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ON THE FUNDAMENTAL GROUP OF A SIMPLE LIE GROUP

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Introduction

Let G be a simply connected simple Lie group and C the center of G, which is isomorphic with the fundamental group of the adjoint group of G. For an element c of C, an element x of the Lie algebra $\mathfrak{g}$ of G is called a representative of c in $\mathfrak{g}$ if $\exp x = c$. Sirota-Solodovnikov [7] found a complete set of representatives of the center C in $\mathfrak{g}$ and studied the group structure of C, and using their results Goto-Kobayashi [1] classified subgroups of the center C with respect to automorphisms of G. The group structure of C was also studied in Takeuchi [8].

Sirota-Solodovnikov's complete representatives were obtained by calculating a free abelian group Z_* modulo a subgroup Z_0 for each simple group. But if G is compact, owing to the classical result of E. Cartan, they are obtained systematically as follows. Let $\mathfrak{h}$ be a maximal abelian subalgebra of $\mathfrak{g}$, Δ the root system of the complexification $\mathfrak{g}^C$ of $\mathfrak{g}$ with respect to the complexification $\mathfrak{h}^C$ of $\mathfrak{h}$ and $\Pi = \{\alpha_1, \cdots, \alpha_l\}$ a fundamental system of Δ. Let $\mu = \sum_{i=1}^{l} n_i\alpha_i$ be the highest root of Δ with respect to Π and Λ_i^* $(1 \leqslant i \leqslant l)$ the dual basis of Π in $\sqrt{-1}\mathfrak{h}$ defined by the relations: $\alpha_i(\Lambda_j^*) = \delta_{ij}$ $(1 \leqslant i, j \leqslant l)$. Then the set $\{0\} \cup \{2\pi\sqrt{-1}\Lambda_i^*;\ 1 \leqslant i \leqslant l,\ n_i = 1\}$ give a complete set of representatives of the center C in $\mathfrak{g}$. Thus it is quite easy to see how an automorphism of G acts on C.

Moreover, by an unpublished result of Murakami (Theorem 2), if G is compact, C is isomorphic with a subgroup of the group of automorphisms of the extended fundamental system Π^*, where Π^* is defined from Π by adding $-\mu$ to it.

In this note we shall generalize the above results to general G (not necessarily compact). A complete set of representatives of the center C in

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$\mathfrak{g}$ is obtained by seeing the fundamental system and the highest root of $\mathfrak{g}$ and those of simple components of a maximal compact subalgebra $\mathfrak{k}$ of $\mathfrak{g}$, in terms of the dual basis of fundamental systems. The group strucutre of the center C is described by means of a group of automorphisms of the "extended fundamental system" of $\mathfrak{k}$. Thus it is immediate to find the action of automorphisms of G on C.

