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AZUMAYA'S CANONICAL MODULE AND COMPLETIONS OF ALGEBRAS

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Introduction

We are concerned with an algebra S over a commutative ring. Precisely S is a non-commutative ring with identity which is also a finitely generated unital R module such that $r(xy) = (rx)y = x(ry)$ for r in R and $x, y \in S$. In section one, we assume A is a commutative, Artinian ring. Following Goro Azumaya (see (1, p. 273)), we define the canonical module F of A to be the injective hull of A modulo the Jacobson radical of A i.e. $F = I(A/J(A))$. Let S be an algebra over A, we call a bi-S module Q, a canonical S module if Q is isomorphic as a bi-S module to $\mathrm{Hom}_A(S, F)$. Azumaya has shown that the canonical bi-S module is uniquely determined, up to isomorphism, by the ring S and is independent of choice of the base ring. In Prop. 1.2 we show that Q as a left S module is the S hull of S modulo $J(S)$. i.e. $Q = I(S/J(S))$. Moreover the left S endomorphism ring of Q is S. (See Prop. 1.3.)

In section 2 we consider an algebra S over a commutative ring R (without chain conditions). For any maximal ideal $\mathfrak{p}$ of R let $J(\mathfrak{p})$ be the two sided ideal of S such that $\mathfrak{p}S \subset J(\mathfrak{p})$ and $J(\mathfrak{p})/\mathfrak{p}S$ is the Jacobson radical of $S/\mathfrak{p}S$. Then $\bigcap_{\mathfrak{p} \text{ max in } R} J(\mathfrak{p}) = J(S)$, the Jacobson radical of S.

In section 3 we assume R is a commutative, Noetherian ring and S is an R algebra. Let $\mathfrak{p}$ be a maximal ideal of R, then Prop. 3.2 states the left S hull of $S/J(\mathfrak{p}), I_{\mathfrak{p}}$, is $\mathrm{Hom}_R(S, I(R/\mathfrak{p}))$.

If we assume R is semilocal, then we show in Prop. 3.4 that $I(S/J(S))$ is countable generated.

In section 4, Prop. 4.1 we show that the left S endomorphism ring of $I_{\mathfrak{p}}$ is the completion of S with respect to the $\mathfrak{p}S$-adic topology. Also $I_{\mathfrak{p}}$ is injective over its endomorphism ring, see Prop. 4.3. If R is semilocal, then the left S endomorphism ring of $I(S/J(S))$ is the completion

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of S with respect to the $J(S)$ adic topology. Furthermore, $I(S/J(S))$ is injective over its endomorphism ring, see Propositions 4.2 and 4.4.

In section 5, we set $E = \bigoplus_{\mathfrak{p} \text{ max in } R} I_{\mathfrak{p}}$. We show that the left S endomorphism ring of E is inv. $\lim S/\mathfrak{U}$ where $\mathfrak{U}$ is a left ideal of S such that $S/\mathfrak{U}$ is Artinian, see Prop. 5.3. In Prop. 5.5 we show the bicommutator of E is the completion of S with respect to the finite topology.

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§ 1. The Canonical Module in the Artinian Case

We assume A is a commutative, Artinian ring and S an algebra over A. The Jacobson radical of S (respectively A) is $J(S)$ (respectively $J(A)$.)

DEFINITION 1.1. The A *canonical module* is the A injective hull of $A/J(A)$. Denote the canonical module by F.

PROPOSITION 1.1. *The A canonical module F is a finitely generated A module. The ring map $A \to \mathrm{End}_A(F)$, which sends $a \in A$ to $(x \to ax)$, $x \in F$ is an isomorphism.*

Proof. See Azumaya (1, Prop. 10, p. 273)

If S is an algebra over A, then S is left and right Artinian.

DEFINITION 1.2. A bi-S module Q is called *a canonical S-module* if Q is isomorphic as a bi-S module to $\mathrm{Hom}_A(S, F)$.

Remark 1.1. We regard $\mathrm{Hom}_A(S, F)$ as a bi-S module by defining $(sf) = (t \to f(ts)), (fs) = (t \to f(st))$ for $f \in \mathrm{Hom}_A(S, F), s, t \in S$.

So with each base ring of S, there is a canonical S module. Azumaya has shown that the canonical two sided S module is uniquely determined, up to isomorphism, by the ring S and is independent of the choice of the base ring (see 1, Thm. 21, p. 276).

