REMARKS TO THE PAPER "ON MONTEL'S THEOREM" BY KAWAKAMI

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1. We take a measurable set E on the positive η-axis and denote by $\mu(r)$ the linear measure of the part of E in the interval $0 < \eta < r$. The lower density of E at $\eta = 0$ is defined by

$$\lambda = \varliminf_{r \to 0} \frac{\mu(r)}{r}.$$

Theorem by Kawakami [1] asserts that if λ is positive, if a function $f(\zeta) = f(\xi + i\eta)$ is bounded analytic in $\xi > 0$ and continuous at E, and if $f(\zeta) \to A$ as $\zeta \to 0$ along E, then $f(\zeta) \to A$ as $\zeta \to 0$ in $|\eta| \leqq k\xi$ for any $k > 0$. He also has shown that one obtains the same conclusion if the assumption $\lambda > 0$ is replaced, in the above conditions, by the assumption that the following quantity is positive:

$$\lambda_\alpha = \varliminf_{r \to 0} r^{\alpha-1} \int_r^1 \frac{d\mu(t)}{t^\alpha},$$

where α is any number not smaller than 2.

We observe that, for any $\alpha > \alpha' > 1$,

$$r^{\alpha-1} \int_r^1 \frac{d\mu(t)}{t^\alpha} \leqq r^{\alpha-\alpha'} r^{\alpha'-1} \int_r^1 \frac{d\mu(t)}{r^{\alpha-\alpha'} t^{\alpha'}} = r^{\alpha'-1} \int_r^1 \frac{d\mu(t)}{t^{\alpha'}},$$

and hence that $\lambda_\alpha > 0$ implies $\lambda_{\alpha'} > 0$ whenever $\alpha > \alpha' > 1$.

In this section we shall prove that, for any $\alpha > 1$, $\lambda > 0$ *is equivalent to* $\lambda_\alpha > 0$.

(i) $\lambda > 0 \to \lambda_\alpha > 0$: First we note that $\mu(r)$ is a continuous non-decreasing function such that

$$(1) \qquad \mu(r_2) - \mu(r_1) \leqq r_2 - r_1$$

for any r_1 and r_2, $0 \leqq r_1 \leqq r_2$.

Received April 19, 1956.

We suppose that there exists a positive constant $\varepsilon < 1$ such that $\mu(r) \geqq \varepsilon x$ for all r $(0 < r < 1)$. By (1), in $0 < r \leqq t \leqq 1$, $\mu(t)$ is not smaller than the following continuous function:

$$p_r(t) = \begin{cases} \mu(r) & \text{for} \quad r \leqq t \leqq \mu(r)/\varepsilon \\ \varepsilon t & \text{for} \quad \mu(r)/\varepsilon \leqq t \leqq r_0, \\ t-(1-\mu(1)) & \text{for} \quad r_0 \leqq t \leqq 1, \end{cases}$$

where r_0 is determined by $\varepsilon r_0 = r_0 - (1-\mu(1))$. Except for the trivial case that $\mu(1) = 1$, we see that $\mu(r)/\varepsilon < r_0$ for sufficiently small r.

Now, for any $\alpha > 1$ and for sufficiently small r,

$$\begin{aligned} r^{\alpha-1}\int_r^1 \frac{d\mu(t)}{t^\alpha} &= r^{\alpha-1}\left[\frac{\mu(t)}{t^\alpha}\right]_r^1 + \alpha r^{\alpha-1}\int_r^1 \frac{\mu(t)}{t^{\alpha+1}}dt \\ &\geqq r^{\alpha-1}\left[\frac{p_r(t)}{t^\alpha}\right]_r^1 + \alpha r^{\alpha-1}\int_r^1 \frac{p_r(t)}{t^{\alpha+1}}dt \\ &= r^{\alpha-1}\int_r^1 \frac{dp_r(t)}{t^\alpha} = \varepsilon r^{\alpha-1}\int_{\mu(r)/\varepsilon}^{r_0} \frac{dt}{t^\alpha} + r^{\alpha-1}\int_{r_0}^1 \frac{dt}{t^\alpha} \\ &= \frac{\varepsilon r^{\alpha-1}}{\alpha-1}\left\{\left(\frac{\varepsilon}{\mu(r)}\right)^{\alpha-1} - \frac{1}{r_0^{\alpha-1}}\right\} + r^{\alpha-1}\int_{r_0}^1 \frac{dt}{t^\alpha} \\ &\geqq \frac{\varepsilon^\alpha}{\alpha-1}\left(\frac{r}{\mu(r)}\right)^{\alpha-1} - \frac{\varepsilon r^{\alpha-1}}{(\alpha-1)r_0^{\alpha-1}} \\ &\geqq \frac{\varepsilon^\alpha}{\alpha-1} - \frac{\varepsilon r^{\alpha-1}}{(\alpha-1)r_0^{\alpha-1}}. \end{aligned}$$

The last quantity tends to $\dfrac{\varepsilon^\alpha}{\alpha-1}$ as $r \to 0$. Thus

$$\lambda_\alpha = \varliminf_{r\to 0} r^{\alpha-1}\int_r^1 \frac{d\mu(t)}{t^\alpha} > 0$$

for any $\alpha > 1$.

