REDUCTION THEOREM FOR CONNECTIONS AND ITS APPLICATION TO THE PROBLEM OF ISOTROPY AND HOLONOMY GROUPS OF A RIEMANNIAN MANIFOLD

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The present paper constitutes, together with [13], a continuation of the study of differential geometry of homogeneous spaces which we started in [11]. Our main result states that if the homogeneous holonomy group of a complete Riemannian manifold is contained in the linear isotropy group at each point, then the Riemannian manifold is symmetric. The converse is of course one of the well known properties of a Riemannian symmetric space [4]. Besides the results already sketched in [12], we add a few applications of the main theorem.

After giving a brief sketch of the general theory of connections and holonomy groups, we first establish a reduction theorem for connections in a principal fiber bundle. Although it is just an exact formulation of E. Cartan's theorem [3] and is supposedly "well known" by now, there is no published proof as yet.

By using this theorem, we prove Theorem 1 on certain invariant affine connections on a homogeneous space. Then applying the method developed in [13] and using Theorem 1, we obtain Theorem 2 which is the main result.

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I. Reduction of a connection

1. Connection in a principal fiber bundle. Although the notion of a connection in a principal fiber bundle is well known [1], [5], [7], we shall give a brief resumé of the definitions which we need for our purpose. By differentiability, we shall always understand that of class C^∞.

Let $P = P(B, G)$ be a differentiable principal fiber bundle over the base manifold B with structure Lie group G [7], [15]. We always assume that P satisfies the second axiom of countability. Then this is true for the base B

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and the structure group G. For G, it means that G is a Lie group (a locally connected group whose identity component G^0 is an analytic group [6]) which has at most a countable number of components. Conversely, if B satisfies the second axiom of countability and if G/G^0 is at most countable, then P satisfies the second axiom of countability.

At each point x of P, let P_x be the tangent space to P and G_x the subspace of P_x which is tangent to the fiber through x. A *connection* Γ in P is a choice of a tangent subspace Q_x at each x of P which satisfies the following conditions:

1) $P_x = G_x + Q_x$ (direct sum);
2) $Q_{xa} = R_a \cdot Q$, where R_a $(a \in G)$ denotes the action of G on P to the right, that is, $R_a \cdot x = xa$;
3) Q_x depends differentiably on x.

Q_x is called the *horizontal subspace* at x, and a tangent vector of P is called *horizontal* if it belongs to some Q_x. The last condition can be explained as follows. Given a vector field X in P, we have at each $x \in P$ a unique decomposition $X_x = Y_x + Z_x$, where $Y_x \in G_x$ and $Z_x \in Q_x$, by 1). The vector field $Z : x \to Z_x$, called the *horizontal component* of X, is required to be differentiable if X is differentiable. In this case, the vector field $Y : x \to Y_x$, called the *vertical component* of X, is also differentiable.

Given a connection Γ in P, we define the parallel displacement of fibers along any (piecewise differentiable) curve in B. Let τ be a curve from u_0 to u_1 in B. Then it can be proved that, for any point x_0 of P lying over u_0, there is one and only one curve τ^* beginning at x_0 which covers τ and whose tangent vectors are all horizontal (such a curve is called a horizontal curve). We call τ^* the *lift* of τ through x_0. Thus τ defines a mapping, denoted by the same letter, which maps x_0 into the end point of the lift τ^* through x_0. From the condition 2), we have $\tau(x \cdot a) = \tau(x) \cdot a$ for any point x over u_0 and $a \in G$. It follows that τ is a differentiable isomorphism of the fiber over u_0 onto the fiber over u_1. This is called the *parallel displacement along the curve* τ.

Let x_0 be any arbitrary but fixed point of P. The set of parallel displacements which correspond to all closed curves at $u_0 = \pi(x_0)$ form a subgroup Ψ of G, which is by definition the *holonomy group* of the given connection Γ. Corresponding to all closed curves which are homotopic to zero, we have a subgroup Ψ^0 of Ψ, called the *restricted holonomy group* of the given connection. It is a connected Lie group which is proved to the identity component of Ψ. Ψ is a Lie group such that Ψ/Ψ^0 is at most countable. For the proof, see [1]. We remark that the fundamental group of B is at most countable as it follows from our assumption on the second axiom of countability.

