

ON THE WEAKNESS OF SOME BOUNDARY COMPONENT

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1. Let D be a domain in the complex z -plane and γ be a boundary component of D consisting of a single point. The component γ is said to be weak if its image under any conformal mapping of D consists of a single point. If γ is not weak, then we say that γ is unstable (Sario [3], [4]).

Let $S_n (n = 1, 2, \dots)$ be a sequence of slits being symmetric and orthogonal to the positive real axis of the complex z -plane and converging to the origin $O : z = 0$. We delete the set $\bigcup_{n=1}^{\infty} S_n \cup \{O\}$ from the z -plane and denote by D the resulting domain. In this note, we treat the weakness of the boundary component O of the domain D .

2. First we prove the following

LEMMA 1. Consider two slits: $x = a_j (> 0)$, $|y| \leq h_j (j = 1, 2)$, ($a_2 < a_1$) which are symmetric and orthogonal to the positive real axis and satisfy the equality $\frac{h_1}{a_1} = \frac{h_2}{a_2} = k$. Construct a doubly connected domain B bounded by two circular arcs $C_j : |z| = \sqrt{a_j^2 + h_j^2}$, $\alpha \leq \arg z \leq 2\pi - \alpha$, ($j = 1, 2$), where $0 < \alpha = \tan^{-1}k (< \frac{\pi}{2})$, and slits S_1, S_2 . Let μ be the module of B and μ^* be the module of the ring domain $R : \sqrt{a_2^2 + h_2^2} < |z| < \sqrt{a_1^2 + h_1^2}$. Then it holds

$$\frac{1}{M(\alpha)} \mu^* \leq \mu \leq M(\alpha) \mu^*,$$

where $M(\alpha)$ is a constant depending only on α .

Proof. Let $z_j = a_j + ih_j$ and $z'_j = a_j - ih_j$ be two endpoints $S_j (j = 1, 2)$. We map the trapezoid $T : (z_2, z'_2, z'_1, z_1)$ onto the quadrilateral (z_2, z'_2, z'_1, z_1) bounded by two minor circular arcs $\widehat{z_2 z'_2}$ on $|z| = \sqrt{a_2^2 + h_2^2}$, $\widehat{z_1 z'_1}$ on $|z| = \sqrt{a_1^2 + h_1^2}$ and two rectilinear segments $\overline{z_2 z_1}$, $\overline{z'_2 z'_1}$ under the topological mapping $\zeta(z) = \sqrt{(1+k^2)x^2 - y^2} + iy = \sqrt{x^2 \sec^2 \alpha - y^2} + iy$, $z = x + iy$. It is obvious that $|\zeta(z)| = \sqrt{1+k^2}x$.

Received June 29, 1960.

Now we put

$$p = \frac{1}{2} \left(\frac{\partial \zeta}{\partial x} - i \frac{\partial \zeta}{\partial y} \right), \quad q = \frac{1}{2} \left(\frac{\partial \zeta}{\partial x} + i \frac{\partial \zeta}{\partial y} \right).$$

By an easy computation, we have

$$\begin{aligned} \frac{|p| + |q|}{|p| - |q|} &\leq \frac{(x \sec^2 \alpha + \sqrt{x^2 \sec^2 \alpha - y^2})^2 + y^2}{x \sec^2 \alpha \sqrt{x^2 \sec^2 \alpha - y^2}} \leq \frac{(\sec^2 \alpha + \sec \alpha)^2 + \tan^2 \alpha}{\sec^2 \alpha} \\ &\leq \frac{4 \sec^4 \alpha + \sec^2 \alpha}{\sec^2 \alpha} = 4 \sec^2 \alpha + 1. \end{aligned}$$

