ON THE DIMENSION OF MODULES AND ALGEBRAS, II

(FROBENIUS ALGEBRAS AND QUASI-FROBENIUS RINGS)

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In this paper we study Frobenius algebras and quasi-Frobenius rings with particular emphasis on their cohomological dimensions. For definitions of these cohomological dimensions we refer the reader to Cartan-Eilenberg [3] or Eilenberg [4].

We consider (in §2) symmetric and Frobenius algebras in a setting more general than that of Brauer and Nesbitt [2], [13], and show (in §3) that for such algebras the cohomological dimension is either 0 or ∞.

These results are applied (§4) to the group ring $\Lambda = K(\Pi)$ where Π is a finite group of order r and K is a commutative ring. It is shown that Λ is symmetric and that $\dim \Lambda = 0$ if $rK = K$ and that $\dim \Lambda = \infty$ if $rK \neq K$.

The phenomenon that the cohomological dimension is either 0 or ∞ is again encountered (§5) in a ring Λ which is *left self-injective* i.e. a ring Λ which when regarded as a left Λ-module is injective. Such rings, under different terminologies have been considered recently in Ikeda [7], Nagao-Nakayama [10] and Ikeda-Nakayama [9] in connection with quasi-Frobenins algebras and rings. We further refine these results by showing (§§6, 7) that the notions "quasi-Frobenius ring" and "left self-injective ring" are equivalent for rings which are (left and right) Noetherian, or satisfy minimum condition for left or right ideals.

All rings considered have a unit element which operates as the identity on all modules considered.

§1. Duality

Let K be a commutative ring. For each K-module A we define the *dual* K-module

$$A^\circ = \mathrm{Hom}_K(A, K).$$

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A K-homomorphism $f: A \to B$ induces a dual homomorphism $f^\circ : B^\circ \to A^\circ$, in the well-known natural fashion.

Given K-modules A and C we define

$$\tau : A \otimes_K C \to \mathrm{Hom}_K (A^\circ, C)$$

by setting

$$[\tau(a \otimes c)]f = (fa)c, \qquad a \in A, \quad c \in C, \quad f \in A^\circ.$$

Clearly τ is an isomorphism if $A = K$. Therefore, by a simple direct sum argument it follows that τ is also an isomorphism if A is K-projective and finitely K-generated.

Taking $C = K$ we obtain a natural homomorphism

$$\tau : A \to A^{\circ\circ}$$

which is again an isomorphism if A is K-projective and finitely K-generated.

For each K-homomorphism

$$\varphi : A \to C^\circ$$

the *transposed* homomorphism

$$\varphi' : C \to A^\circ$$

is defined by composition

$$C \xrightarrow{\tau} C^{\circ\circ} \xrightarrow{\varphi^\circ} A^\circ.$$

It is easily seen that

$$(\varphi' c)a = (\varphi a)c.$$

This shows that $\varphi'' = \varphi$. If C is K-projective and finitely K-generated and if φ is an isomorphism then φ' is an isomorphism.

Now assume that Λ is a K-algebra and that A is a left (right) Λ-module. Then A° is a right (left) Λ-module and it follows from the naturality of τ that τ is a Λ-homomorphism. If A is a left Λ-module, C is a right Λ-module and φ above is a Λ-homomorphism then φ' also is a Λ-homomorphism.

As usual we shall regard Λ as a two-sided Λ-module and consequently Λ° is also a two-sided Λ-module. It should however be noted that in defining the structure of Λ° as a left (right) Λ-module we utilize the structure of Λ as a right (left) Λ-module.

§ 2. Frobenius algebras

DEFINITION Let Λ be a K-algebra which is K-projective and finitely K-generated. We shall say that Λ is a *Frobenius algebra* if there exists a left Λ-module isomorphism

$$\Phi : \Lambda \approx \Lambda^\circ.$$

If there exists a two-sided Λ-module isomorphism $\Phi : \Lambda \approx \Lambda^\circ$ then Λ is called a *symmetric algebra.*

Starting with a left Λ-module isomorphism Φ as above we obtain a right Λ-module isomorphism

$$\Phi' : \Lambda \approx \Lambda^\circ,$$

by taking the transposed

$$[\Phi'\lambda](\gamma) = [\Phi\gamma](\lambda), \qquad \lambda, \gamma \in \Lambda.$$

Since $\Phi\lambda = \lambda\Phi 1$ and $\Phi'\lambda = (\Phi' 1)\lambda$ it follows readily that

$$(*) \qquad [\Phi\gamma](\lambda) = \varphi(\lambda\gamma) = [\Phi'\lambda](\gamma)$$

where

$$\varphi = \Phi 1 = \Phi' 1 : \Lambda \to K.$$

Conversely we may begin with a K-homomorphism $\varphi : \Lambda \to K$ and define Φ (and Φ') using $(*)$. Then Φ is a left Λ-homomorphism $\Lambda \to \Lambda^\circ$. The conditions that Φ be a monomorphism and epimorphism are respectively

(l. 1) $\quad \varphi(\lambda\gamma) = 0$ for all $\lambda \in \Lambda$ implies $\gamma = 0$,

(l. 2) $\quad$ for each $f \in \Lambda^\circ$, there is a $\gamma \in \Lambda$ such that $f\lambda = \varphi(\lambda\gamma)$.