2. Reduction of the structure group. Given two principal fiber bundles $P'(B', G')$ and $P(B, G)$ over the same base manifold B, a differentiable mapping f of P' into P is called a *homomorphism* if there exists a homomorphism f of G' into G such that $f(x'a') = f(x')f(a')$, where $x' \in P'$ and $a' \in G'$, and if it induces the identity transformation of the base B onto itself. If it is one-to-one, it is called an *isomorphism* of P' into P. In the case where G' is a Lie subgroup of G, an isomorphism f which corresponds to the injection of G' into G is called an *injection* of $P'(B', G')$ into $P(B, G)$.

The structure group of a principal fiber bundle $P(B, G)$ is said to be *reducible* to a Lie subgroup G' of G if there exists a differentiable principal fiber bundle $P'(B, G')$ and an injection f into $P(B, G)$. In this case, $P'(B, G')$ together with the injection f is called the *reduced bundle*.

The structure group G of $P(B, G)$ is reducible to G' if and only if there exists a suitable covering $\{U_\alpha\}$ of B and an isomorphism of $\pi^{-1}(U_\alpha)$ with $U_\alpha \times G$ expressed by $x \in \pi^{-1}(U_\alpha) \to (\pi(x), \varphi_\alpha(x)) \in U_\alpha \times G$ such that the corresponding transit functions $\psi_{\beta\alpha}(u) = \varphi_\beta(x) \cdot \varphi_\alpha(x)^{-1}$, $u = \pi(x)$, take their values in the subgroup G'. It should be remarked that if such a covering and transit functions exist, then $\psi_{\beta\alpha}(u)$ are differentiable mappings from $U_\alpha \cap U_\beta$ into the Lie subgroup G'. This may be proved in the same way as Proposition 1, p. 95, of [6]. We can construct the reduced bundle $P'(B, G')$ as follows. For each α, consider a space X_α which is homeomorphic with $U_\alpha \times G'$, the homeomorphism being denoted by g_α. Let $X = \bigcup_\alpha X_\alpha$ be the topological sum of X_α and introduce an equivalence relation R in X by

$$g_\alpha(u, a') = g_\beta(u, \psi_{\beta\alpha}(u)\, a'), \qquad u \in U_\alpha \cap U_\beta$$

Let P' be the quotient space X/R with quotient topology. G' acts on P' by [class of $g_\alpha(u, a')$] $\cdot\, b'$ = [class of $g_\alpha(u, a'b')$]. It is easy to verify that P' is a differentiable principal fiber bundle $P'(B, G')$ with an obvious injection into $P(B, G)$.

3. Reduction of a connection. We first prove

PROPOSITION 1. *A homomorphism f of $P'(B, G')$ into $P(B, G)$ maps a connection in P' into a connection in P.*

Proof. To establish this, we first define the horizontal subspace at any point of $f(P')$. Let $f(x') = x$. We define Q_x as the image by f of the horizontal subspace $Q'_{x'}$ at x' of the given connection in P'. It is independent of the choice of x' such that $f(x') = x$. In fact, if $f(x') = f(y') = x$, then $y' = x'a'$ for some $a' \in G'$ and $Q'_{y'} = R_{a'} \cdot Q_{x'}$. We have $f \cdot R_{a'} = R_a \cdot f$, where $a = f(a')$ is indeed the identity of G because $f(x'a') = f(x')f(a') = f(x')$ implies that $f(a') = e$. From this we have $f \cdot Q'_{y'} = f \cdot R_{a'} \cdot Q'_{x'} = f \cdot Q'_{x'}$.

For any other point y of P, we define the horizontal subspace at y by $Q_y = R_a \cdot Q_x$ if $y = xa$ with $x \in f(P')$ and $a \in G$ (any point of P is written in such a form). Q_y is defined independently of the representation $y = x_1 a_1 = x_2 a_2$, where x_1 and x_2 belong to $f(P')$. It is easy to verify that $y \to Q_y$ defines a connection in P.