PROPOSITION 1.2. *If Q is the canonical two sided S module, then Q as a left S module (respectively as a right S module) is the left (respectively the right) injective hull of $S/J(S)$ regarding $S/J(S)$ as a left S module (respectively as a right S module). Thus the left (or right) S hull of S/J is a bi-S module.*

Proof. For any base ring A of S, as a two sided S module, $Q \simeq \mathrm{Hom}_A(S, F)$. Now by (3, Prop. 6.1a, p. 30) $\mathrm{Hom}_A(S, F)$ is left and right S injective. It is well known that an injective S module is the hull of its socle. It is also clear that $r_Q(J) = \{q \in Q \mid Jq = 0\}$ is the socle of Q. Now $r_Q(J) = \mathrm{Hom}_A(S/J, F)$ by (1, Lemma 3, p. 275). We decompose $S/J = \bar{S} = \overline{Se}_1 + \cdots + \overline{Se}_n$, where the $\overline{Se}_i$'s are simple subrings and $\bar{e}_i$'s are orthogonal idempotents. Then $r_Q(J) = \bigoplus_{i=1}^{n} \mathrm{Hom}_A(\bar{S}e_i, F) = \bigoplus_{i=1}^{n} \bar{e}_i\bar{S} = S/J$ by (1, Lemma 2, p. 274). Thus the socle of Q as a left (or right S) module is S/J. So as a left (or right S) module Q is the injective hull of S/J.

PROPOSITION 1.3. *Let S be an algebra over a commutative, Artinian ring, then the left S injective hull of $S/J, I$, is finitely generated and contains a copy of every simple S-module. Moreover, the map S to $\mathrm{End}_S I$ which sends s to $(x \to xs), x \in I, s \in S$ is an isomorphism of rings. We can replace left by right in the above.*

Proof. As a bi-S module, I is of QF type (1, Thm. 19, p. 275). Since S is left and right Artinian, we have established (iii) of Theorem 6 (1, p. 259), which is equivalent to (i) of Theorem 6 (1, p. 259). But (i) Theorem 6 is our result.

§ 2. The Jacobson Radical of an Algebra

We assume R is an arbitrary commutative ring and S an R algebra.

PROPOSITION 2.1. *Let M be a non-zero simple left S module. Then there exists a unique maximal ideal $\mathfrak{p}$ of R such that $\mathfrak{p}M = 0$. Thus if $\mathfrak{P}$ is a left maximal ideal of S there exists a unique maximal ideal $\mathfrak{p}$ of R such that $\mathfrak{p}S \subset \mathfrak{P}$. Moreover, $\mathfrak{p} = \{r \in R \mid r \cdot 1_S \subset \mathfrak{P}\}$, if $R \subset$* center *of S, then $\mathfrak{p} = R \cap \mathfrak{P}$.*

Proof. Follows easily from Azumaya (2, Theorem 5, p. 123).

PROPOSITION 2.2. *For any algebra S over R, let $J(\mathfrak{p})$ be, for each maximal ideal $\mathfrak{p}$ of R, the two sided ideal of S such that $\mathfrak{p}S \subset J(\mathfrak{p})$ and $J(\mathfrak{p})/\mathfrak{p}S$ is the Jacobson radical of the residue class algebra $S/\mathfrak{p}S$. Then the radical J of S is the intersection of all the $J(\mathfrak{p})$'s i.e. $J(S) = \bigcap_{\mathfrak{p}\ \text{maximal in}\ R} J(\mathfrak{p})$. So $J(R) \cdot S \subset J(S)$. Moreover, if $\mathfrak{p} \neq \mathfrak{q}$ are maximal ideals of R, then $J(\mathfrak{p}) + J(\mathfrak{q}) = S = \mathfrak{p}S + \mathfrak{q}S$.*

Proof. The first statement is the corollary of Lemma 2 (2, p. 125). If $\mathfrak{p} \neq \mathfrak{q}$, then $S = R \cdot S = (\mathfrak{p} + \mathfrak{q})S \subset \mathfrak{p}S + \mathfrak{q}S \subset J(\mathfrak{p}) + J(\mathfrak{q}) \subset S$. So $S = \mathfrak{p}S + \mathfrak{q}S = J(\mathfrak{p}) + J(\mathfrak{q})$.

§ **3.** From now on we assume R is a commutative, Noetherian ring and S is an R algebra. Thus S is left and right Noetherian. Let $\mathfrak{p}$ be a maximal ideal of R.

Remark 3.1. Let S, R and $\mathfrak{p}$ be as above and $i \geq 1$, then $R/\mathfrak{p}^i$ is a local, Artinian ring, $S/\mathfrak{p}^i S$ is an algebra over $R/\mathfrak{p}^i$ and the radical of $S/\mathfrak{p}^i S$ is $J(\mathfrak{p})/\mathfrak{p}^i S$.

Proof. Now $S/\mathfrak{p}S$ is finite dimensional over $R/\mathfrak{p}$, so $S/\mathfrak{p}S$ is Artinian. Thus the Jacobson radical is nilpotent i.e. for some $k > 0$, $J(\mathfrak{p})^k \subset \mathfrak{p}S$. So $J(\mathfrak{p})^{ik} \subset \mathfrak{p}^i S$, but $S/J(\mathfrak{p})$ is semisimple and so has no non-zero nilpotent ideals. Thus $J(\mathfrak{p})/\mathfrak{p}^i S$ is the Jacobson radical of $S/\mathfrak{p}^i S$.