Applying the same reasoning to Φ' we find the conditions

(r. 1) $\quad \varphi(\lambda\gamma) = 0$ for all $\gamma \in \Lambda$ implies $\lambda = 0$,

(r. 2) $\quad$ for each $f \in \Lambda^\circ$ there is a $\lambda \in \Lambda$ such that $f\gamma = \varphi(\lambda\gamma)$.

The two sets of conditions are thus equivalent.

If Λ is symmetric and $\Phi : \Lambda \approx \Lambda^\circ$ is a two-sided Λ-isomorphism, then the relation $\Phi 1 = \Phi' 1$ implies $\Phi = \Phi'$, or equivalently

$$(\text{s}) \qquad \varphi(\lambda\gamma) = \varphi(\gamma\lambda).$$

Conversely if φ satisfies this condition (in addition to conditions (l. 1), (l. 2) or

(r. 1), (r. 2)) then $\Phi = \Phi'$ is a two-sided isomorphism $\Lambda \approx \Lambda^\circ$, and Λ is symmetric.

Remark. Condition (1. 1) asserts that the "hyperplane" in Λ given by $\varphi = 0$ does not contain any left ideals except zero.

Remark. If K is a field, then Λ and Λ° are vector spaces of the same finite degree over K. Thus if Φ is a monomorphism, it is necessarily an isomorphism. Consequently in this case condition (1. 2) is a consequence of (1. 1). Similarly (r. 2) follows from (r. 1). Conditions (1. 1) and (r. 1) are then equivalent.

PROPOSITION 1. Let Λ_1, Λ_2 be K-algebras and $\Lambda = \Lambda_1 + \Lambda_2$ their direct product. Then Λ is a Frobenius (or a symmetric) algebra if and only if the same holds for Λ_1 and Λ_2.

Proof. Clearly Λ is K-projective and finitely K-generated if and only if Λ_1 and Λ_2 are. The rest follows from the isomorphism $(A_1 + A_2)^\circ \approx A_1^\circ + A_2^\circ$ for the direct sum of any K-modules A_1 and A_2.

PROPOSITION 2. If Λ_1 and Λ_2 are Frobenius (or symmetric) K-algebras, then so is $\Lambda = \Lambda_1 \otimes_K \Lambda_2$.

Proof. Clearly Λ is K-projective and finitely K-generated.

For any K-modules A_1 and A_2 consider the mapping

$$\zeta : A_1^\circ \otimes_K A_2^\circ \to (A_1 \otimes_K A_2)^\circ$$

given by

$$[\zeta(f_1 \otimes f_2)](a_1 \otimes a_2) = f_1 a_1 \otimes f_2 a_2 .$$

Clearly ζ is an isomorphism for $A_1 = A_2 = K$. It follows that ζ is an isomorphism also if A_1 and A_2 are K-projective and finitely K-generated. Thus

$$\zeta : \Lambda^\circ \approx \Lambda_1^\circ \otimes_K \Lambda_2^\circ$$

and, since ζ is natural, this is an isomorphism of two-sided Λ-modules. This yields the conclusion.

PROPOSITION 3. *A* full matrix algebra Λ over a commutative ring K is symmetric.

Proof. The K-homomorphism $\Lambda \to K$ given by the trace is easily seen to satisfy (1. 1), (1. 2) and (s).

PROPOSITION 4. A division algebra Λ over a field K with $(\Lambda : K) < \infty$ is symmetric.

Proof. For each K-algebra Λ let $[\Lambda, \Lambda]$ denote the subgroup generated by the commutators $\lambda\gamma - \gamma\lambda$, for $\lambda, \gamma \in \Lambda$. Automatically $[\Lambda, \Lambda]$ is a K-module.

Assume now that Λ is a division algebra over the field K with $(\Lambda : K) < \infty$. Any non-zero K-homomorphism $\varphi : \Lambda \rightarrow K$ satisfies (1.1), so that Λ is a Frobenius algebra. In order that φ satisfy also condition (s) it is necessary and sufficient that $\varphi[\Lambda, \Lambda] = 0$. Thus it suffices to prove that $[\Lambda, \Lambda] \neq \Lambda$. This last statement being independent of the ground-field we may assume that K is the center of Λ. Let L be an extension of K which is a splitting field for Λ. Then the L-algebra $\Lambda_L = \Lambda \otimes_K L$ is a full matrix algebra over L and thus, by Prop. 3, Λ_L is a symmetric L-algebra. It follows that $[\Lambda_L, \Lambda_L] \neq \Lambda_L$. Since $[\Lambda_L, \Lambda_L] = [\Lambda, \Lambda]_L$, we deduce that $[\Lambda, \Lambda] \neq \Lambda$.