In the case where f is an injection, we say that a given connection Γ in P is *reducible* to a connection Γ' in P' if f maps Γ' into Γ in the manner of Proposition 1.

Now we are in a position to establish

PROPOSITION 2 (*reduction theorem for connections*). *Let $P(B, G)$ be a principal fiber bundle with a connection Γ, and let Ψ be the holonomy group of Γ with reference point at x_0 of P. Then the structure group G is reducible to Ψ, and the connection Γ is reducible to a connection in the reduced bundle $P'(B, \Psi)$. Moreover, the reduced bundle may be regarded as a subbundle of P consisting of points which can be joined to x_0 by a horizontal curve.*

Proof. Let $\pi(x_0) = u_0$. We construct a covering $\{U_\alpha\}$ of B and a set of transit functions all taking values in Ψ. For this purpose, we take any covering $\{U_\alpha\}$ of B, each U_α being a cube $|u^i| < \delta_\alpha$, $\delta_\alpha > 0$, with center $u_\alpha = (0, 0, \ldots, 0)$ with respect to some local coordinate system $(u^1, \ldots, u^n)$. We choose once for all a family of curves τ_α in B all starting from u_0 and each ending at u_α. Let $x_\alpha = \tau_\alpha(x_0)$. Now for any point x in $\pi^{-1}(U_\alpha)$ we take the ray ρ_α in U_α (with respect to the given local coordinates) from $u = \pi(x)$ to u_α. We have $\rho_\alpha(x) = x_\alpha \cdot a$ for some $a \in G$ and we define this element a to be $\varphi_\alpha(x)$. Then we easily see that $x \to (\pi(x), \varphi_\alpha(x))$ gives an isomorphism of $\pi^{-1}(U_\alpha)$ with $U_\alpha \times G$. If $x \in \pi^{-1}(U_\alpha \times U_\beta)$, a simple manipulation shows that $\psi_{\beta\alpha}(x) = \varphi_\beta(x) \cdot \varphi_\alpha^{-1}(x)$ belongs to Ψ. Namely, we have transit functions all taking values in Ψ.

This shows that G is reducible to Ψ. The reduced bundle $P'(B, \Psi)$, which is constructed from the transit functions $\psi_{\beta\alpha}$ as we have indicated in 2, may be mapped into P in the following manner. We map a point of P' represented by the class (u, a), $u \in U_\alpha$, $a \in \Psi$, into the point $x = \rho_\alpha^{-1}(x_\alpha \cdot a)$ of P. This mapping, denoted by f, is an injection. We show that $f(P')$ coincides with the set of points P_0 which can be joined to x_0 by a horizontal curve. If we write $x \sim y$ when x and y can be joined by a horizontal curve, then $x_0 \sim x_\alpha$ by the definition of x_α, and hence $x_0 a \sim x_\alpha a$. Since $a \in \Psi$, we have $x_0 \sim x_0 \cdot a$. Then $x = \rho_\alpha^{-1} \cdot (x_\alpha \cdot a) \sim x_\alpha \cdot a \sim x_0$. We have thus shown that any point $f(P')$ can be joined to x_0 by a horizontal curve. The converse is also easy to prove.

We shall now show that the given connection in P may be reduced to a connection in the reduced bundle P_0. If $x \in P_0$, then the horizontal subspace Q_x of the given connection in P is tangent to P_0. In fact, any horizontal vector at x is the tangent vector to a certain horizontal curve, which must belong to P_0 by the definition of P_0. Thus $x \in P_0 \to Q_x$ defines a connection in P_0 and the original connection in P is induced from this connection in P_0 in the manner of Proposition 1. We have thereby concluded the proof of Proposition 2.

It might be remarked that the holonomy group of P_0 is exactly Ψ.

II. Holonomy and isotropy groups

4. A theorem on G/H. Let G be a connected Lie group and H a closed subgroup. The homogeneous space G/H may be considered as the base space of a principal fiber bundle G on which H acts to the right in the natural fashion. G acts on G itself to the left.

