we denote any one of inclusions f, g, h.

$H^*(\Omega K;\ Z_3)$ in low degrees are described as follows:

$$
(1)\qquad
\begin{aligned}
&H^*(\Omega F_4;\ Z_3) = Z_3[u_2,\ u_{10}] \quad \text{in deg. } \leqq 10,\\
&H^*(\Omega E_6;\ Z_3) = Z_3[u_2,\ u_8,\ u_{10}] \quad \text{in deg. } \leqq 10,\\
&H^*(\Omega E_7;\ Z_3) = Z_3[u_2,\ u_{10},\ u_{14}] \quad \text{in deg. } \leqq 14,\\
&H^*(\Omega E_8;\ Z_3) = Z_3[u_2,\ u_{14}] \quad \text{in deg. } \leqq 14,
\end{aligned}
$$

where suffixes of generators denote degrees. u_2 is defined as the dual class of P_{1*}. Then, since $\Omega k_*(P_{1*}) = P_{1*}$ by (II. 12), (II. 16) and (II. 20) we see that

$$(2)\qquad \Omega k^*(u_2) = u_2$$

where Ωk^* denotes the cohomology map mod 3 induced by Ωk. By a discussion of Bott-Samelson [10], Chap. II, §13, we know that

$$(3)\qquad u_2^3 \neq 0$$

for $K = E_6$, E_7 and E_8. Since $\Omega f^* \mid H^6(\Omega E_6;\ Z_3)$ is bijective by (II. 12), we see that (3) is true also for $K = F_4$. By (3) and the Poincaré polynomials of compact exceptional groups (*e.g.*, cf. [8]), we can see that (1) is true.

2. $H^*(K;\ Z_3)$ are known by [3, 5, 6]. All elements of them have height $\leqq 3$ and their systems of generators of type (M) are as follows:

$$
(4)\qquad
\begin{aligned}
&F_4 : x_3,\ x_7,\ x_8,\ x_{11},\ x_{15};\\
&E_6 : x_3,\ x_7,\ x_8,\ x_9,\ x_{11},\ x_{15},\ x_{17};\\
&E_7 : x_3,\ x_7,\ x_8,\ x_{11},\ x_{15},\ \ldots;\\
&E_8 : x_3,\ x_7,\ x_8,\ x_{15},\ \ldots .
\end{aligned}
$$

The omitted generators are not needed in our present discussion. The following relations about Steenrod reduced powers and Bockstein operation hold:

$$(5)\qquad x_7 = \mathscr{P}^1 x_3, \quad x_8 = \delta_3^* x_7 \qquad \text{for all groups } K,$$

(6) $$x_{15} = \mathscr{P}^1 x_{11} \qquad \text{for } K = F_4.$$

3. In the spectral sequences mod 3 associated with usual fibrations of loop spaces of K (K = base space, ΩK = the fibre), the generators of fibre cohomologies described in (1) are all transgressive and the transgression image of u_i is represented by x_{i+1}.

This is proved by discussing these spectral sequences one by one, and of course not generally true. For example, $H^*(\Omega G_2; Z_3)$ has generators u_2, u_6, $\ldots$, with $u_2^3 = 0$, and the second generator u_6 is not transgressive.

The transgression image of u_2 is the well-defined element x_3. Hence, by (2) we see that $k^*(x_3) = x_3$. Consequently, by (5) we see that

(7) $$k^*(x_i) = x_i \qquad \text{for } i = 3,\ 7,\ 8.$$

The generators x_3, x_7 and x_8 are universally transgressive. Hence, by making use of Theorem WCSS of Chap. I we see that

(8) $$H^*(B_K;\ Z_3) = Z_3[y_4,\ y] \otimes \Lambda_3(y_9) \qquad \text{in deg. } \leqq 9$$

for all $K = F_4$, E_6, E_7 and E_8, where y_4 is the image of x_3 by the universal transgression and $y_8 = \mathscr{P}^1 y_4$, $y_9 = \delta_3^* y_8$.