Combining Prop. 1, 2, 3 and 4 we obtain (cf. [11], footnote 51):

PROPOSITION 5. A semi-simple algebra Λ over a field K with $(\Lambda : K) < \infty$ is symmetric.

PROPOSITION 6. Let Λ be an algebra over a field K with $(\Lambda : K) < \infty$, and let L be an extension field of K. The algebra $\Lambda_L = \Lambda \otimes_K L$ over L is a Frobenius (or a symmetric) algebra if and only if Λ is so.

Proof. This follows easily from our definition of Frobenius and symmetric algebras and the fact (E. Noether's lemma) that two (left, right or two-sided) Λ-modules A_1, A_2 with $(A_1 : K) < \infty$, $(A_2 : K) < \infty$ are isomorphic if and only if the Λ_L-modules $(A_1)_L = A_1 \otimes_K L$ and $(A_2)_L = A_2 \otimes_K L$ are isomorphic (cf. e.g. M. Deuring, Galoissche Theorie und Darstellungstheorie, Math. Ann. 107 (1933), p. 144).

§ 3. Dimension in Frobenius algebras

PROPOSITION 7. If Λ is a Frobenius algebra, over a commutative ring K, then we have a natural isomorphism

$$\mathrm{Ext}^q_\Lambda(A, \Lambda \otimes_K C) \approx \mathrm{Ext}^q_K(A, C) \tag{3.1}$$

for each left Λ-module A and each K-module C.

Proof. Assuming a fixed isomorphism $\Phi' : \Lambda \approx \Lambda^\circ$ (of right Λ-modules) and utilizing the map τ we have

$$\mathrm{Hom}_\Lambda(A, \Lambda\otimes_K C) \approx \mathrm{Hom}_\Lambda(A, \mathrm{Hom}_K(\Lambda^\circ, C)) \approx \mathrm{Hom}_\Lambda(A, \mathrm{Hom}_K(\Lambda, C))$$

$$\approx \mathrm{Hom}_K(\Lambda\otimes_\Lambda A, C) \approx \mathrm{Hom}_K(A, C).$$

This gives the desired isomorphism (3.1) for $q=0$. Now replace A by a Λ-projective resolution X. Since Λ is K-projective, X also is a K-projective resolution of A. Passing to homology yields (3.1) in virtue of the definitions of Ext_Λ and Ext_K.

Remark. The isomorphism (3.1) although "natural" is not necessarily unique, as it depends upon the choice of Φ'. With $\Phi' : \Lambda \approx \Lambda^\circ$ fixed, the isomorphisms (3.1) obtained, commute properly with maps induced by Λ-homomorphisms of A and K-homomorphisms of C.

COROLLARY 8. If Λ is a Frobenius K-algebra and C is a K-module, then $\mathrm{l.\,inj.\,dim}_\Lambda(\Lambda\otimes_K C) \leqq \mathrm{inj.\,dim}_K C$.

COROLLARY 9. If Λ is a Frobenius algebra and K is self-injective then Λ is both left and right self-injective.

THEOREM 10. *If Λ is a Frobenius K-algebra and A is a left Λ-module satisfying $l.\,dim_\Lambda A < \infty$, then*

$$\mathrm{l.\,dim}_\Lambda A = \mathrm{l.\,dim}_K A.$$

Proof. Let $\mathrm{l.\,dim}_\Lambda A = n < \infty$, and let C be a left Λ-module such that $\mathrm{Ext}_\Lambda^n(A, C) \neq 0$. Consider an exact sequence $0 \to B \to F \to C \to 0$ with F Λ-free. Since $\mathrm{Ext}_\Lambda^{n+1}(A, B) = 0$, it follows from exactness that $\mathrm{Ext}_\Lambda^n(A, F) \to \mathrm{Ext}_\Lambda^n(A, C)$ is an epimorphism and therefore $\mathrm{Ext}_\Lambda^n(A, F) \neq 0$. Since $F \approx \Lambda\otimes_K H$ where H is a free K-module, it follows from Prop. 7 that $\mathrm{Ext}_K^n(A, H) \neq 0$. Thus $\mathrm{l.\,dim}_\Lambda A \leqq \mathrm{dim}_K A$. The opposite inequality is trivial since Λ is K-projective.

THEOREM 11. *If Λ is a Frobenins K-algebra then $dim\,\Lambda = 0, \infty$.*

Proof. Clearly Λ^* (the algebra opposite to Λ) also is a Frobenius K-algebra. Thus it follows from Prop. 2 that $\Lambda^e = \Lambda\otimes_K\Lambda^*$ also is a Frobenius K-algebra. Consequently, if $\dim\Lambda < \infty$, we have, by Theorem 10,

$$\dim\Lambda = \mathrm{l.\,dim}_{\Lambda^e}\Lambda = \mathrm{l.\,dim}_K\Lambda = 0.$$

The special case of Theorem 11 with K a field has been established in [8]

(§5, Corollary); cf. also Corollary 21 (and Corollary 19) below. A further particular case is a result of Hochschild [6] (see also [4]) that if Λ is a semi-simple algebra over a field K with $(\Lambda : K) < \infty$ then either Λ is separable (i.e. $\dim \Lambda = 0$) or $\dim \Lambda = \infty$.