PROPOSITION 3. *In order that there exist a connection in the principal fiber bundle $G(G/H, H)$ which is invariant by the left translations of G, it is necessary and sufficient that G/H is reductive in the sense of* [11], *that is, there is a subspace $\mathfrak{m}$ of the Lie algebra $\mathfrak{g}$ of G such that $\mathfrak{g} = \mathfrak{m} + \mathfrak{h}$ (direct sum), $\mathfrak{h}$ being the subalgebra determined by H, and such that $ad(H) \cdot \mathfrak{m} = \mathfrak{m}$. More precisely, by taking such a subspace $\mathfrak{m}$ as the horizontal subspace at the identity element e of G, we can define an invariant connection in G, and vice versa.*

Proof. We regard $\mathfrak{g}$ as the tangent space of G at the identity element [6]. If there is a connection Γ in $G(G/H, H)$, the horizontal subspace at e is a subspace $\mathfrak{m}$ of $\mathfrak{g}$ such that $\mathfrak{g} = \mathfrak{m} + \mathfrak{h}$, where $\mathfrak{h}$ is considered as the subspace tangent to the fiber H through e. By the condition 2) for a connection, we see that the horizontal subspace at a point $h \in H$ is the image of $\mathfrak{m}$ by the right translation R_h of G. If Γ is invariant by the left translations of G, then the horizontal subspace at h must be the image of $\mathfrak{m}$ by the left translation L_h. Hence we must have $R_h \cdot \mathfrak{m} = L_h \cdot \mathfrak{m}$, that is, $ad(h) \cdot \mathfrak{m} = \mathfrak{m}$. Since this is true for every $h \in H$, we have $ad(H) \cdot \mathfrak{m} = \mathfrak{m}$.

Conversely, if there exists a subspace m satisfying $\mathfrak{g} = \mathfrak{m} + \mathfrak{h}$ and $ad(H) \cdot \mathfrak{m} = \mathfrak{m}$, we take $\mathfrak{m}$ as the horizontal subspace at e in G. For any element $a \in G$, the horizontal subspace Q_a at a is defined to be the image of $\mathfrak{m}$ by the left translation L_a. For any element h of H, we have then $R_h \cdot Q_a = R_h \cdot L_a \cdot \mathfrak{m} = L_a \cdot R_h \cdot \mathfrak{m} = L_a \cdot L_h \cdot \mathfrak{m} = L_{ah} \cdot \mathfrak{m} = Q_{ah}$, which is the condition 2) for connections. The condition 3) is easy to verify. Proposition 3 is hence proved.

From now on, we assume that G is effective on G/H, that is, H does not contain any invariant subgroup ($\neq e$) of G. If furthermore the canonical homomorphism of H onto the linear isotropy group $\tilde{H}$, is one-to-one, then the principal fiber bundle $G(G/H, H)$ admits an injection in the bundle of frames $P(G/H, GL(n, R))$ over G/H. Namely, we fix an arbitray frame x_0 at P_0 of G/H and map $a \in G$ into the frame which is the image of x_0 by the transformation of G/H induced by a. This mapping f is an injection and is commutative with the transformations of G acting on G to the left and on P in the natural fashion.

If $G(G/H, H)$ admits a connection Γ invariant by the left translations of G, then the injection f induces, in virtue of Proposition 1, a connection Γ' in P which is invariant by G. In this way, we obtain an affine connection on G/H invariant by G.

Let X be any element of $\mathfrak{m}$ and let x_t be the one-parameter group of G generated by X. Its tangent vector x_t' at the point x_t is obtained from X by the left translation L_{x_t} and belongs to the horizontal subspace at x_t. Therefore, x_t is a horizontal curve in G. If h is an arbitrary element of H, then the curve $R_h \cdot x_t = L_{x_t} \cdot h$ is also horizontal. Let $\tilde{x}_t$ be the image of x_t by the canonical projection of G onto G/H. Then the parallel displacement along the curve $\tilde{x}_t$

of the fiber H is the same as the left translation of H by x_t.