(ii) $\lambda_\alpha > 0 \to \lambda > 0$: Suppose that

$$\lambda = \varliminf_{r\to 0} \frac{\mu(t)}{r} = 0.$$

Then we can choose $1 > r_n \downarrow 0$ such that

$$\frac{\mu(r_n)}{r_n} < \frac{1}{n^2}. \tag{2}$$

Let us define in $[r_n/n, 1]$ the following function:

$$q_n(t) = \begin{cases} \mu(r_n/n) + t - r_n/n & \text{for } r_n/n \leqq t \leqq \rho_1, \\ \mu(r_n) & \text{for } \rho_1 \leqq t \leqq r_n, \\ \mu(r_n) + t - r_n & \text{for } r_n \leqq t \leqq \rho_2, \\ \mu(1) & \text{for } \rho_2 \leqq t \leqq 1, \end{cases}$$

where ρ_1 is determined by $\mu(r_n/n) + \rho_1 - r_n/n = \mu(r_n)$ and ρ_2 is determined by $\mu(r_n) + \rho_2 - r_n = \mu(1)$. By (1), it follows that $q_n(t) \geqq \mu(t)$ in $r_n/n \leqq t \leqq 1$. For any $\alpha > 1$,

$$\begin{aligned}
\left(\frac{r_n}{n}\right)^{\alpha-1} \int_{r_n/n}^{r_n} \frac{d\mu(t)}{t^\alpha} &= \left(\frac{r_n}{n}\right)^{\alpha-1} \left[\frac{\mu(t)}{t^\alpha}\right]_{r_n/n}^{r_n} + \alpha\left(\frac{r_n}{n}\right)^{\alpha-1} \int_{r_n/n}^{r_n} \frac{\mu(t)}{t^{\alpha+1}}\,dt \\
&\leqq \left(\frac{r_n}{n}\right)^{\alpha-1} \left[\frac{q_n(t)}{t^\alpha}\right]_{r_n/n}^{r_n} + \alpha\left(\frac{r_n}{n}\right)^{\alpha-1} \int_{r_n/n}^{r_n} \frac{q_n(t)}{t^{\alpha+1}}\,dt = \left(\frac{r_n}{n}\right)^{\alpha-1} \int_{r_n/n}^{r_n} \frac{dq_n(t)}{t^\alpha} \\
&= \left(\frac{r_n}{n}\right)^{\alpha-1} \int_{r_n/n}^{\rho_1} \frac{dt}{t^\alpha} \leqq \left(\frac{r_n}{n}\right)^{\alpha-1} \int_{r_n/n}^{(r_n/n)+\mu(r_n)} \frac{dt}{t^\alpha} \\
&= \frac{1}{\alpha-1}\left(\frac{r_n}{n}\right)^{\alpha-1} \left[\left(\frac{n}{r_n}\right)^{\alpha-1} - \frac{1}{\left\{\dfrac{r_n}{n} + \mu(r_n)\right\}^{\alpha-1}}\right] \\
&= \frac{1}{\alpha-1}\left[1 - \frac{1}{\left\{1 + n\dfrac{\mu(r_n)}{r_n}\right\}^{\alpha-1}}\right] \leqq \frac{1}{\alpha-1}\left\{1 - \frac{1}{\left(1+\dfrac{1}{n}\right)^{\alpha-1}}\right\},
\end{aligned}$$

where we use (2). The last quantity tends to 0 as $n \to \infty$. We also see that

$$\left(\frac{r_n}{n}\right)^{\alpha-1} \int_{r_n}^{1} \frac{d\mu(t)}{t^\alpha} \leqq \left(\frac{r_n}{n}\right)^{\alpha-1} \int_{r_n}^{1} \frac{dq_n(t)}{t^\alpha} \leqq \left(\frac{r_n}{n}\right)^{\alpha-1} \int_{r_n}^{1} \frac{dt}{t^\alpha} \leqq \frac{1}{\alpha-1} \cdot \frac{1}{n^{\alpha-1}}.$$

These two evaluations give

$$\left(\frac{r_n}{n}\right)^{\alpha-1} \int_{r_n/n}^{1} \frac{d\mu(t)}{t^\alpha} \to 0 \qquad \text{as } n \to \infty.$$

That is,

$$\lambda_\alpha = \lim_{r\to 0} r^{\alpha-1} \int_r^1 \frac{d\mu(t)}{t^\alpha} = 0$$

for any $\alpha > 1$.