§4. The algebra of a finite group

Let Π be a finite group of order r and let K be a commutative ring. The group algebra $\Lambda = K(\Pi)$ is defined as usual. This is a supplemented algebra under the map $\varepsilon : \Lambda \to K$ given by $\varepsilon x = 1$ for all $x \in \Pi$. Using this map, we may regard K as a (right or left) Λ-module.

We first show that $\Lambda = K(\Pi)$ is symmetric. To this end we define a K-homomorphism $\varphi : \Lambda \to K$ by setting $\varphi 1 = 1$, $\varphi x = 0$ for $x \in \Pi$, $x \neq 1$. If $\gamma = \sum_{x \in \Pi} a(x)x$ for $a(x) \in K$, then $\varphi(x^{-1}\gamma) = a(x)$. Thus $\varphi(\lambda\gamma) = 0$ for all $\lambda \in \Lambda$ implies $\gamma = 0$, so that (1.1) is satisfied. Next let $f : \Lambda \to K$ be a K-homomorphism and let $\gamma = \sum f(x)x^{-1}$, $x \in \Pi$. Then for $y \in \Pi$

$$\varphi(y\gamma) = \varphi(\sum f(x)yx^{-1}) = f(y)$$

so that (1.2) also is satisfied. Finally condition (s) holds trivially. Thus Λ is symmetric.

Note that the map φ defined above yields an isomorphism and in consequence, as in the proof of Prop. 7 yields an isomorphism

$$\rho : \mathrm{Hom}_K(A, C) \approx \mathrm{Hom}_\Lambda(A, \Lambda \otimes_K C)$$

for a left Λ-module A and a K-module C. This isomorphism ρ may be written directly as

$$(\rho f)a = \sum_{x \in \Pi} x \otimes f(x^{-1}a), \qquad f \in \mathrm{Hom}_K(A, C), \quad a \in A.$$

We may now apply the results of the preceding section. First we observe that since each K-module may also be regarded as a Λ-module (using $\varepsilon : \Lambda \to K$), Corollaries 8 and 9 may be strengthened as follows

COROLLARY 8′. $\mathrm{l.\,inj.\,dim}_\Lambda(\Lambda \otimes_K C) = \mathrm{l.\,inj.\,dim}_K C$

COROLLARY 9′. Λ is left (or right) self-injective if and only K is self-injective.

THEOREM 12. *If $rK = K$ (where r is the order of the group Π), then*

$$dim\ \Lambda = l.\ dim_\Lambda K = 0.$$

If $rK \neq K$, *then*

$$dim\ \Lambda = l.\ dim_\Lambda K = \infty.$$

Proof. The equality $\dim \Lambda = \mathrm{l.dim}_\Lambda K$ has been proved in [3] (Ch. X, §6) without any case subdivision and is valid also for infinite groups Π. Since $\dim_K K = 0$ it follows from Theorem 10 that $\mathrm{l.dim}_\Lambda K$ is either 0 or ∞. Thus everything reduces to the question as to when K is Λ-projective. Clearly this is the case if and only if there exists a Λ-homomorphism $\psi : K \to \Lambda$, such that $\varepsilon\psi =$ identity. Such a ψ must satisfy $\psi 1 = \sum_{x \in \Pi} \alpha x$ for some $\alpha \in K$. The condition $\varepsilon\psi 1 = 1$ then yields $x\alpha = 1$. The existence of ψ is thus equivalent with $rK = K$.

Remark. The above argument shows that if Π is an infinite group then K is not Λ-projective.

Remark. The proof above utilized the equality $\dim \Lambda = \mathrm{l.dim}_\Lambda K$ established by general methods. Actually, in the case considered here an ad hoc argument can be applied. Indeed, since Λ is symmetric, it follows from Theorem 11 that $\dim \Lambda = 0, \infty$. This we must find out when $\dim \Lambda = 0$ i.e. when Λ is Λ^e-projective, where $\Lambda^e = \Lambda \otimes_K \Lambda^*$. We have the Λ^e-epimorphism $\eta : \Lambda^e \to \Lambda$ given by $\eta(x \otimes y^*) = xy$, $x, y \in \Pi$. Thus Λ is Λ^e-projective if and only if there is a Λ^e-homomorphism $\psi : \Lambda \to \Lambda^e$ with $\eta\psi =$ identity. The map ψ has the form

$$\psi(x) = \sum_{y,z} k(x, y, z) y \otimes z^* \qquad k(x, y, z) \in K$$

with summation extended over all $y, z \in \Pi$. The condition that ψ is a Λ^e-homomorphism becomes