§1. Fundamental group of a semi-simple group

Let G be a connected semi-simple Lie group and $\mathfrak{g}$ its Lie algebra. Let $\mathfrak{k}$ be a maximal compact subalgebra of $\mathfrak{g}$, $\mathfrak{k}'$ the derived algebra of $\mathfrak{k}$ and K (resp. K') the connected subgroup of G generated by $\mathfrak{k}$ (resp. $\mathfrak{k}'$). Then we have $\pi_1(K) \cong \pi_1(G)$ since G is diffeomorphic with the product of K and a Euclidean space (Helgason [22], p. 214). We take a Cartan subalgebra $\mathfrak{h}_1$ of $\mathfrak{k}$. Then $\mathfrak{h}_1$ contains a regular element of $\mathfrak{g}$ (Murakami [5]) so that $\mathfrak{h}_1$ can be extended uniquely to a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$. Let $\mathfrak{g}^C$ (resp. $\mathfrak{k}^C$, $\mathfrak{k}'^C$, $\mathfrak{h}^C$) denote the complexification of $\mathfrak{g}$ (resp. $\mathfrak{k}, \mathfrak{k}', \mathfrak{h}$) and let $\mathfrak{h}^C_+ = \mathfrak{h}^C \cap \mathfrak{k}^C$, $\mathfrak{h}'^C = \mathfrak{h}^C \cap \mathfrak{k}'^C$. Then $\mathfrak{h}^C$ (resp. $\mathfrak{h}^C_+, \mathfrak{h}'^C$) is a Cartan subalgebra of $\mathfrak{g}^C$ (resp. $\mathfrak{k}^C, \mathfrak{k}'^C$). Let $\mathfrak{h}_0$ be the real part of $\mathfrak{h}^C$ and put $\mathfrak{h}_+ = \mathfrak{h}_0 \cap \mathfrak{h}^C_+$, $\mathfrak{h}' = \mathfrak{h}_0 \cap \mathfrak{h}'^C$, $\mathfrak{c} =$ the orthogonal complement of $\mathfrak{h}'$ in $\mathfrak{h}_+$ with respect to the Killing form $(\ , \ )$ of $\mathfrak{g}^C$. Then the Weyl group W (resp. W') of $\mathfrak{k}^C$ (resp. $\mathfrak{k}'^C$) on $\mathfrak{h}^C_+$ (resp. $\mathfrak{h}'^C$) is considered as a group of orthogonal transformations of $\mathfrak{h}_+$ (resp. $\mathfrak{h}'$) with respect to the Killing form of $\mathfrak{g}^C$ and W acts trivially on $\mathfrak{c}$ and coincides with W' on $\mathfrak{h}'$. In the following we shall identify the dual space of $\mathfrak{h}_0$ with $\mathfrak{h}_0$ by means of the Killing form of $\mathfrak{g}^C$, so that the root system Δ (resp. Δ') of $\mathfrak{g}^C$ (resp. $\mathfrak{k}'^C$) with respect to $\mathfrak{h}^C$ (resp. $\mathfrak{h}'^C$) is contained in $\mathfrak{h}_0$ (resp. $\mathfrak{h}'$). Let $\Pi' = \{\beta_1, \cdots, \beta_{l'}\}$ be a fundamental system of Δ' and $>$ the lexicographic order of Δ' associated with Π'. Now we put

$$Z = \frac{1}{2\pi\sqrt{-1}} \, kernel \, \{\exp : \sqrt{-1}\,\mathfrak{h}_+ \longrightarrow K\}$$

and let $t(z)$ denote the translation $h \longmapsto h + z$ of $\mathfrak{h}_+$ by an element z of Z. Then $W \cap t(Z) = \{1\}$ and W normalizes $t(Z)$ since W leaves Z invariant and $wt(z)w^{-1} = t(wz)$ for $w \in W$ and $z \in Z$. Thus we have a group $\tilde{W}$ of isometries of the Euclidean space $\mathfrak{h}_+$ defining that

$$\tilde{W} = t(Z)W.$$

The groups Z and $\tilde{W}$ for the universal covering group of G or the adjoint group of G will be denoted by Z_0 and $\tilde{W}_0$ or Z_* and $\tilde{W}_*$. Then we have

$$Z_* = \{h \in \mathfrak{h}_+;\ (\alpha, h) \in \boldsymbol{Z} \textit{ for any root } \alpha \textit{ of } \Delta\},$$

$$Z_0 = \sum_{i=1}^{l'} \boldsymbol{Z}\beta_i^*, \quad \textit{where } \beta_i^* = (2/(\beta_i, \beta_i))\beta_i.$$

The latter equality follows from the fact that the righthand side is the dual group of the group of weights of $\mathfrak{k}'^{C}$. It is clear that $Z_0 \subset Z \subset Z_*$ and $\tilde{W}_0 \subset \tilde{W} \subset \tilde{W}_*$. If we denote by K_0 the simply connected subgroup of the universal covering group G_0 of G generated by $\mathfrak{k}$ and by φ the covering homomorphism of K_0 onto K, then the map $\gamma : Z/Z_0 \longrightarrow G_0$ defined by $z \bmod Z_0 \longmapsto \exp_{G_0} 2\pi\sqrt{-1}\, z$ induces the isomorphism of Z/Z_0 onto the kernel of φ, which is isomorphic with $\pi_1(K) \cong \pi_1(G)$. Thus

$$Z/Z_0 \cong \pi_1(G).$$

LEMMA 1. *$wz \equiv z \pmod{Z_0}$ for $w \in W$ and $z \in Z$.*

Proof. There exists an element k of the normalizer in K_0 of $\mathfrak{h}_+$ such that $\mathrm{Ad}\,k$ restricted to $\mathfrak{h}_+$ coincides with w. Since the kernel of the above covering homomorphism φ is contained in the center of K_0, the element k centralizes the kernel of φ, which yields Lemma. q.e.d.