PROPOSITION 3.1. *Let $\mathfrak{p}$ be a prime ideal of a commutative, Noetherian ring R, call the injective hull of $R/\mathfrak{p}, I$, and let $A_i = \{x \in I \mid \mathfrak{p}^i x = 0\}$, then A_i is a submodule of $I, A_i \subset A_{i+1}$ and $I = \bigcup_i A_i$. Moreover, if $\mathfrak{p}$ is a maximal ideal, then each A_i is finitely generated R-module, thus I is a countable generated R-module.*

Proof. See Matlis (4, Theorem 3.4, p. 520) and (4, Theorem 3.11, p. 525).

PROPOSITION 3.2. *Let $\mathfrak{p}$ be a maximal ideal of a commutative, Noetherian ring and S an algebra over R. Then the left S injective hull of $S/J(\mathfrak{p})$, which we call $I_\mathfrak{p}$, is* $\mathrm{Hom}_R(S, I(R/\mathfrak{p}))$. *Thus $I_\mathfrak{p}$ becomes in the natural way a bi-S module. Moreover,* $\mathrm{Hom}_R(S, I(R/\mathfrak{p}))$ *is the union of the canonical $S/\mathfrak{p}^i S$ modules i.e. $I_\mathfrak{p} = \bigcup_i \mathrm{Hom}_R(S, A_i)$. We can replace left by right in the above.*

Proof. Since S is a finitely generated R module $\mathrm{Hom}_R(S, I(R/\mathfrak{p})) = \bigcup_i \mathrm{Hom}_R(S, A_i)$. Now for each $i > 0$, $\mathrm{Hom}_R(S, A_i) = \mathrm{Hom}_{R/\mathfrak{p}^i}(S/\mathfrak{p}^i S, A_i)$, let $\bar{S} = S/\mathfrak{p}^i S$ and $\bar{R} = R/\mathfrak{p}^i$ we observe $\bar{R}$ is commutative, Artinian and $\bar{S}$ is an algebra over $\bar{R}$. By (1, Thm. 17, p. 272) A_i is the $\bar{R}$ injective hull of $R/\mathfrak{p}$. Thus for each $i > 0$, $\mathrm{Hom}_R(S, A_i) = \mathrm{Hom}_{\bar{R}}(\bar{S}, I_{\bar{R}}(R/\mathfrak{p})) = Q_i$ which is the canonical $\bar{S}$ module. We know by Proposition 1.2 and Remark 3.1, that as a left $\bar{S}$ module Q_i is the injective hull of $S/J(\mathfrak{p})$.

Also $Q_i \subseteq Q_{i+1}$, for $A \subset A_{i+1}$, thus $S/J(\mathfrak{p})$ is a large S submodule of $\bigcup_i Q_i = \mathrm{Hom}_R(S, I(R(\mathfrak{p}))$. But $\mathrm{Hom}(S, I(R/\mathfrak{p}))$ is injective by (3, Prop. 6.1a, p. 30.). Thus $\mathrm{Hom}_R(S, I(R/\mathfrak{p}))$ is the left S injective hull of $S/J(\mathfrak{p})$. For B a subset of S, let $r(B) = \{y \in I_\mathfrak{p} \mid By = 0\}$ and $l(B) = \{y \in I_\mathfrak{p} \mid yB = 0\}$.

PROPOSITION 3.3. *The notation as in Prop.* 3.2, *then* $I_\mathfrak{p} = \bigcup_i r(\mathfrak{p}^i S) = \bigcup_i r(J(\mathfrak{p})^i) = \bigcup_i l(\mathfrak{p}^i S) = \bigcup_i l(J(\mathfrak{p})^i)$.

Proof. Let $i > 0$ and regard Q_i as an S-module, then the S hull of Q_i is $I_\mathfrak{p}$. Now $r(\mathfrak{p}^i S) = Q_i$ as an $S/\mathfrak{p}^i S$ module (see 1, Cor. Thm. 17, p. 273). So $I_\mathfrak{p} = \bigcup_i r(\mathfrak{p}^i S) = \bigcup_i l(\mathfrak{p}^i S)$. Also $S/\mathfrak{p}S$ is Artinian, so for some $k, J(\mathfrak{p})^k \subset \mathfrak{p}S$. Thus $I_\mathfrak{p} = \bigcup_i r(J(\mathfrak{p})^i) = \bigcup_i l(J(\mathfrak{p})^i)$.

We call R semilocal, if R is commutative Noetherian ring with only a finite number of maximal ideals, $\mathfrak{p}_1, \cdots, \mathfrak{p}_t$.

PROPOSITION 3.4. *Let R be a semilocal ring and S an R-algebra. Then the left S injective hull of $S/J(S)$ is* $\mathrm{Hom}_R(S, I(R/J(R))$. *Thus $I(S/J(S))$ becomes a bi-S module in the natural way. We can replace left by right in the above.*

Proof. By Prop. 2.2 and the Chinese Remainder Theorem, $S/J(S) = S/J(\mathfrak{p}_1) \oplus \cdots \oplus S/J(\mathfrak{p}_t)$, so $I_S(S/J(S)) = I_S(S/J(\mathfrak{p}_1)) \oplus \cdots \oplus I_S(S/J(\mathfrak{p}_t)) = \mathrm{Hom}_R(S, I(R/\mathfrak{p}_1)) \oplus \cdots \oplus \mathrm{Hom}_R(S, I(R/\mathfrak{p}_t)) = \mathrm{Hom}_R(S, I(R/J(R))$.