$$k(xz, y, 1) = k(x, y, z) = k(yx, 1, z)$$

so that setting $\gamma(x) = k(x, 1, 1)$ we have

$$\psi(x) = \sum_{y,z} \gamma(yxz) y \otimes z^*.$$

Conversely any map $\gamma : \Pi \to K$ defines a Λ^e-homomorphism ψ.
Then

$$\eta\psi(1) = \sum_{y,z} \gamma(yz) yz = r \sum_x \gamma(x) x$$

and the condition $\eta\psi =$ identity is equivalent with

$$r\gamma(1) = 1, \qquad \gamma(x) = 0 \quad \text{for } x \neq 1.$$

Hence Λ is Λ^e-projective if and only if $rK = K$.

As a corollary of Theorem 12 we obtain the following known result (see [6], p. 948):

Corollary 13. Let K be a field of characteristic p. If $p = 0$ or $(p, r) = 1$ then $\Lambda = K(\Pi)$ is separable. If $(p, r) \neq 1$ then $\dim \Lambda = \mathrm{l.dim}_\Lambda K = \infty$.

In the case $rK = K$ we have a more detailed result which may be of interest.

Proposition 14. If $rK = K$ then for each pair of left Λ-modules A and C the K-module $\mathrm{Ext}_\Lambda^q(A, C)$ is isomorphic with a direct summand of $\mathrm{Ext}_\Lambda^q(A, C)$. This implies

$$\mathrm{l.dim}_\Lambda A = \dim_K A,$$
$$\mathrm{l.inj.dim}_\Lambda C \leqq \mathrm{inj.dim}_K C,$$
$$\mathrm{l.gl.dim}\, \Lambda = \mathrm{gl.dim}\, K.$$

Proof. Let $\alpha \in K$ be such that $r\alpha = 1$. Consider the homomorphisms

$$\mathrm{Hom}_\Lambda(A, C) \xrightarrow{j} \mathrm{Hom}_K(A, C) \xrightarrow{i} \mathrm{Hom}_\Lambda(A, C)$$

where i is the inclusion, while

$$(jf)a = \alpha \sum_x x f(x^{-1}a).$$

If f is a Λ-homomorphisms then $xf(x^{-1}a) = f(a)$ and $jf = f$. Thus $ji =$ identity. Now replace A by a projective resolution X of A and pass to homology. There result homomorphisms

$$\mathrm{Ext}_\Lambda^q(A, C) \longrightarrow \mathrm{Ext}_K^q(A, C) \longrightarrow \mathrm{Ext}_\Lambda^q(A, C)$$

whose composition is the identity. This yields the desired conclusion.

Incidentally, the case $rK = K$ with K not semi-simple gives an example of an algebra $\Lambda = K(\Pi)$ with $\dim \Lambda = 0$ and $\mathrm{l.gl.dim}\, \Lambda = \mathrm{gl.dim}\, K > 0$.

§ 5. Self-injective rings

Let again K be a commutative ring.

Theorem 15. *Let Λ be a K-algebra which is K-projective, finitely K-generated* *and left self-injective. Then each K-projective and finitely K-generated left Λ-module A such that* $l.\dim_\Lambda A < \infty$ *is projective and injective.*

Proof. Since Λ is left self-injective, every free (left) Λ-module on a finite base is injective. Therefore every finitely generated projective left Λ-module is injective.

Assume $\mathrm{l.dim}_\Lambda A = n$, $0 < n < \infty$ and assume the result already proved for $n-1$. Consider the exact sequence

$$0 \longrightarrow B \longrightarrow \Lambda \otimes_K A \xrightarrow{\alpha} A \longrightarrow 0$$

where $\alpha(\lambda \otimes a) = \lambda a$, $B = \mathrm{Ker}\, \alpha$. Since A is K-projective, $\Lambda \otimes_K A$ is Λ-projective and therefore $\mathrm{l.dim}_\Lambda B = n-1$. The K-homomorphism $\zeta : A \to \Lambda \otimes_K A$ given by $\zeta a = 1 \otimes a$ satisfies $\alpha\zeta =$ identity and shows that the exact sequence splits over K. Since both Λ and A are K-projective and finitely K-generated, the same is true for $\Lambda \otimes_K A$ and therefore also for B which is a K-direct summand of $\Lambda \otimes_K A$. Thus by the inductive assumption B is Λ-projective and Λ-injective. Therefore the exact sequence splits (over Λ!) and thus A is Λ-projective, and hence also Λ-injective.

THEOREM 16. *Let Λ be a left Noetherian ring (i.e. a ring satisfying maximum condition for left ideals) which is left self-injective. Then each left Λ-module A such that* $l.\dim_\Lambda A < \infty$ *is projective and injective. In particular* $l.gl.\dim \Lambda = 0, \infty$.