This property of the parallel displacement is transfered to the corresponding invariant affine connection Γ' on G/H. Hence Γ' is what we have called the canonical affine connection of the second kind on the reductive homogeneous space G/H [11]. The torsion and curvature tensor fields being invariant by G, they are also invariant by the parallel displacement. This means that their covariant derivatives are zero.

Conversely, we start with a homogeneous space G/H which admits an invariant affine connection. The canonical homomorphism of H onto $\tilde{H}$ is one-to-one [10]. If furthermore the homogeneous holonomy group Ψ is contained in $\tilde{H}$, then the affine connection is reducible, in virtue of Proposition 2, to a connection in the reduced bundle $P'(G/H, \Psi)$ and, a fortiori, to a connection Γ in $G(G/H, H)$ which contains $P'(G/H, \Psi)$. Γ is certainly invariant by the left translations of G. We have thereby proved

THEOREM 1. *Let G/H be a homogeneous space of a connected Lie group G over a closed subgroup H, where H does not contain any invariant subgroup $(\neq e)$ of G. If G/H admits an invariant affine connection whose homogeneous holonomy group is contained in the linear isotropy group $\tilde{H}$, then G/H is reductive and the covariant derivatives of the torsion and curvature tensor fields are all zero.*

5. The main theorem. Let M be a Riemannian manifold. We denote by G the largest connected group of isometries of M. By the isotropy group H_p at a point p of M we mean the subgroup of G consisting of isometries which leave the point p invariant. The linear isotropy group $\tilde{H}_p$ is the group of linear transformations of the tangent space T_p induced by the elements of H_p. We shall now prove

THEOREM 2. *Let M be a complete Riemannian manifold. If the restricted homogeneous holonomy group Ψ_p^0 is contained in the linear isotropy group $\tilde{H}_p$ at each point p of M, then M is Riemannian symmetric, that is, the covariant derivatives of the curvature tensor field are zero.*

Proof. The proof is divided into three steps.

1) First, we show that it is sufficient to prove the theorem in the case where M is simply connected. Let $\tilde{M}$ be the universal covering manifold

of M with a natural Riemannian metric induced by the projection π. Of course, M is symmetric if and only if $\tilde{M}$ is symmetric. We show that the assumption of Theorem 2 remains valid for $\tilde{M}$. The homogeneous holonomy group of $\tilde{M}$ is isomorphic with the restricted homogeneous holonomy group of M in the manner precised in [13]. On the other hand, if $\tilde{p}$ is any point of $\tilde{M}$ and if $\pi(\tilde{p}) = p$, every isometry of M belonging to H_p^0 can be lifted to an isometry of $\tilde{M}$ which belongs to the isotropy group at $\tilde{p}$ of $\tilde{M}$ (Lemma 6, [13]). This proves our assertion.

2) Assume that M is simply connected and complete. We show that it is sufficient to prove Theorem 2 in the case where M is irreducible. For this purpose, consider the canonical decomposition $M = M_0 \times M_1 \times \ldots \times M_r$ given by G. de Rham [13], [14]. M is symmetric if and only if every factor M_i is symmetric (the Euclidean part M_0 is trivially so). We show that the assumption of Theorem 2 is valid for each M_i.

Let $p = (p_0, p_1, \ldots, p_r) \in M_0 \times M_1 \times \ldots \times M_r$. The homogeneous holonomy group Ψ_p of M is then the direct product $\Psi_0 \times \Psi_1 \times \ldots \times \Psi_r$ where each Ψ_i is the homogeneous holonomy group of M_i and acts trivially on T_j if $j \neq i$ [13], [14]. On the other hand, let φ be any isometry of M belonging to H_p. Then φ leaves invariant each T_i [13] and hence induces an isometry φ_i of M_i. Obviously, $\varphi_i(p_i) = p_i$. If we consider the isometry ψ of M defined by $\psi(q) = (\varphi_0(q_0), \varphi_1(q_1), \ldots, \varphi_r(q_r))$ for $q = (q_0, q_1, \ldots, q_r) \in M_0 \times M_1 \times \ldots \times M_r$, then the linear transformation of T_p induced by ψ coincides with the one induced by φ. Hence $\psi = \varphi$. It is also clear that each φ_i belongs to the isotropy group H_i at p_i of M_i. Hence $H_p = H_0 \times H_1 \times \ldots \times H_r$.