Let $\mathfrak{P}$ be a left maximal ideal of S, we know there exists a unique maximal ideal $\mathfrak{p}$ of R such that $\mathfrak{p}S \subset \mathfrak{P}$. Moreover, if R is contained in the center of S, then $\mathfrak{p} = R \cap \mathfrak{P}$.

PROPOSITION 3.5. *Let $\mathfrak{P}$ be a left maximal ideal of an algebra S over a commutative noetherian ring R. Call the left S injective hull of $S/\mathfrak{P}, I$. Let r $(\mathfrak{p}^i S)$ be $\{x \in I \mid (\mathfrak{p}^i S)x = 0\}$. Then $I = \bigcup_i r(\mathfrak{p}^i s) = \bigcup_i r(J(\mathfrak{p})^i)$.*

Proof. Since $S/\mathfrak{P}$ is a simple left S module, it is a simple left $S/J(\mathfrak{p})$ module. Also $S/J(\mathfrak{p})$ is completely reducible, so $S/\mathfrak{P}$ is isomorphic to a direct summand of $S/J(\mathfrak{p})$. Thus I is a direct summand of $I_\mathfrak{p} = \bigcup_i r(\mathfrak{p}^i S)$. So $I = \bigcup_i r_I(\mathfrak{p}^i S)$.

PROPOSITION 3.6. *Let R, $\mathfrak{p}$, S and $\mathfrak{P}$ be as above. Then the left S injective hull of $S/\mathfrak{P}$ and $S/J(\mathfrak{p})$ are countable generated.*

Proof. Propositions 3.3, 3.5 and 1.3.

PROPOSITION 3.7. *If R is a semilocal ring, then the left* (*or right*) *S injective hull of $S/J(S)$ is countable generated.*

Proof. Propositions 3.6 and 3.4.

§4. We fix a maximal ideal $\mathfrak{p}$ of a commutative, Noetherian ring R. Let S be an R-algebra with the "$\mathfrak{p}S$-adic" topology. We define the completion of S with respect to the $\mathfrak{p}S$-adic topology to be inv. lim $S/\mathfrak{p}^iS$, denoted by $\hat{S}_\mathfrak{p}$. Now $I_\mathfrak{p}$ is a right $\hat{S}_\mathfrak{p}$ module. For let $\hat{s} = (s_i + \mathfrak{p}^iS) \in \hat{S}_\mathfrak{p}$ and $x \in I_\mathfrak{p}$. Then for $k > 0$, $x(\mathfrak{p}^kS) = 0$, (by Prop. 3.3) define $x\hat{s} = xs_k$. If $x(\mathfrak{p}^jS) = 0$, assume $j < k$, then $s_k - s_j \in \mathfrak{p}^jS$ so $x(s_k - s_j) = 0$ or $xs_k = xs_j$. Since $I_\mathfrak{p}$ is a bi S-module (Prop. 3.2), $I_\mathfrak{p}$ becomes a bi-$S - \hat{S}_\mathfrak{p}$ module.

We also consider S with the $J(\mathfrak{p})$-adic topology. We call inv. lim $S/J(\mathfrak{p})^i$, the completion of S with respect to the $J(\mathfrak{p})$-adic topology, denoted by $\hat{S}_{J(\mathfrak{p})}$. As above, $I_\mathfrak{p}$ becomes a bi-$S - \hat{S}_{J(\mathfrak{p})}$ module. Since $\mathfrak{p}S \subset J(\mathfrak{p})$ and $J(\mathfrak{p})^k \subset \mathfrak{p}S$, then $\hat{S}_\mathfrak{p} = \hat{S}_{J(\mathfrak{p})}$.

PROPOSITION 4.1. *The S endomorphism ring of $I_\mathfrak{p}$* (*as either a left or right S module*) *is the completion of S with respect to the $\mathfrak{p}S$-adic or $J(\mathfrak{p})$-adic topologies i.e.* $\mathrm{End}_S I_\mathfrak{p} = \hat{S}_\mathfrak{p}$.

Proof. Since $(\bigcap_i \mathfrak{p}^iS)\cdot I_\mathfrak{p} = 0, I_\mathfrak{p}$ is a left $S/\bigcap_i \mathfrak{p}^iS$ module. In other words, we may assume S is Hausdorff in the $\mathfrak{p}S$-adic topology. Now $I_\mathfrak{p} = \bigcup_i r(\mathfrak{p}^iS)$. So for $f \in \mathrm{End}_S(I_\mathfrak{p})$ $f|_{r(\mathfrak{p}^i)S} \in \mathrm{End}_{S/\mathfrak{p}^iS}(r(\mathfrak{p}^iS))$, where $f|_{r(\mathfrak{p}^iS)}$ means f restricted to $r(\mathfrak{p}^iS)$. It follows that $\mathrm{End}_S I_\mathfrak{p} =$ inv. lim $\mathrm{End}_{S/\mathfrak{p}^iS}(r(\mathfrak{p}^iS))$. We now find for each $i > 0$, $\mathrm{End}_{S/\mathfrak{p}^iS}(r(\mathfrak{p}^iS))$.