Note that $(z, \beta) \in \boldsymbol{Z}$ for $z \in Z$ and $\beta \in \Delta'$, since β is obtained as the orthogonal projection to $\mathfrak{h}_+$ of some root of Δ and $Z \subset Z_*$. This fact will be used sometimes in the following.

The subset

$$D = \{h \in \mathfrak{h}_+;\ (h, \beta) \in \boldsymbol{Z} \textit{ for some root } \beta \textit{ of } \Delta'\}$$

of $\mathfrak{h}_+$ is called the *diagram* of $\mathfrak{k}$ on $\mathfrak{h}_+$ and a connected component of $\mathfrak{h}_+ - D$ is called a *cell* of $\mathfrak{k}$ on $\mathfrak{h}_+$. Then $\tilde{W}$ leaves D invariant since $W(\Delta') \subset \Delta'$ and $(Z, \Delta') \subset \boldsymbol{Z}$. It follows that $\tilde{W}$ acts on the set of cells of $\mathfrak{k}$ on $\mathfrak{h}_+$. A classical theorem of E. Cartan (cf. Helgason [2], p. 265) says that $\tilde{W}_0$ acts simply transitively on the set of cells. (An algebraic proof of this theorem is seen in Iwahori-Matsumoto [3].) Let $\mathscr{C}'$ be the positive Weyl chamber of $\mathfrak{k}'$ on $\mathfrak{h}'$ with respect to Π', that is, $\mathscr{C}' = \{h \in \mathfrak{h}';\ (h, \beta_i) > 0$ for any root β_i of $\Pi'\}$, and S the unique cell of $\mathfrak{k}$ in $\mathfrak{h}_+$ such that the closure $\bar{S}$ of S contains 0 and $S \cap \mathscr{C}' \neq \phi$. We put

$$\tilde{W}(S) = \{\tau \in \tilde{W};\ \tau S = S\}.$$

THEOREM 1. *The group $\tilde{W}(S)$ is isomorphic with the fundamental group $\pi_1(G)$ of G.*

Proof. Let us consider the map of $\tilde{W}$ to Z/Z_0 defined by $t(z)w \longmapsto z \bmod Z_0$ for $z \in Z$ and $w \in W$. Since we have $(t(z_1)w_1)(t(z_2)w_2) = t(z_1 + w_1 z_2)(w_1 w_2)$, the map is a homomorphism in view of Lemma 1. The kernel of this homomorphism is just the group $\tilde{W}_0$. It follows that $\tilde{W}_0$ is a normal subgroup of $\tilde{W}$ and $\tilde{W}/\tilde{W}_0 \cong Z/Z_0 \cong \pi_1(G)$. On the other hand, the theorem of E. Cartan yields that $\tilde{W}$ is the semi-direct product of $\tilde{W}(S)$ and $\tilde{W}_0$. It follows that $\tilde{W}(S) \cong \tilde{W}/\tilde{W}_0 \cong \pi_1(G)$. q.e.d.

COROLLARY. *The corresponding group $\tilde{W}_*(S)$ for centerless group G is isomorphic with the center C of the universal covering group of G.*

Proof. Obvious since $\pi_1(G)$ is isomorphic with C. An explicit isomorphism is given by $\tilde{W}_*(S) \cong Z_*/Z_0 \overset{\gamma}{\cong} C$. q.e.d.

REMARK. The group $\tilde{W}(S)$ may be described in terms of covering transformations of the universal covering space of an open submanifold of K (cf. Takeuchi [8], Helgason [2]).

If we put $S' = \mathfrak{h}' \cap S$, we have $S = \mathfrak{c} \times S'$. Now we define certain groups on $\mathfrak{h}'$ similarly to those on $\mathfrak{h}_+$. Let

$$Z'_* = \{h \in \mathfrak{h}';\ (h, \beta) \in \boldsymbol{Z} \text{ for any root } \beta \text{ of } \Delta'\},$$

$$\tilde{W}'_* = t'(Z'_*)W', \text{ where } t'(z')h' = z' + h' \text{ for } h' \in \mathfrak{h}'.$$

Then Z'_* contains $Z \cap \mathfrak{h}'$ and $\tilde{W}'_*$ leaves $D' = \mathfrak{h}' \cap D$ invariant so that $\tilde{W}'_*$ acts on connected components of $\mathfrak{h}' - D'$, which are called *cells* of $\mathfrak{k}'$ on $\mathfrak{h}'$. S' is the unique cell of $\mathfrak{k}'$ on $\mathfrak{h}'$ such that $\bar{S}'$ contains 0 and $S' \cap \mathscr{C}' \neq \phi$. Put

$$\tilde{W}'_*(S') = \{\tau' \in \tilde{W}'_*;\ \tau' S' = S'\}.$$

The same argument as above shows that $\tilde{W}'_*(S')$ is isomorphic with the fundamental group of the adjoint group of $\mathfrak{k}'$ and with the center of the universal covering group of K'.