§ 2. $f^* : H^*(E_6;\ Z_3) \to H^*(F_4;\ Z_3)$

1. We choose the generators of $H^*(F_4;\ Z_3)$ to satisfy the Prop. I. 2. Consider the spectral sequences mod 3 of loop spaces of F_4 and E_6, and the homomorphism of them induced by f.

Ωf^*-image of u_{10} of ΩE_6 must be transgressive. If $\Omega f^*(u_{10}) = 0$, then u_{10} of ΩF_4 is not in the images of Ωf^*. This contradicts the Prop. II. 8 because we see that Ωf^* is surjective in degree 10 by this proposition. Hence we can choose u_{10} of ΩE_6 such that

$$\Omega f^*(u_{10}) = u_{10}.$$

Then, by No. 1. 3, we see that $f^*(x_{11}) = x_{11} + a x_3 x_9$ with $a \in Z_3$.

We can choose x_{11} of E_6 such that it is d_r-cocycles for $r \leqq 4$ in the universal spectral sequence. Then, using the Prop. I. 2 we see that

(7′) $$f^*(x_{11}) = x_{11}.$$

Now, by (5′) $f^*(\mathscr{P}^1 x_{11}) = x_{15}$ is indecomposable in $H^*(F_4;\ Z_3)$. Hence $\mathscr{P}^1 x_{11}$ is not decomposable in $H^*(F_6:\ Z_3)$, and we can choose the generator x_{15} of

$H^*(E_6; Z_3)$ to satisfy

(6′) $$x_{15} = \mathscr{P}^1 x_{11}.$$

Then $f^* x_{15} = x_{15}$, and by (4), (7) and (7′) we see that f^* mod 3 is surjective. Hence we have the

PROPOSITION 1. *F_4 is totally non homologous zero mod 3 in E_6.*

2. PROPOSITION 2. *In the universal spectral sequence mod 3 of E_6 the behaviors of generators of $H^*(E_6; Z_3)$ except x_{17} are as follows: x_3, x_7, x_8 and x_9 are universally transgressive; x_{11} and x_{15} are never universally transgressive and they can be chosen such that*

$$d_i \kappa_i^2 x_{11} = d_i \kappa_i^2 x_{15} = 0 \qquad for\ 2 \leqq i < 9,$$
$$d_9 \kappa_9^2 x_{11} = \kappa_9^2(y_9 \otimes x_3) \neq 0,$$
$$d_9 \kappa_9^2 x_{15} = \kappa_9^2(y_9 \otimes x_7) \neq 0.$$

The fact that x_9 can be chosen to be universally transgressive is easily seen. Then the Prop. I. 2, (6′) and (7′) prove this proposition.

COROLLARY. *$H^*(B_{E_6}; Z_3) = Z_3[y_4, y_8, y_{10}] \otimes \Lambda_3(y_9)/(y_4 y_9, y_8 y_9)$ in deg. $\leqq 17$ where $y_{10} \in H^{10}(B_{E_6}; Z_3)$ is a representative of the transgression image of x_9.*

Remark. The behavior of x_{17} in the universal spectral sequence is not determined. About this we have two possibilities: x_{17} can be chosen to be universally transgressive or not. If x_{17} is never universally transgressive, then we can choose it such that $d_i \kappa_i^2 x_{17} = 0$ for $2 \leqq i < 10$ and $d_{10} \kappa_{10}^2 x_{17} = \kappa_{10}^2(y_{10} \otimes x_8) \neq 0$.

3. The coproduct $\phi^* : H^*(E_6; Z_3) \to H^*(E_6; Z_3) \otimes H^*(E_6; Z_3)$ is determined in the same way as Theorem I. 1.

PROPOSITION 3. *For generators of $H^*(E_6; Z_3)$ which behave in the universal spectral sequence as stated in the Prop. 2 (and the remark to it) their coproducts are as follows:*

a) *x_3, x_7, x_8 and x_9 are primitive,*

b) $\phi^* x_{11} = 1 \otimes x_{11} + x_{11} \otimes 1 + x_8 \otimes x_3,$

c) $\phi^* x_{15} = 1 \otimes x_{15} + x_{15} \otimes 1 + x_8 \otimes x_7,$

d) $\phi^* x_{17} = 1 \otimes x_{17} + x_{17} \otimes 1 + \varepsilon x_9 \otimes x_8, \qquad \varepsilon \in Z_3,$

where $\varepsilon = 0$ or 1 according as x_{17} can be chosen to be universally transgressive *or not.*

Then the Pontrjagin ring $H_*(E_6;\ Z_3)$ can be discussed in the same way as Theorem I.2.