In the proof of Prop. 3.3, we showed $r(\mathfrak{p}^iS)$ as a left $S/\mathfrak{p}^iS$ module is the $S/\mathfrak{p}^iS$ hull of $S/J(\mathfrak{p})$. Using Prop. 1.3, we conclude $\mathrm{End}_{S/\mathfrak{p}^iS}(r(\mathfrak{p}^iS)) = S/\mathfrak{p}^iS$, the isomorphism given by right multiplication. Since the following diagram commutes

$$\begin{array}{ccc} \mathrm{End}_S(r(\mathfrak{p}^iS)) & \longleftarrow & \mathrm{End}_S(r(\mathfrak{p}^{i+k}S)) \\ \wr\| & & \wr\| \\ S/\mathfrak{p}^iS & \longleftarrow & S/\mathfrak{p}^{i+k}S \end{array}$$

we conclude that $\mathrm{End}_S(I_\mathfrak{p}) =$ inv. lim $\mathrm{End}_{S/\mathfrak{p}^iS}(r(\mathfrak{p}^iS)) =$ inv. lim $S/\mathfrak{p}^iS$.

By a semilocal ring R, we mean a commutative, Noetherian ring with only a finite number of maximal ideals, $\mathfrak{p}_1, \cdots, \mathfrak{p}_t$.

PROPOSITION 4.2. *Let R be a semilocal ring and S an algebra over R. Then the endomorphism ring of the injective hull of $S/J(S), I(S/J(S))$, is the completion of S with respect to the $J(S)$-adic topology.*

Proof. We have seen (Prop. 3.4) $I(S/J(S)) = \bigoplus_{i=1}^{t} I(S/J(\mathfrak{p}^i))$. Let $\mathfrak{p} \neq \mathfrak{q}$ be maximal ideals of R, we show for $f \in \mathrm{Hom}_S(I_\mathfrak{p}, I_\mathfrak{q})$, then $f = 0$. Let $x \in I_\mathfrak{p}$, then $(\mathfrak{p}^k S)x = 0$ and $(\mathfrak{q}^l S)f(x) = 0$ for $k, l > 0$, by Prop. 3.3. Since $\mathfrak{p}^k + \mathfrak{q}^l = R$, there exists $a \in \mathfrak{p}^k$, $b \in \mathfrak{q}^l$ such that $a + b = 1$. So $f(x) = f(ax + bx) = f(ax) + bf(x) = 0$. Thus $f \equiv 0$. We conclude $\mathrm{End}_S(I(S/J(S)) = \bigoplus_{i=1}^{t} \mathrm{End}_S(I_{\mathfrak{p}_i}) = \bigoplus_{i=1}^{t} \text{inv.}\lim S/\mathfrak{p}_i^j S = S \bigotimes_R \left(\bigoplus_{i=1}^{t} \text{inv.}\lim R/\mathfrak{p}_i^j\right) = S \bigotimes_R \text{inv.}\lim R/J(R)^i = \text{inv.}\lim S/J(R)^i S$.

Now $S/J(R)\cdot S$ is an algebra over the commutative, Artinian ring $R/J(R)$. So $S/J(R)S$ is Artinian, thus its Jacobson radical is nilpotent of index k, so $J(S)^k \subset J(R)S$. Also $J(R)S \subset J(S)$, thus $\text{inv.}\lim S/J(R)^k S = \text{inv.}\lim S/J(S)^k$.

Returning to a commutative, Noetherian ring R, $\mathfrak{p}$ a maximal ideal of R and S an R algebra, we call the left S endomorphism ring of $I_\mathfrak{p}$, $H_\mathfrak{p}$. We have seen (Prop. 4.1) that $H_\mathfrak{p}$ is $\hat{S}$, the completion of S with respect to the $J(\mathfrak{p})$-adic topology. Let $\widehat{J(\mathfrak{p})} = \text{inv.}\lim J(\mathfrak{p})/J(\mathfrak{p})^i$, then $\hat{S}_\mathfrak{p}/\widehat{J(\mathfrak{p})}$ is $S/J(\mathfrak{p})$ as left S modules.