LEMMA 2. 1) *Let Z'' be the image of $\bar{S} \cap Z$ by the orthogonal projection of $\mathfrak{h}_+$ onto $\mathfrak{c}$. Then $\bar{S} \cap Z \subset Z'' \times (\bar{S}' \cap Z'_*)$.*

2) *Let $\xi(\tau) = \tau(0)$ for $\tau \in \tilde{W}(S)$. Then the map ξ gives a bijection of $\tilde{W}(S)$*

onto $\bar{S}\cap Z$. *The set* $2\pi\sqrt{-1}\,(\bar{S}\cap Z_*)$ *is a complete set of representatives in* $\mathfrak{g}$ *of the center* C *of the universal covering group of* G.

3) *Let* $\xi'(\tau')=\tau'(0)$ *for* $\tau'\in\tilde{W}'_*(S')$. *Then the map* ξ' *gives a bijection of* $\tilde{W}'_*(S')$ *onto* $\bar{S}'\cap Z'_*$. *The set* $2\pi\sqrt{-1}\,(\bar{S}'\cap Z'_*)$ *is a complete set of representatives in* $\mathfrak{k}'$ *of the center of the universal covering group of* K'.

Proof. 1) Let $z=(z'',z')$ be an element of $\bar{S}\cap Z=(\mathfrak{c}\times\bar{S}')\cap Z$, where $z''\in Z''$ and $z'\in\bar{S}'$. Then for any root β of Δ' we have $(z',\beta)=(z,\beta)-(z'',\beta)=(z,\beta)\in\boldsymbol{Z}$ so that $z'\in\bar{S}'\cap Z'_*$.

2) For any element $\tau=t(z)w$ of $\tilde{W}(S)$, where $z\in Z$ and $w\in W$, we have $\xi(\tau)=\tau(0)=z\in Z$. It follows that $\xi(\tau)\in\bar{S}\cap Z$ since $0\in\bar{S}$. We shall show first that ξ is surjective. In view of 1), any element z of $\bar{S}\cap Z$ can be written as $z=z''+z'$, where $z''\in Z''$ and $z'\in\bar{S}'\cap Z'_*$. Then $t(z)^{-1}S=\mathfrak{c}\times t(z')^{-1}S'$ and $t(z')^{-1}S'$ is a cell of $\mathfrak{k}'$ on $\mathfrak{h}'$ such that its closure contains 0. Since W' acts transitively on Weyl chambers of $\mathfrak{k}'$ on $\mathfrak{h}'$, we have an element w of W such that $w^{-1}t(z')^{-1}S'=S'$. It follows that $w^{-1}t(z)^{-1}S=\mathfrak{c}\times S'=S$ so that $\tau=t(z)w\in\tilde{W}(S)$ and $\xi(\tau)=z$. We shall show next that ξ is injective. Let $\tau_i=t(z_i)w_i$ $(i=1,2)$ be elements of $\tilde{W}(S)$ such that $\xi(\tau_1)=\xi(\tau_2)$. Then we have $z_1=z_2$ and $\tau_2^{-1}\tau_1=w_2^{-1}w_1\in W\cap\tilde{W}(S)\subset\tilde{W}_0\cap\tilde{W}(S)$. But since $\tilde{W}_0\cap\tilde{W}(S)=\{1\}$ by the theorem of E. Cartan, we have $\tau_1=\tau_2$. The second statement follows from the first statement and Corollary of Theorem 1.

3) is proved similarly to the above. q.e.d.

LEMMA 3. 1) Z'' *is a subgroup of* $\mathfrak{c}$. *The corresponding group* Z''_* *for centerless group* G *is a lattice of* $\mathfrak{c}$.