Proof. Since Λ is left Noetherian the direct sum of injective left Λ-modules is injective (see [3], Ch. I, Exer. 8). Therefore every free Λ-module is injective and consequently every projective left Λ-module is injective.

Now let $\mathrm{l.dim}_\Lambda A = n < \infty$. As in the proof of Theorem 10 we have $\mathrm{Ext}^n_\Lambda(A, F) \neq 0$ for some free Λ-module F. Since F is injective it follows that $n = 0$ and A is projective.

PROPOSITION 17. Let Λ be a left Noetherian, left self-injective and non-semisimple ring. If Λ is an algebra over a semisimple commutative ring K then $\dim \Lambda = \infty$.

Proof. Since K is semisimple we have $\mathrm{l.gl.dim}\, \Lambda \leqq \dim \Lambda$ (see [3] or [4]), and by the preceding theorem $\mathrm{l.gl.dim}\, \Lambda = \infty$.

PROPOSITION 18. Let Λ be a left Noetherian ring which is left self-injective. Then for a left ideal $\mathfrak{l}$ of Λ the following conditions are equivalent:

(i) $\mathrm{l.dim}_\Lambda \Lambda/\mathfrak{l} < \infty$,

(ii) $\mathrm{l.dim}_\Lambda \mathfrak{l} < \infty$,

(iii) $\mathfrak{l}$ is projective,

(iv) $\mathfrak{l}$ is injective,

(v) $\mathfrak{l} = \Lambda e$ with an idempotent elemnt e in Λ,

(vi) $\Lambda/\mathfrak{l}$ is projective.

Proof. The equivalences (i)$\Longleftrightarrow$(vi) and (ii)$\Longleftrightarrow$(iii) follow from Theorem 16. Thus it suffices to establish the equivalence of (iii), (iv), (v) and (vi).

(iii)$\Longrightarrow$(iv) follows from Theorem 16.

(iv)$\Longrightarrow$(v). Since $\mathfrak{l}$ is injective there exists a projection $p : \Lambda \to \mathfrak{l}, p\lambda = \lambda$ for $\lambda \in \mathfrak{l}$. Then $\lambda = p(\lambda) = p(\lambda 1) = \lambda p(1)$ for $\lambda \in \mathfrak{l}$. Thus $e = p1$ is idempotent and $\mathfrak{l} = \Lambda e$.

(v)$\Longrightarrow$(vi). If $\mathfrak{l} = \Lambda e$ then $\Lambda/\mathfrak{l} \approx \Lambda(1-e)$ and thus $\Lambda/\mathfrak{l}$ is isomorphic to a direct summand of Λ.

(vi)$\Longrightarrow$(iii) follows from the exact sequence $0 \to \mathfrak{l} \to \Lambda \to \Lambda/\mathfrak{l} \to 0$ which splits if $\Lambda/\mathfrak{l}$ is projective.

§ 6. Quasi-Frobenius rings

For each subset X of a ring Λ we denote by $l(X)$ (or $r(X)$) the set of all left (or right) annihilators of X. Clearly $l(X)$ is a left ideal while $r(X)$ is a right ideal. If X is a two-sided ideal then so are $l(X)$ and $r(X)$.

Quasi-Frobenius rings were defined by Nakayama [12] p. 8. For the purposes of this paper we adopt the following alternative definition ([12], p. 9, Th. 6): A ring Λ is a quasi-Frobenius ring if it satisfies minimum conditions for left and right ideals and if the relations

$$(\alpha) \qquad l(r(\mathfrak{l})) = \mathfrak{l}, \qquad r(l(\mathfrak{r})) = \mathfrak{r}$$

hold for all left ideals $\mathfrak{l}$ and all right ideals $\mathfrak{r}$.

Actually, in the presence of (α) it suffices to assume that Λ is left (or right) Noetherian, i.e. that Λ satisfies maximum condition for left (or right) ideals. The minimum conditions for both left and right ideals then follow.

THEOREM 18. *For each ring Λ the following conditions are equivalent:*

(i) *Λ is a quasi-Frobenius ring.*

(ii) *Λ is left Noetherian and the relations*

$$r(\mathfrak{l}_1 \cap \mathfrak{l}_2) = r(\mathfrak{l}_1) + r(\mathfrak{l}_2), \qquad r(l(\mathfrak{r})) = \mathfrak{r}$$

hold for left ideals $\mathfrak{l}_1$, $\mathfrak{l}_2$ and all right ideals $\mathfrak{r}$.

(iii) *Λ is (left and right) Noetherian and left self-injective.*

(iv) *Λ satisfies minimum condition for right ideals and is left self-injective.*

(v) *Λ satisfies minimum condition for left ideals and is left self-injective.*

Since (i) is symmetric with respect to "left" and "right," it follows that conditions obtained from (ii)-(v) by interchanging "left" and "right" may be added to the list.

The theorem may be looked upon as a generalization of the main theorem of Ikeda [7], Ikeda-Nakayama [9]. The proof of the theorem is postponed to the next section.