PROPOSITION 4.3. *The notation as above, then $I_\mathfrak{p}$ is an injective $H_\mathfrak{p}$ module. In fact, $I_\mathfrak{p}$ is the $H_\mathfrak{p}$ injective hull of $\hat{S}_\mathfrak{p}/\widehat{J(\mathfrak{p})}$. Moreover, $\hat{A}_k = \{x \in I_\mathfrak{p} \mid x\widehat{J(\mathfrak{p})}^k = 0\}$ and $A_k = \{x \in I_\mathfrak{p} \mid xJ(\mathfrak{p})^k = 0\}$ are equal for all $k > 0$.*

Proof. Denote the right $\hat{S}$ module $\hat{S}/\widehat{J(\mathfrak{p})}$ by C. Let D be the right $\hat{S}$ hull of C. We show C is an essential S submodule of D. Now $\hat{S}$ is a left and right Noetherian ring, since it is an algebra over $\text{inv.}\lim R/\mathfrak{p}^i$. So $D = \bigcup_i D_i$, where $D_i = \{x \in D \mid x\widehat{J(\mathfrak{p})}^i = 0\}$. Let $0 \neq d \in D$ so $d \in \hat{A}_k$ for some k. Also there exists $\hat{s} = (s_i + J(\mathfrak{p})^i) \in \hat{S}_\mathfrak{p}$ such that $0 \neq d\hat{s} \in C$; hence $0 \neq ds_k \in C$. So C is an essential right S module of D. Also by Prop. 3.2, $I_\mathfrak{p}$ is a right S injective module.

Thus we can find a right S map h such that $hg = i$, where $g = (S/J(\mathfrak{p}) \simeq \hat{S}/\widehat{J(\mathfrak{p})} \subseteq D)$ and i are viewed as right S maps.

$$\begin{tikzcd} & 0 \arrow[d] & \\ 0 \arrow[r] & S/J(\mathfrak{p}) \arrow[r, "g"] \arrow[d, hook, "i"] & D \arrow[dl, "h"] \\ & I_{\mathfrak{p}} & \end{tikzcd}$$

Now h is one to one for $S/J(\mathfrak{p})$ is an essential right S module. Since $I_{\mathfrak{p}}$ is a right $\hat{S}$ module and D is an injective $\hat{S}_{\mathfrak{p}}$ module, D is a direct summand of $I_{\mathfrak{p}}$. However, i is essential, so $D = I_{\mathfrak{p}}$. The equality of $\hat{A}_k$ and A_k follows from $\widehat{J(\mathfrak{p})} = J(\mathfrak{p})\hat{S}$.

§ **5.** As usual we assume R is commutative Noetherian and S is an R-algebra. The direct sum (as left S modules) of the $I_{\mathfrak{p}}$'s, $\mathfrak{p}$ ranging over all maximal ideals of R, we call the canonical cogenerator, E. i.e. $E = \oplus I_{\mathfrak{p}}$. Now E is the left S hull of F, where F is the direct sum of the $S/J(\mathfrak{p})$'s. Moreover, since S is a finitely generated R-module, $E = \mathrm{Hom}_R\left(S, \bigoplus_{\mathfrak{p} \text{ max in } R} I_R(R/\mathfrak{p})\right)$. Thus E becomes in the natural way a bi-S module and the right S hull of F. Because E contains a copy of each simple left (right) S module, E is left (right) S cogenerator; hence, E is faithful as a left (right) S module.

We denote by $\boldsymbol{P}$ the totality of all products of powers of maximal ideal of R. If $\mathfrak{p}_1^{t_1}\cdots\mathfrak{p}_m^{t_n} \in \boldsymbol{P}$, then $\mathfrak{p}_1^{t_1} \cap \cdots \cap \mathfrak{p}_m^{t_n} = \mathfrak{p}_1^{t_1}\cdots\mathfrak{p}_n^{t_n}$.

For B a subset of S, we call $r(B) = \{x \in E \mid Bx = 0\}$ and $l(B) = \{x \in E \mid xB = 0\}$.

PROPOSITION 5.1. $E = \bigcup_{w \in \boldsymbol{P}} r(wS) = \bigcup_{w \in \boldsymbol{P}} l(ws)$

Proof. Let $x \in E$, then $x = x_1 + \cdots + x_n, x_i \in I_{\mathfrak{p}_i}; i = 1, \cdots, n$. By Proposition 3.3, $(\mathfrak{p}_1^{k_1}S)x_1 = 0; \cdots; (\mathfrak{p}_n^{k_n}S)x_n = 0$. So $\mathfrak{p}_1^{k_1}\cdots\mathfrak{p}_n^{k_n} = w \in \boldsymbol{P}$ and $(wS)x = 0$.

The n-adic topology of S has as a basis of neighborhoods of zero ideals of the form $wS, w \in \boldsymbol{P}$. We partially order $\boldsymbol{P}$ by inclusion. In fact, $\boldsymbol{P}$ is a direct set. We call $S^* = \mathrm{inv.}\lim_{w \in \boldsymbol{P}} S/wS$, the completion of S with respect to the n-adic topology. Furthermore, E is a bi-$S - S^*$ module. Let $s^* = (s_w + wS) \in S^*, s_w \in S, w \in \boldsymbol{P}$ and $x \in E$, then $0 = x(vS)$ for $v \in \boldsymbol{P}$, define $xs^* = xs_v$. If $x(wS) = 0$ for $w \in \boldsymbol{P}$, then $x((vw)S) = 0$. Thus $s_v - s_{vw} \in vS$ and $s_w - s_{vw} \in wS$, so $xs_v = xs_{vw} = xs_w$. We conclude the multiplication is well defined.