If we put $4 \sec^2 \alpha + 1 = M(\alpha)$, then

$$\sup_{z \in T} \frac{|p| + |q|}{|p| - |q|} \leq M(\alpha).$$

Hence $\zeta(z)$ is a quasiconformal mapping with bounded dilatation. Therefore, if we define

$$\zeta = \varphi(z) = \begin{cases} \zeta(z), & z \in T \\ z, & z \in B - T, \end{cases}$$

then $\zeta = \varphi(z)$ is a quasiconformal mapping of B onto R with bounded dilatation. Thus we have the required inequality

$$\frac{1}{M(\alpha)} \mu^* \leq \mu \leq M(\alpha) \mu^*.$$

3. Suppose that S_n ($n = 1, 2, \dots$) are segments: $x = a_n$ (> 0), $|y| \leq h_n$ satisfying $0 < a_{n+1} < a_n$, $\lim_{n \rightarrow \infty} a_n = 0$ and

$$h_n \leq a_n \tan \alpha = h'_n$$

for some fixed α ($0 < \alpha < \frac{\pi}{2}$).

Let S'_n be a segments $x = a_n$, $|y| = h'_n$. Denote by D (or D') the domain obtained by deleting segments S_n (or S'_n) ($n = 1, 2, \dots$) and the origin $z = 0$ from the complex z -plane. It is obvious that $D \supset D'$.

We construct doubly connected domains B_n ($n = 1, 2, \dots$) in D' bounded by S'_n , S'_{n+1} and by two circular arcs $C_j: |z| = \sqrt{a_j^2 + h_j'^2}$, $\alpha \leq \arg z \leq 2\pi - \alpha$, ($j = n, n+1$). Evidently, $B_n \subset D$ and $B_n \cap B_m = \emptyset$ if $n \neq m$. Let μ_n be the module of B_n . By Lemma 1, we have

$$\frac{1}{M(\alpha)} \mu_n^* \leq \mu_n \leq M(\alpha) \mu_n^*,$$

where $\mu_n^* = \log \frac{a_n}{a_{n+1}}$ is the module of the ring domain $\sqrt{a_{n+1}^2 + h_{n+1}'^2} < |z| < \sqrt{a_n^2 + h_n'^2}$. Hence it follows that

$$\sum_{n=1}^{\infty} \mu_n^* \leq M(\alpha) \sum_{n=1}^{\infty} \mu_n.$$

Since $\lim_{n \rightarrow \infty} a_n = 0$, the left hand side of the above inequality is divergent. By Savage's criterion [5] we see that the origin O is a weak boundary component of D . Thus we obtain the following

THEOREM 1. *If S_n ($n = 1, 2, \dots$) are segments: $x = a_n$ (> 0), $|y| \leq h_n$ satisfying $0 < a_{n+1} < a_n$, $\lim_{n \rightarrow \infty} a_n = 0$ and*

$$(*) \quad h_n \leq a_n \tan \alpha = h_n'$$

for some fixed α ($0 < \alpha < \frac{\pi}{2}$), then O is a weak boundary component of the domain obtained by deleting $\bigcup_{n=1}^{\infty} S_n \cup \{O\}$ from the z -plane.

4. Here we show that in the case when segments in our Theorem 1 do not satisfy the condition $(*)$ the origin O is not always weak.

First we prove the following

LEMMA 2. *Consider two slits $S_j: x = a_j$ (> 0), $|y| \leq h_j$ ($j = 1, 2$), ($a_2 < a_1$) which are symmetric and orthogonal to the real axis. Let Ω be the doubly connected domain obtained by deleting S_1 and S_2 from the z -plane and let Q be the rectangle: $(a_2 + ih, a_2 - ih, a_1 - ih, a_1 + ih)$, where $h = \min(h_1, h_2)$. If μ is the module of Ω , then it holds*

$$\mu \leq \frac{\pi(a_1 - a_2)}{h}.$$

Proof. We denote by $\{\gamma\}$ a family of rectifiable curves in Ω separating S_1 from S_2 and by $\{\gamma'\}$ a family whose elements consist of rectifiable curves joining the upper side $a_2 + ih, a_1 + ih$ to the lower side $a_2 - ih, a_1 - ih$ of Q in Q . It is obvious that each $\gamma \in \{\gamma\}$ contains a curve $\gamma' \in \{\gamma'\}$. Denoting by $\lambda\{\gamma\}$, $\lambda\{\gamma'\}$ the extremal lengths of these families in the sense of Ahlfors-Beurling [1], we get the following inequality:

$$\lambda\{\gamma'\} \leq \lambda\{\gamma\}.$$

From the relation $\lambda\{\gamma\} = \frac{2\pi}{\mu}$ and $\lambda\{\gamma'\} = \frac{2h}{a_1 - a_2}$, we have

$$\mu \leq \frac{\pi(a_1 - a_2)}{h}.$$

Now we denote by S_n ($n = 1, 2, \dots$) segments in the z -plane

$$x = \frac{1}{n}, \quad |y| \leq h_n = c \left(\frac{1}{n-1} \right)^p, \quad (0 < p < 1),$$

where c is a positive constant and $z = x + iy$. Let D be a domain obtained by deleting $\bigcup_{n=1}^{\infty} S_n \cup \{O\}$ from the z -plane and let B_j ($j = 1, 2, \dots$) be any sequence of doubly connected domains in D separating O from the infinity and converging to O . We suppose that B_{j+1} lies in a domain G_j which is a component, containing O , of the complementary sets of $\overline{B_j}$ with respect to the z -plane.

Let $S_{m(j)}$ be the segment such that, for any $n > m(j)$, $S_n \subset G_j$ and that $S_{m(j)} \not\subset G_j$. Then B_j separates S_n ($n > m(j)$) from $S_{m(j)}$.

Without loss of generality, we may assume that $\{B_j\}$ ($j = k_l + 1, \dots, k_{l+1}$) are all the doubly connected domains separating $S_{m(k_l+1)}$ ($= \dots = S_{m(k_{l+1})}$) from $S_{m(k_{l+1})+1}$, where $k_0 = 0$, $m(1) = 1$ and

$$m(k_{l+1}) < m(k_l + 1) + 1 \leq m(k_{l+1} + 1).$$

Denote by $\mathcal{Q}_l$ the domain obtained by deleting $S_{m(k_l+1)}$ and $S_{m(k_{l+1})+1}$ from the z -plane. Then B_j ($j = k_l + 1, \dots, k_{l+1}$) are contained in $\mathcal{Q}_l$. The well-known Teichmüller's inequality implies that

$$\sum_{j=k_l+1}^{k_{l+1}} \mu_j \leq \mu_l^*,$$

where μ_j ($j = k_l + 1, \dots, k_{l+1}$) are the moduli of B_j ($j = k_l + 1, \dots, k_{l+1}$) and μ_l^* is that of $\mathcal{Q}_l$.

Thus, using Lemma 2, we obtain

$$\begin{aligned} \sum_{j=1}^{\infty} \mu_j &= \sum_{l=0}^{\infty} \sum_{j=k_l+1}^{k_{l+1}} \mu_j \leq \sum_{l=0}^{\infty} \mu_l^* \\ &\leq \pi \sum_{l=0}^{\infty} \frac{1}{m(k_l+1)} - \frac{1}{m(k_l+1)+1} \leq \pi \sum_{n=1}^{\infty} \frac{1}{n} - \frac{1}{n+1} \leq \frac{\pi}{c} \sum_{n=1}^{\infty} n^{p-2}. \end{aligned}$$

Since $\sum_{n=1}^{\infty} n^{p-2}$ ($0 < p < 1$) is convergent, we see that the series $\sum_{j=1}^{\infty} \mu_j$ is convergent for any sequence $\{B_j\}$. By using Oikawa's theorem [2], i.e., the converse of Savage's criterion, we have the following

THEOREM 2. *If S_n ($n = 1, 2, \dots$) are segments: $x = \frac{1}{n}$, $|y| \leq c \left(\frac{1}{n-1} \right)^p$, ($0 < p < 1$), then the origin O is an unstable boundary component of the domain obtained by deleting $\bigcup_{n=1}^{\infty} S_n \cup \{O\}$ from the z -plane.*

Recently Oikawa has treated the case that the number of boundary components converging to the origin is not countable and obtained interesting results, some of which contain our results.

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