THEOREM 1. *The Pontrjagin ring $H_*(E_6;\ Z_3)$ is non-commutative and has a 3-simple system of generators $v_3,\ v_7,\ v_8,\ v_9,\ v_{11},\ v_{15}$ and v_{17} satisfying the following relations*

$$v_i \vee v_i = 0 \text{ for } i \neq 8,\ v_8 \vee v_8 \vee v_8 = 0,$$
$$[v_i,\ v_j] = 0 \text{ for } i < j \text{ and } (i,\ j) \neq (3,\ 8),\ (7,\ 8),\ (8,\ 9).$$
$$[v_8,\ v_3] = v_{11},\ [v_8,\ v_7] = v_{15},\ [v_9,\ v_8] = \varepsilon v_{17}$$

where $[v_i,\ v_j] = v_i \vee v_j - (-1)^{ij} v_j \vee v_j$ and ε is the same with that in the Prop. 3.

Remark. The generators v_i, $i = 3,\ 7,\ 8,\ 11,\ 15$ of $H_*(E_6;\ Z_3)$ can be chosen as the f_*-images of generators v_i of $H_*(F_4;\ Z_3)$. Then the relations of non-commutativity $[v_8,\ v_3] = v_{11}$, $[v_8,\ v_7] = v_{15}$ are inherited from the corresponding ones in $H^*(F_4;\ Z_3)$.

4. $H^*(\mathrm{Ad}\,E_6;\ Z_3)$ has a system of generators $x_1,\ x_2,\ x_3,\ x_7,\ x_8,\ x_9,\ x_{11},\ x_{15}$ of type (M) [3]. Let $\pi : E_6 \to \mathrm{Ad}\,E_6$ be the projection. The above generators can be chosen to satisfy

$$\pi^*(x_i) = x_i \qquad \text{for } i = 3,\ 7,\ 8,\ 11,\ 15.$$

Since coproducts commute with the projection, from the Prop. 3 we see that the coproducts $\phi^* x_{11}$ and $\phi^* x_{15}$ in $H^*(\mathrm{Ad}\,E_6;\ Z_3)$ are not symmetric (by a definition of [5], p. 283). Then by [5], Prop. 2.5, we have the

THEOREM 2. *The Pontrjagin ring $H_*(\mathrm{Ad}\,E_6;\ Z_3)$ is not commutative.*

§3. $g^* : H^*(E_7;\ Z_3) \to H^*(E_6;\ Z_3)$

1. By the Prop. II.9 we see that Ωg^* mod 3 is injective in deg. 10. Hence $\Omega g^*(u_{10}) \neq 0$ and is transgressive in the fibration of loop space of E_6 by No. 1.3. Hence we can choose u_{10} of ΩE_7 such that

$$\Omega g^*(u_{10}) = u_{10}.$$

Then by the same discussion as No. 2.1 we can choose the generators x_{11} and x_{15} of $H^*(E_7;\ Z_3)$ to satisfy

$$g^*(x_{11}) = x_{11}, \tag{7''}$$

(6'') $$x_{15} = \mathscr{P}^1 x_{11}.$$

Now $f^* g^* x_{15} = x_{15}$, and by (4), (7), (7') and (7'') we see that $f^* g^*$ mod 3 is surjective. In another word

PROPOSITION 4. *F_4 is totally non homologous zero mod 3 in E_7. $f^* g^*$ mod 3 is bijective in deg. $\leqq 18$.*

Then by Theorem WCSS the induced cohomology map of classifying spaces of F_4 and E_7 is bijective in deg. $\leqq 19$. Hence by the Cor. to the Prop. I.2 we see the

COROLLARY. $H^*(B_{E_7}; Z_3) = Z_3[y_4, y_8] \otimes \Lambda_3(y_9)/(y_4 y_9, y_8 y_9)$ *in deg.* $\leqq 19$.