For any $B \subset S$, let $l_F(B) = \{x \in F \mid Bx = 0\}$ and $l_E(B) = \{x \in E \mid Bx = 0\}$, $l_F(B) \subset l_E(B)$. For a fixed $w \in \boldsymbol{P}$, let $\bar{S} = S/wS$ and $\bar{R} = R/w$, $\bar{S}$ is an algebra over the commutative, Artinian ring $\bar{R}$. Thus $\bar{S}$ is both left and right Artinian.

PROPOSITION 5.2. *The notation as above. If $Q = r_E(wS)$, then Q is the canonical bi-$\bar{S}$ module.*

Proof. Since E is the left S hull of $F, r_E(wS)$ is the left $\bar{S}$ hull of $r_F(wS)$. (See 1, Thm. 17, p. 272). Now let $w = \mathfrak{p}_1^{k_1} \cdots \mathfrak{p}_t^{k_t}$, $\mathfrak{p}_1, \cdots, \mathfrak{p}_t$ maximal ideals of R. We show $r_F(wS) = S/J(\mathfrak{p}_1) \oplus \cdots \oplus S/J(\mathfrak{p}_t)$. Since $\mathfrak{p}_1 S \subset J(\mathfrak{p}_1), \cdots, \mathfrak{p}_t S \subset J(\mathfrak{p}_t)$, we have $r_F(wS) \supseteq S/J(\mathfrak{p}_1) \oplus \cdots \oplus S/J(\mathfrak{p}_t)$. Let $x \in r_F(wS)$, so $x = \bar{x}_1 + \cdots + \bar{x}_n$, $0 \neq \bar{x}_i = x_i + J(\mathfrak{q}_i)$, for $x_i \in S$ and $\mathfrak{q}_i$ a maximal ideal of R for $i = 1, \cdots, n$. Now $(wS)x = 0$ implies $(wS)x_1 \subset J(\mathfrak{q}_1), \cdots, (wS)x_n \subset J(\mathfrak{q}_n)$. If $\mathfrak{q}_1 \neq \mathfrak{p}_1, \cdots, \mathfrak{p}_t$, then $\mathfrak{q}_1 + w = R$. Thus $x_1 \in x_1(\mathfrak{q}_1 + w)S \subset x_1(\mathfrak{q}_1 S) + x_1(wS) \subset J(\mathfrak{q}_1)$ or $\bar{x}_1 = 0$. However, we assumed $\bar{x}_1 \neq 0$, thus $\mathfrak{p}_1 = \mathfrak{q}_1$ (after renumbering) continuing we see $\mathfrak{q}_i = \mathfrak{p}_i$ (after renumbering) and $t \geqq n$. Thus $r_F(wS) = S/J(\mathfrak{p}_1) \oplus \cdots \oplus S/J(\mathfrak{p}_t)$ so $r_E(wS) = I_{\bar{S}}(r_F(wS)) = I_{\bar{S}}(S/J(\mathfrak{p}_1) \oplus \cdots \oplus S/J(\mathfrak{p}_t)) = I_{\bar{S}}(\bar{S}/J(\bar{S})) = \operatorname{Hom}_{\bar{R}}(\bar{S}, I_{\bar{R}}(\bar{R}/J(\bar{R}))$ by Prop. 1.2. Thus $r_E(wS)$ as a bi-$\bar{S}$ module is the canonical $\bar{S}$ module.

PROPOSITION 5.3. *The endomorphism ring of E is the completion of S with respect to the n-adic topology.*

Proof. Since $E = \bigcup_{w \in \boldsymbol{P}} r(wS)$ (Prop. 6.1) $\operatorname{End}_S E = \text{inv.} \lim_{w \in \boldsymbol{P}} \operatorname{End}_{S/wS} (r_E(wS))$. By Propositions 5.2, 1.2 and 1.3 $S/wS = \operatorname{End}(r_E(wS))$ by $(a + wS) \rightarrow (x \rightarrow xs)$, $a \in S$, $x \in r(wS)$. If $wS \subset vS$, then the following diagram commutes

$$\begin{array}{ccc} \operatorname{End}(r(wS)) & \xrightarrow{\text{restriction}} & \operatorname{End}(r(vS)) \\ \wr\| & & \wr\| \\ S/wS & \longrightarrow & S/vS \end{array}$$

So $\operatorname{End}_S(E) = \text{inv.} \lim_{w \in \boldsymbol{P}} S/wS$.

The question arises: is E injective over its endomorphism ring? F. L. Sandomierski has shown that as long as E has an infinite number of direct summands, then E is not injective over its endomorphism ring. (See Sandomierski) (5, Thm. 1, p. 244).