The equivalence has thus been proved. It is now seen that the theorem by Kawakami is concluded if $\lambda_\alpha > 0$ for a certain $\alpha > 1$.

In a letter, Professor Kawakami raised the following question: Can we draw the same conclusion from the assumption that

$$\lambda'_\alpha = \varliminf_{r\to 0} r^{\alpha-1} \int_0^r \frac{d\mu(t)}{t^\alpha} > 0$$

for α between 0 and 1?

By a similar but simpler calculation, we can in fact prove that, for any α, $0<\alpha<1$, also $\lambda'_\alpha>0$ is equivalent to $\lambda>0$.

2. Theorem 4 in the preceding paper [2] by the present writer is concerned with the same problem as the theorem by Kawakami, although the domains are different.[1] In [2], the domain is the strip $B: 0<x<+\infty,\ 0<y<1$ and the closed set F on the positive real axis along which the function tends to a limit is required to have the following property:

Denoting by $F_a(x)$ the part of F in the interval $[x-a,\ x+a]$, there exist $x_0>0$, $a>0$ and $d>0$ such that the linear measure $m(F_a(x))>d$ for all $x>x_0$.

Then F is said in [2] to have positive average linear measure near $x=+\infty$. What does this mean of the image F' of F on the positive η-axis if B is mapped onto the half plane $\xi>0$ $(\zeta=\xi+i\eta)$ in a one-to-one conformal manner in such a way that $\zeta=0$ corresponds to $x=+\infty$?

In this section we shall show that it simply means the positiveness of the lower density at $\eta=0$ of F'.

We map B onto the right half of the disc $|Z|<1$ in the Z-plane $(Z=X+iY)$ by $Z=ie^{-\pi z}$, so that $Z=0$ corresponds to $x=+\infty$ and the image F_1 of F lies on the positive Y-axis. It is easy to see that the lower density of F_1 at $Y=0$ is positive if and only if that of F' stated above is positive. So we shall prove that the lower density of F_1 at $Y=0$ is positive if and only if F has positive average linear measure near $x=+\infty$.

First we suppose that F satisfies the required condition. Then

$$\frac{m(F_1\cap(0,\ Y))}{Y}=\pi\int_{F\cap[x,+\infty)}e^{\pi(x-t)}dt \geqq \pi\int_{F_a(x+a)}e^{\pi(x-t)}dt>\pi e^{-2\pi a}d>0,$$

where $x=-\dfrac{1}{\pi}\log Y$ is taken so that it is greater than x_0. Thus the lower density of F_1 at $Y=0$ is positive.

Next suppose that, for every $a>0$, there is a sequence of points $x_n(a)\to+\infty$ such that $m(F_a(x_n(a)))\to 0$ as $n\to\infty$. Then if we set $Y_n(a)=e^{-\pi(x_n(a)-a)}$, it follows that

[1] We both gave talks on the same subject at the annual meeting of the Math. Soc. of Japan held in Tokyo in May, 1955, without knowing one another's work.

$$\frac{m(F_1 \cap (0,\ Y_n(a)))}{Y_n(a)} = \pi \int_{F \cap [x_n(a)-a,\ +\infty)} e^{\pi(x_n(a)-a-t)}\,dt$$
$$\leqq \pi \int_{F_a(x_n(a))} dt + \pi \int_{x_n(a)+a}^{\infty} e^{\pi(x_n(a)-a-t)}\,dt = \pi m(F_a(x_n(a))) + e^{-2\pi a}.$$

This value is smaller than any assigned positive value, if we take first a and then n sufficiently large. Thus the lower density of F_1 at $Y=0$ is zero.

On account of this equivalence, the theorem by Kawakami follows from Theorem 4 in [2] and, by Theorem 5 in [2], it is seen that the metrical condition $\lambda > 0$ in the theorem by Kawakami is in a sense the best possible.

Bibliography

[1] Y. Kawakami: On Montel's theorem, Nagoya Math. J., **10** (1956), pp. 125-127.
[2] M. Ohtsuka: Generalizations of Montel-Lindelöf's theorem on asymptotic values, ibid., pp. 129-163.

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