ON INFINITESIMAL HOLONOMY AND ISOTROPY GROUPS

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We have proved in [2] that if the restricted homogeneous holonomy group of a complete Riemannian manifold is contained in the linear isotropy group at every point, then the Riemannian manifold is locally symmetric, that is, the covariant derivatives of the curvature tensor field are zero. The proof of this theorem, however, depended on an insufficiently stated proposition (Theorem 1, [2]). In the present note, we shall give a proof of a more general theorem of the same type.

1. Let M be a differentiable manifold with an affine connection of class C^∞. For tensor calculus on M, we shall use the notation in [3] (for a full account, see also [4]). In particular, T and R stand for the torsion and curvature tensor fields respectively, and ∇_X denotes covariant differentiation with respect to a vector field X.

At each point p of M, we denote by L_p the group of all linear transformations of the tangent space $T_p(M)$ at p. The *infinitesimal holonomy group* H'_p is, by definition [1], the Lie subgroup of L_p whose Lie algebra is spanned by the set of all endomorphisms of $T_p(M)$ of the forms $R(A, B)$, $(\nabla_C R)(A, B)$, $(\nabla_D \nabla_C R)(A, B)$, . . . (all successive covariant derivatives), where A, B, C, D, . . . are arbitrary tangent vectors at p. On the other hand, we define here the *infinitesimal linear isotropy group* K_p at p as the subgroup of L_p consisting of all linear transformations of $T_p(M)$ which leave the torsion tensor T at p, the curvature tensor R at p, and all their successive covariant differentials ∇T, $\nabla^2 T$, . . . , ∇R, $\nabla^2 R$, . . . at p invariant. For example, $\phi \in L_p$ leaves ∇T invariant if and only if $\phi(\nabla_X T)(X, Y) = (\nabla_{\phi Z} T)(\phi X, \phi Y)$ for all $X, Y, Z \in T_p(M)$. If S is an endomorphism of $T_p(M)$, then the 1-parameter group $\phi_t = \exp tS$ of L_p generated by S leaves the tensor ∇T invariant if and only if $S \cdot (\nabla_Z T)(X, Y) = (\nabla_{SZ} T)(X, Y) + (\nabla_Z T)(SX, Y) + (\nabla_Z T)(X, SY)$ for all $X, Y, Z \in T_p(M)$.

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Let G be the largest connected group of affine transformations of M and H_p the subgroup of G consisting of all elements of G which leave the point p fixed. Then it is clear that the linear isotropy group determined by H_p is contained in the infinitesimal linear isotropy group K_p.

2. Our main result is the following

THEOREM. *Let M be an affinely-connected manifold of class C^∞. Then $\nabla T = 0$ and $\nabla R = 0$ if the following two conditions are satisfied:*

1) *H'_p is contained in K_p at every point p of M;*

2) *H'_p is irreducible at every point p of M.*

In the case where M and its affine connection are real analytic, the infinitesimal holonomy group H'_p coincides with the restricted homogeneous holonomy group of M [1]. Hence we obtain, among others, the following corollary which is a correction to Theorem 1, [2].

COROLLARY. *Let G/H be a homogeneous space of a connected Lie group which admits an invariant affine connection. If the restricted homogeneous holonomy group is irreducible and contained in the linear isotropy group determined by H, then $\nabla T = 0$ and $\nabla R = 0$.*