From these considerations, it follows that if Ψ_p is contained in $\tilde{H}_p$, then Ψ_i is contained in $\tilde{H}_i$ for each i. This proves that the assumption of Theorem 2 is valid for each M_i.

3) Finally, we assume that M is simply connected and irreducible non-Euclidean. The orbit $G(p)$ of G of an arbitray point p is a submanifold of M; more precisely, the injection of $G(p)$ provided with the differentiable structure as a homogeneous space G/H_p into M is of maximum rank at every point of $G(p)$. Let T_p' be the subspace of T_p tangent to $G(p)$. Since T_p' is obviously invariant by $\tilde{H}_p$, it is invariant by Ψ_p which is irreducible by assumption. Therefore we have two cases: either $T_p' = (0)$ or $T_p' = T_p$. If $T_p' = (0)$ at every

point p of M, it means that $G(p)=p$ for every p. H_p and hence Ψ_p consists of the identity, that is, M is Euclidean. This is contradictory to the assumption.

Hence there exists at least one point p such that $T'_p = T_p$. This means that $G(p)$ contains a neighborhood of p and hence is an open set in M. Then $G(p)$ is complete as a Riemannian homogeneous space G/H_p. Therefore we must have $M = G(p) = G/H_p$. We conclude the proof of Theorem 2 by applying Theorem 1.

Remark. In Theorem 2, it is sufficient to assume that the homogeneous holonomy group is contained in the linear isotropy group at each point of a (non-empty) open set U of M. Indeed, in the above proof, we have only to check the following point in 3). If $T'_p = (0)$ at every point p of U, then $G(p) = p$ in U. Since any isometry which is the identity on an open set is the identity on the whole manifold, we see that G consists of the identity only.

6. Applications. Let M be a complete Riemannian manifold of dimension n. Let r and s be the dimension of the largest connected group of isometries G and that of the restricted homogeneous holonomy group Ψ_p^0, respectively.

THEOREM 3. *Let M be a complete Riemannian manifold whose restricted homogeneous holonomy group is irreducible. If $r = s + n$, then either M is a Kählerian manifold whose Ricci curvature is zero or M is symmetric.*

Proof. Let p be an arbitray point of M and let H_p be the isotropy group at p. G/H_p being the orbit of p by G, we have $r - \dim H_p^0 \leqq n$. On the other hand, if M is not a Kählerian manifold whose Ricci curvature is zero, the connected component of the normalizor of Ψ_p^0 coincides with Ψ_p^0 [9]. Therefore, $\tilde{H}_p^0$ is contained in Ψ_p^0 and hence $\dim \tilde{H}_p^0 \leqq s = r - n$. From these two inequalities, we get $\dim \tilde{H}_p^0 = r - n = \dim \Psi_p^0$ and hence $\tilde{H}_p^0 = \Psi_p^0$. This holds at every point p. M is then symmetric in virtue of Theorem 2.

For any Riemannian manifold whose Ψ_p^0 is irreducible and which is not a Kählerian manifold with Ricci curvature equal to zero, we have always $r \leqq s + n$ [8]. Theorem 3 gives another characteristic property of symmetric spaces among the class of Riemannian manifolds in consideration.

THEOREM 4. *Let M be a compact Riemannian manifold whose restricted homogeneous holonomy group is irreducible. If $r = s + n$, then M is symmetric.*

Proof. We have only to show that the Ricci curvature is not zero. If it is zero, then the covariant derivatives of any Killing vector field are zero [2]. Since M is irreducible, such a vector field is zero. This means that G consists of the identity only and $r=0$, which is contradictory to the assumption $r=s+n$.

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