2) *Let* F *be the subset of* $\tilde{W}'_*(S')$ *corresponding to* $\bar{S}'\cap Z$ *under the bijection* $\xi'\colon \tilde{W}'_*(S')\longrightarrow\bar{S}'\cap Z'_*$ *and let* $\pi''(\tau)=z''$ *for an element* $\tau=t(z''+z')w$ *of* $\tilde{W}(S)$, *where* $z''\in Z''$, $z'\in\bar{S}'\cap Z'_*$ *and* $w\in W$. *Then* F *is a subgroup of* $\tilde{W}'_*(S')$ *and the map* $\pi''\colon\tilde{W}(S)\longrightarrow Z''$ *is a homomorphism. Moreover we have a split exact sequence*:

$$0\longrightarrow F\longrightarrow\tilde{W}(S)\xrightarrow{\pi''}Z''\longrightarrow 0.$$

Thus we have an isomorphism: $\tilde{W}(S)\cong Z''\times F$.

Proof. For elements $\tau_i=t(z''_i+z'_i)w_i$ of $\tilde{W}(S)$ $(i=1,2)$, we have $\tau_1\tau_2=t((z''_1+z''_2)+(z'_1+w_1z'_2))\,(w_1w_2)$ so that π'' is a homomorphism of $\tilde{W}(S)$ into $\mathfrak{c}$. Since $\pi''\tilde{W}=Z''$ in view of Lemma 2, Z'' is a subgroup of $\mathfrak{c}$.

If $\pi''(\tau)=0$ for an element $\tau=t(z)w$ of $\tilde{W}(S)$, then $z\in\mathfrak{h}'\cap Z\subset Z'_*$. It follows that τ is identity on $\mathfrak{c}$, its restriction τ' to $\mathfrak{h}'$ belongs to $\tilde{W}'_*(S')$ and $\xi'(\tau')\in\bar{S}'\cap Z$. Conversely if τ' is an element of $\tilde{W}'_*(S')$ with $\xi'(\tau')\in\bar{S}'\cap Z$, then the trivial extension τ of τ' to $\mathfrak{h}_+$ satisfies $\tau\in\tilde{W}(S)$ and $\tilde{w}''(\tau)=0$. It follows that F is a subgroup of $\tilde{W}'_*(S')$ and isomorphic with the kernel of π''. So we have the desired exact sequence, which splits because Z'' is free.

If G is centerless, then K is compact so that $Z_*\cap\mathfrak{c}$ is a lattice of $\mathfrak{c}$. Since Z''_* contains $Z_*\cap\mathfrak{c}$, Z''_* is also a lattice of $\mathfrak{c}$. q.e.d.

Now we want to describe the structure of the group F. Let $\mathfrak{k}'=\sum_{i=1}^{r}\mathfrak{k}'_i$ be the decomposition of $\mathfrak{k}'$ into simple factors. Then $\mathfrak{h}', \Delta', \Pi', Z'_*, S', \bar{S}'\cap Z'_*$, W', $\tilde{W}'_*$ and $\tilde{W}'_*(S')$ are the direct products of corresponding objects for simple factors $\mathfrak{k}_i$, which will be denoted by the same symbol with the suffix i. Let μ'_i be the highest root of Δ'_i and $\Pi'^*_i=\Pi'_i\cup\{-\mu'_i\}$. Let $\mathrm{Aut}(\Pi'^*_i)$ denote the group of orthogonal transformations of $\mathfrak{h}'_i$ preserving Π'^*_i and let

$$\Pi'^*=\bigcup_{i=1}^{r}\Pi'^*_i,$$

$$\mathrm{Aut}\,(\Pi'^*)=\prod_{i=1}^{r}\mathrm{Aut}\,(\Pi'^*_i).$$

THEOREM 2. 1) *Let $\pi'(\tau')=w'$ for an element $\tau'=t'(z')w'$ of $\tilde{W}'_*(S')$, where $z'\in Z'_*$ and $w'\in W'$. Then $\pi'(\tau')\in\mathrm{Aut}\,(\Pi'^*)$ for any element τ' of $\tilde{W}'_*(S')$ and the map $\pi':\tilde{W}'_*(S')\longrightarrow\mathrm{Aut}\,(\Pi'^*)$ is an injective homomorphism. The image $\pi'\tilde{W}'_*(S')$ of π' will be denoted by $\mathscr{F}(\mathfrak{k}')$, which is isomorphic with the fundamental group of the adjoint group of $\mathfrak{k}'$.*