Combining the theorem with Corollary 9 we obtain:

COROLLARY 19. If Λ is a Noetherian Frobenius K-algebra over a self-injective (commutative) ring K then Λ is a quasi-Frobenius ring.

If K is Noetherian and Λ is finitely K-generated then also Λ is Noetherian. Thus we obtain:

COROLLARY 20. *A* Frobenius K-algebra Λ over a quasi-Frobenius (commutative) ring K is a quasi-Frobenius ring.

As a further application we obtain the following result in [8] (Corollary to Main Theorem):

COROLLARY 21. Let Λ be an algebra over a field K with $(\Lambda : K) < \infty$. If Λ is a quasi-Frobenius ring then either Λ is separable (i.e. $\dim \Lambda = 0$) or $\dim \Lambda = \infty$.

Proof. Assume $\dim \Lambda < \infty$. Then from [4] we know that $\dim \Lambda = \mathrm{l.dim}_{\Lambda^e} (\Lambda/N)$ where N is the radical of Λ. Since, by Theorem 18, Λ is left self-injective, it follows from Theorem 16 that $\mathrm{l.dim}_{\Lambda} (\Lambda/N) = 0, \infty$. Thus $\dim \Lambda = 0$.

§ 7. Proof of Theorem 18

It is known ([3]; cf. also [1]) that a ring Λ is left self-injective if and only if the following condition holds:

(a) If $\varphi : \mathfrak{l} \to \Lambda$ is a Λ-homomorphism of a left ideal $\mathfrak{l}$ then there exists an element $\lambda \in \Lambda$ such that $\varphi\xi = \xi\lambda$ for all $\xi \in \mathfrak{l}$.

We shall denote by (a*) the same condition restricted to finitely generated left ideals. We shall also consider conditions

(b) $$r(\mathfrak{l}_1 \cap \mathfrak{l}_2) = r(\mathfrak{l}_1) + r(\mathfrak{l}_2),$$

(c) $$r(l(\mathfrak{r})) = \mathfrak{r}$$

concering respectively pairs $\mathfrak{l}_1$, $\mathfrak{l}_2$ of left ideals of Λ and right ideals $\mathfrak{r}$ of Λ. We shall designate by (c*) condition (c) restricted to finitely generated right ideals.

In a recent paper Ikeda- Nakayama [9] (cf. also Ikeda [7]) the following implications were proved

(I-N) $$(a) \Longrightarrow [(b) \ \& \ (c^*)] \Longrightarrow (a^*).$$

These will be used in the sequel; otherwise our proof of Theorem 18 will be self-contained except for basic and well known facts from the theory of rings with minimum conditions.

To prove the theorem we shall establish implications

$$(i) \Longrightarrow (ii) \Longrightarrow (iii) \Longrightarrow (iv) \Longrightarrow (i),$$
$$(i) \Longrightarrow (v) \Longrightarrow (iii).$$

(i)$\Longrightarrow$(ii). We only need to establish (b). Since r establishes an anti-isomorphism of ordered sets, lattice operations must be dualized, so that (b) holds.

(ii)$\Longrightarrow$(iii). From (I-N) we deduce (a*). Since Λ is left Noetherian, we obtain (a); thus Λ is left self-injective. Since Λ satisfies maximum condition for left ideals the condition $r(l(\mathfrak{r})) = \mathfrak{r}$ implies the minimum condition for right ideals. Thus Λ is also right Noetherian.

(iii)$\Longrightarrow$(iv). From (I-N) we deduce (c*), and since Λ is right Noetherian, we have also (c). Since Λ is left Noetherian it follows from $r(l(\mathfrak{r})) = \mathfrak{r}$ that Λ satisfies also minimum condition for right ideals.

(i)$\Longrightarrow$(v). Clearly Λ satisfies minimum condition for left ideals. Further in view of the already proved implications (i)$\Longrightarrow$(ii)$\Longrightarrow$(iii), Λ is left self-injective.

(v)$\Longrightarrow$(iii). Since Λ satisfies minimum condition for left ideals it is left

Noetherian. From (I-N) we deduce (c^*), so that $r(l(\mathfrak{r}))=\mathfrak{r}$ for finitely generated right ideals. It follows that maximum condition holds for finitely generated right ideals. This implies that each right ideal is finitely generated i.e. that Λ is right Noetherian.

(iv)$\Longrightarrow$(i). It suffices to establish the annihilator conditions (α), as left minimum condition follows automatically. From (I-N) we have (c^*), and since Λ is right Noetherian we have also (c), i.e. the annihilator condition for right ideals. This implies that Λ satisfies condition

(d) If $\mathfrak{r}_1 \supsetneqq \mathfrak{r}_2$ are right ideals, then $l(\mathfrak{r}_1) \subsetneqq l(\mathfrak{r}_2)$.