Let $\boldsymbol{U}$ be the collection $\{U\}$ of left ideals of S such that S/U is left

Artinian. We order $\boldsymbol{U}$ by inclusion; since the intersection of two ideals of $\boldsymbol{U}$ is in $\boldsymbol{U}$, $\boldsymbol{U}$ is directed. We call the $\operatorname{inv.}\lim_{v \in U} S/U$ the completion of S with respect to $\boldsymbol{U}$ topology. Now S/U has a composition series $S/U = M_0 \supset M_1 \supset \cdots \supset M_n = 0$ for $U \in \boldsymbol{U}$.

By Prop. 2.2 there exists a unique maximal p_i of R such that $p_i M_i \subset M_{i+1}$ for $i = 0, \cdots, n-1$. Now $p_{n-1} \cdots p_0(S/U) = 0$ i.e. if $w = p_{n-1} \cdots p_0$, then $wS \subset U$ and $w \in \boldsymbol{P}$. Furthermore, by the Jordan-Hölder Theorem w is unique. Thus we show for each $U \in \boldsymbol{U}$ there exists a $w \in P$ such that $wS \subset U$ i.e. $\{wS \mid w \in \boldsymbol{P}\}$ is cofinal in $\boldsymbol{U}$.

PROPOSITION 5.4. *The endomorphism ring of E (as a left S module) is the completion of S with respect to the $\boldsymbol{U}$ topology.*

Proof. We have seen $\{wS \mid w \in \boldsymbol{P}\}$ is cofinal in $\boldsymbol{U}$. Thus $\operatorname{End}_S E = \operatorname{inv.}\lim_{w \in \boldsymbol{P}} S/wS = \operatorname{inv.}\lim_{U \in \boldsymbol{U}} S/U$.

The finite topology on S has basic neighborhoods of zero of the form $U_{x_1 \cdots x_n}(0) = \{s \in S \mid sx_1 = \cdots = sx_n = 0\}$ for $x_1, \cdots, x_n \in E$. Since E is faithful the finite topology is Hausdorff. Moreover, by an argument similar to the proof of Prop. 5.4 for each $U_{x_1 \cdots x_n}(0), x_1, \cdots, x_n \in E$ there exists a $w \in \boldsymbol{P}$ such that $wS \subset U_{x_1 \cdots x_n}(0)$. Thus the finite topology is coarser than the n-adic topology and the n-adic topology is Hausdorff.

By the bicommutator of E (Bic(E)) we mean the set of all endomorphisms of E as an Abelian group which commutes with every element of $H(= \operatorname{End}_S E)$.

PROPOSITION 5.5. *The bicommutator of E is the completion of S with respect to the finite topology.*

Proof. Let $x_1, \cdots, x_n \in E$ and $U = U_{x_1 \cdots x_n}(0)$, we have a $w \in \boldsymbol{P}$ such that $wS \subseteq U$. So S/U can be regarded as a module over an artinian ring S/wS. We define a product on $S/U \times (x_1 H + \cdots + x_n H) \to E$, by $(s + U, \sum^n x_i h_i) \to \sum_{i=1}^n s x_i h_i \in E$. It is easy to see that S/U and $x_1 H + \cdots + x_n H$ form an orthogonal pair with respect to E. See (1, p. 254). Now E is a quasi-Frobenius bi-$S - H$ module because E is left S injective and contains a copy of every simple left S module (See (1, Thm. 4, p. 257)). Furthermore S/U has a composition series as a left S/wS module; hence, S/U has a composition series as a left S module for $wS \subseteq U$. Thus by (1, Prop. 2, p. 254) $x_1 H + \cdots + x_n H$ has a composition series as a right H module and $S/U = \mathrm{Hom}_S(x_1H + \cdots + x_nH, E)$ by $(s + U) \to (\sum x_ih_i \to \sum sx_ih_i))$. If $x_1H_1 + \cdots + x_nH \subseteq y_1H + \cdots + y_tH, x_1, \cdots, x_n, y_1, \cdots, y_t \in E$, then $U_{x_1\cdots x_n}(0) \supseteq U_{y_1\cdots y_t}(0)$. The following diagram commutes

$$\begin{array}{ccc} S/U_{y_1\cdots y_t}(0) & \longrightarrow & S/U_{x_1\cdots x_n}(0) \\ \wr\| & & \wr\| \\ \mathrm{Hom}_h(y_1H + \cdots + y_tH, E) & \longrightarrow & \mathrm{Hom}_H(x_1H + \cdots + x_nH, E) \end{array}$$

Thus

$$\begin{aligned} \text{inv. lim } S/U_{y_1\cdots y_n}(0) &= \text{inv. lim } \mathrm{Hom}_H(y_1H + \cdots + y_tH, E) \\ &= \mathrm{Hom}_H(\text{dir lim } y_1H + \cdots + y_nH, E) \\ &= \mathrm{Hom}_H(E, E) . \end{aligned}$$

PROPOSITION 5.6. *If R is a commutative, Noetherian ring, then the completion of R with respect to the n-adic topology equals the completion of R with respect to the finite topology.*

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