Now, in order to prove the theorem, let S be the tensor field $R(A, B)$ of type (1, 1), where A and B are arbitrary vector fields on M which we shall fix for the moment. The value of S at p is an endomorphism of $T_p(M)$ and, by assumption 1), leaves invariant the torsion tensor T at p. Since this is true at every $p \in M$, we have

$$S \cdot T(X, Y) - T(SX, Y) - T(X, SY) = 0 \tag{1}$$

for any vector fields X and Y, that is, the tensor field of type (1, 2) which associates to (X, Y) the vector field $S \cdot T(X, Y) - T(SX, Y) - T(X, SY)$ is identically zero. By taking the covariant derivative of this tensor field with respect to an arbitrary vector field Z, we obtain

$$\begin{aligned}(\nabla_Z S) \cdot T(X, Y) - T((\nabla_Z S) \cdot X, Y) - T(X, (\nabla_Z S) \cdot Y) \\ + S \cdot (\nabla_Z T)(X, Y) - (\nabla_Z T)(SX, Y) - (\nabla_Z T)(X, SY) = 0,\end{aligned} \tag{2}$$

where $\nabla_Z S$ is the covariant derivative of S with respect to Z, which is a tensor field of type (1, 1). Since we have

$$(3) \qquad \nabla_Z \cdot S = (\nabla_Z R)(A, B) + R(\nabla_Z A, B) + R(A, \nabla_Z B),$$

and since every term of the right hand side belongs to the Lie algebra of H'_p at every point p, we see that the tensor field $\nabla_Z S$ leaves the torsion tensor T invariant at every point p. Therefore, the terms of (2) which do not contain $\nabla_Z T$ are cancelled out and we have

$$(4) \qquad S \cdot (\nabla_Z T)(X, Y) - (\nabla_Z T)(SX, Y) - (\nabla_Z T)(X, SY) = 0.$$

On the other hand, we know that S leaves the tensor field ∇T invariant, which implies that

$$(5) \quad S \cdot (\nabla_Z T)(X, Y) - (\nabla_Z T)(SX, Y) - (\nabla_Z T)(X, SY) = (\nabla_{SZ} T)(X, Y)$$

for all vector fields X, Y and Z. From (4) and (5), we get $(\nabla_{SZ} T)(X, Y) = 0$ whatever X and Y may be. Thus we have

$$(6) \qquad \nabla_{SZ} T = 0 \quad \text{for every} \quad Z \in T_p(M).$$

Now the above argument can be applied to each of the tensor fields $(\nabla_C R)(A, B)$, $(\nabla_D \nabla_C R)(A, B)$, . . . (all successive covariant derivatives, where A, B, C, D, . . . are arbitrary vector fields), because the formula similar to (3) holds for each of them. Thus we see that (6) holds for every S in the Lie algebra of H'_p.

We now use assumption 2). If H'_p consists of the identity only, the irreducibility means that $\dim M = 1$, in which case we have $T = 0$ and $R = 0$. Except for this trivial case, there is an endomorphism $S \neq 0$ of $T_p(M)$ belonging to the Lie algebra of H'_p. The subspace of $T_p(M)$ formed by all $Z \in T_p(M)$ such that $\nabla_Z T = 0$ is not (0) in virtue of (6). It is obviously invariant by the Lie algebra of H'_p which is irreducible on $T_p(M)$. Hence it coincides with $T_p(M)$, that is, $\nabla T = 0$ at p. This holds at every point p of M.

The proof for $\nabla R = 0$ is similar. If $S = R(A, B)$, then S leaves the curvature tensor field R invariant at every point. We have then $S \cdot R(X, Y) \cdot W - R(SX, Y) \cdot W - R(X, SY) \cdot W - R(X, Y) \cdot S \cdot W = 0$ for arbitrary vector fields Y, Y and W. Starting from this equation which corresponds to (1), we apply the same argument as in the above proof for $\nabla T = 0$.

3. The statement of Theorem 1, [2], is incorrect in the sense that G/H is not in general reductive under the hypothesis. This was pointed out by H. C. Wang; the affine space $R_n (n > 1)$ deprived of the origin may be regarded as a homogeneous space of the general linear group $GL(n, R)$ with invariant flat affine connection, but this homogeneous space is not reductive. According to a result of J. Hano (unpublished), if the restricted homogeneous holonomy group of G/H is assumed to be irreducible in the statement, then G/H is either of dimension 1 or G/H is reductive, and, at any rate, $\nabla T=0$ and $\nabla R=0$.

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