To prove the annihilator condition for left ideals, consider a (left) Λ-homomorphism $\varphi: l(r(\mathfrak{l})) \to \Lambda$ such that $\varphi(\mathfrak{l})=0$. Since Λ is left self-injective, condition (a) holds and therefore there exists an element $\lambda \in \Lambda$ such that $\varphi\xi=\xi\lambda$ for all $\xi \in l(r(\mathfrak{l}))$. Since $\varphi(\mathfrak{l})=0$ we have $\mathfrak{l}\lambda=0$ so that $\lambda \in r(\mathfrak{l})$. This implies $l(r(\mathfrak{l})) \subset l(\lambda)$ so that $\varphi=0$. We have thus proved that

$$\operatorname{Hom}_\Lambda(l(r(\mathfrak{l}))/\mathfrak{l}, \Lambda)=0.$$

The remainder of the argument then follows from the following lemma.

LEMMA 22. *Let Λ be a ring satisfying minimum condition for right (or left) ideals in which* (d) *holds. Then for each non zero left Λ-module A we have $Hom_\Lambda(A, \Lambda) \neq 0$.*

Proof. Let N denote the radical of Λ. Since N is nilpotent we have $A \neq NA$; set $B=A/NA$. Since $NB=0$ it follows that B is completely reducible and thus admits an epimorphism $B \to C$ onto some irreducible left Λ-module C. We thus obtain an epimorphism $A \to C$. This reduces the proof of the lemma to the case when A is irreducible.

Consider the semisimple ring $\Gamma=\Lambda/N$ which we shall regard as a two-sided Λ-module. Let

$$1=E_1+\dots+E_k$$

be a decomposition of the unit element in Λ into mutually orthogonal idempotents such that

$$\Gamma_i=E_i\Gamma=E_i\Gamma E_i=\Gamma E_i$$

are the simple components of Γ $(i=1, 2, \dots, k)$. Then

$$F_i = N + (1 - E_i)\Lambda = N + \Lambda(1 - E_i)$$

are maximal two-sided ideals in Λ. we have

$$\begin{aligned} l(F_i) &= l(N + (1 - E_i)\Lambda) = l(N) \cap l(1 - E_i) \\ &= l(N) \cap \Lambda E_i = l(N)E_i \end{aligned}$$

and similarly

$$r(F_i) = E_i r(N).$$

Thus $l(N)E_i$ and $E_i r(N)$ are two-sided ideals.

If $\mathfrak{z}$ is any two-sided ideal with $\mathfrak{z} \subsetneqq E_i r(N)$ then by condition (d)

$$l(\mathfrak{z}) \supsetneqq l(E_i r(N)) = l(r(F_i)) \supset F_i.$$

Since F_i is a maximal two-sided ideal, it follows that $l(\mathfrak{z}) = \Lambda$ i.e. $\mathfrak{z} = 0$. Thus the two-sided ideal $E_i r(N)$ is either minimal or is zero.

Consider a composition series

$$\Lambda \supsetneqq F_i \supsetneqq \mathfrak{z}_1 \supsetneqq \mathfrak{z}_2 \supsetneqq \ldots \supsetneqq 0$$

of two-sided ideals in Λ. Then by (d)

$$0 = l(\Lambda) \subsetneqq l(F_i) \subsetneqq l(\mathfrak{z}_1) \subsetneqq l(\mathfrak{z}_2) \subsetneqq \ldots \subsetneqq \Lambda.$$

Since the lengths of the series are equal, the latter also is a composition series and therefore $l(F_i) = l(N)E_i$ is a minimal two-sided ideal. Since N is nilpotent we have $Nl(N)E_i \subsetneqq l(N)E_i$ and therefore $Nl(N)E_i = 0$. Thus $l(N)E_i \subset r(N)$, and since this holds for each $i = 1, 2, \ldots, k$, it follows that $l(N) \subset r(N)$. Now consider the direct sum decompositions

$$l(N) = \sum_{i=1}^{k} l(N)E_i, \qquad r(N) = \sum_{i=1}^{k} E_i r(N).$$

In the first, the two-sided ideals are minimal, while in the second one they are minimal or 0. Thus $l(N) \subset r(N)$ implies that

$$l(N) = r(N)$$

and that each of the two-sided ideals $E_i r(N)$ is minimal (and not zero). (Moreover, the two decompositions coincide, i.e. there exists a permutation π of the indices $1, \ldots, k$ such that $E_i r(N) = r(N)E_{\pi(i)}$ (or equivalently $E_i r(N)E_{\pi(i)} \neq 0$)).

Since $NE_ir(N)=0$ it follows that $E_ir(N)$ is completely reducible as a left Λ-module. Let B_i be a minimal left subideal of $E_ir(N)$. Then $B_i \approx \Gamma e_i$ for some primitive idempotent e_i of Γ. Since $B_i \subset E_ir(N)$ we have $B_i = E_iB_i \approx E_i\Gamma e_i$. Thus $e_i \in \Gamma_i$. It follows that Λ contains an isomorphic image of every irreducible left Λ-module. This concludes the proof.

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