2. The behaviors of generators x_{11}, x_{15} of $H^*(E_7; Z_3)$ in the universal spectral sequences and their coproducts are entirely the same as those of corresponding ones of $H^*(E_6; Z_3)$. So that we omit to write them down.

As a corollary of the Prop. 4 we see the following

THEOREM 3. *The Pontrjagin ring $H_*(E_7; Z_3)$ in deg. $\leqq 18$, has 3-simple system of generators v_i, $i = 3, 7, 8, 11, 15$, satisfying the relations described in Theorem I.2. $H_*(E_7; Z_3)$ is not commutative.*

Remark. The relations of non-commutativity $[v_8, v_3] = v_{11}$, $[v_8, v_7] = v_{15}$ are inherited from those in F_4 and in E_6.

§4. $h^* : H^*(E_8; Z_3) \to H^*(E_7; Z_3)$

1. By the Prop. II.10 we see that Ωh^* mod 3 is injective in deg. 14. Then, in the same way with No. 3.1 we can choose the generators $u_{14} \in H^*(\Omega E_8; Z_3)$ and $x_{15} \in H^*(E_8; Z_3)$ to satisfy

(9) $$\Omega h^*(u_{14}) = u_{14}, \qquad h^*(x_{15}) = x_{15}.$$

Then, by similar discussions with those in preceding sections we have the following results.

PROPOSITION 5. *The generator x_{15} of $H^*(E_8; Z_3)$ behave in the universal spectral sequence as follows:*

$$d_i \kappa_i^2 x_{15} = 0 \ \textit{for} \ 2 \leqq i < 9, \ d_9 \kappa_9^2 x_{15} = \kappa_9^2(y_9 \otimes x_7) \neq 0.$$

The coproduct $\psi^* x_{15}$ in $H^*(E_8; Z_3)$ is as follows:

$$\psi^* x_{15} = 1 \otimes x_{15} + x_{15} \otimes 1 + x_8 \otimes x_7.$$

COROLLARY. *$H^*(B_{E_8};\ Z_3) = Z_3[y_4,\ y_8] \otimes \Lambda_3(y_9)/(y_8 y_9)$ in deg. $\leqq 19$.*

THEOREM 4. *The Pontrjagin ring $H_*(E_8;\ Z_3)$ is not commutative and has non-zero elements v_3, v_7, v_8, v_{15}, which generates $H_*(E_8;\ Z_3)$ in deg. $\leqq 18$ and satisfy the following relation of non-commutativity*

$$[v_8,\ v_7] = v_{15}.$$

Remark. The relation $[v_8,\ v_7] = v_{15}$ is inherited from the corresponding one in F_4, E_6, and E_7.

§ 5. On homotopy abelians

Let $G \supset H$ be a group and a subgroup. The maps

$$\lambda,\ \mu : H \times H \rightarrow G$$

are defined by $\lambda(x,\ y) = xy = \mu(y,\ x)$ for $x,\ y \in H$. When λ and μ are homotopic, then H is called homotopy abelian in G, [19]. When $G = H$ and G is homotopy abelian in G, then we say simply that G is homotopy abelian.

As is well-known, if G is homotopy-abelian, then the Pontrjagin products of G over any coefficient field are commutative; if H is homotopy abelian in G, then the Pontrjagin products of H into G over any coefficient field are commutative.

Hence, by Theorems 1, 2, 3 and 4 and the remarks to them we can see the following Theorems.

THEOREM 5. *E_6, $\mathrm{Ad}\,E_6$, E_7, $\mathrm{Ad}\,E_7$ and E_8 are not homotopy abelian.*

THEOREM 6. *In the inclusions*

$$F_4 \subset E_6 \subset E_7 \subset E_8,$$

every subgroup is not homotopy abelian in any group containing it.

Remark. Theorem 5, jointly with the results of James-Thomas [19], proves that every compact simple Lie group is not homotopy abelian.

The author knows another proof of Theorem 5 in the same line with the James-Thomas' proof [17] of homotopy non abelian of other compact simple Lie groups, cf. [20]. Nevertheless, I think the Theorem 6 is a new result.

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