2) *If $\mathfrak{k}'$ is simple, the group $\mathscr{F}(\mathfrak{k}')$ is obtained as follows. Let $M^*_i\in\mathfrak{h}'$ $(1\leqslant i\leqslant l')$ be the dual basis of Π', that is, $(M^*_i,\beta_j)=\delta_{ij}$ $(1\leqslant i,\ j\leqslant l')$ and $P_i=(1/m_i)\,M^*_i$ $(1\leqslant i\leqslant l')$, where m_i is the i-th coefficient of the highest root $\mu'=\sum_{i=1}^{l'}m_i\beta_i$ of Δ'. We put $\beta_0=-\mu'$, $M^*_0=P_0=0$ and $m_0=1$. Then*

a) *$\{P_0,P_1,\cdots,P_{l'}\}$ is the set of vertices of $\bar{S}'$.*

b) *$\bar{S}'\cap Z'_*=\{M^*_i;\ 0\leqslant i\leqslant l',\ m_i=1\}$ and the set $\{2\pi\sqrt{-1}\,M^*_i;\ 0\leqslant i\leqslant l',\ m_i=1\}$ is a complete set of representatives of the center of the simply connected Lie group with the Lie algebra $\mathfrak{k}'$.*

c) *Let τ'_i be the element of $\tilde{W}'_*(S')$ with $\xi'(\tau'_i)=M^*_i$ and π_i the element of the symmetric group of $(l'+1)$ letters $\{0,1,\cdots,l'\}$ defined by $\tau'_iP_j=P_{\pi_i(j)}$ $(0\leqslant j\leqslant l')$. Then $\pi'(\tau'_i)\beta_j=\beta_{\pi_i(j)}$ $(0\leqslant j\leqslant l')$.*

d) $\pi'(\tau_i')$ *is characterized by the property*:

$$\{\beta \in \Delta';\ \beta > 0,\ \pi'(\tau_i')^{-1}\beta < 0\} = \{\beta \in \Delta';\ (\beta, M_i^*) > 0\}.$$

Proof. They were proved in a more general situation in Takeuchi [8] except 2), d) and the last statement was contained together with the other in Iwahori-Matsumoto [3], but we prove them again here for the sake of completeness.

Since we have $\tau_1'\tau_2' = t'(z_1' + w_1'z_2')(w_1'w_2')$ for $\tau_i' = t'(z_i')w_i' \in \tilde{W}_*'(S')$ $(i = 1, 2)$, π' is a homomorphism of $\tilde{W}_*'(S')$ to W'. To prove the statements that $\pi'\tilde{W}_*'(S) \subset \mathrm{Aut}\,(\Pi'^*)$ and π' is injective, we may assume that $\mathfrak{k}'$ is simple. But in this case they are true in view of 2), c).

2) a) follows from

$$S' = \{h' \in \mathfrak{h}';\ (h', \beta_i) > 0\ (1 \leqslant i \leqslant l'),\ (h', \mu') < 1\},$$
$$\bar{S}' = \{h' \in \mathfrak{h}';\ (h', \beta_i) \geqslant 0\ (1 \leqslant i \leqslant l'),\ (h', \mu') \leqslant 1\}.$$

b) The first statement follows from a) and that $Z_*' = \sum_{i=1}^{l'} \boldsymbol{Z} M_i^*$. The second follows from Lemma 2, 3).

c) We shall show first that $m_j = m_{\pi_i(j)}$ $(0 \leqslant j \leqslant l')$. Since $\pi'(\tau_i') = t'(\xi'(\tau_i'))^{-1}\tau_i'$, we have $\pi'(\tau_i')P_j = P_{\pi_i(j)} - \xi'(\tau_i') = (1/m_{\pi_i(j)})M_{\pi_i(j)}^* - \xi'(\tau_i')$ and therefore

$$(*) \qquad \pi'(\tau_i')M_j^* = (m_j/m_{\pi_i(j)})M_{\pi_i(j)}^* - m_j\xi'(\tau_i').$$

Hence $(m_j/m_{\pi_i(j)})M_{\pi_i(j)}^* \in Z_*'$. It follows from the equality: $Z_*' = \sum_{k=1}^{l'} \boldsymbol{Z} M_k^*$ that $m_j/m_{\pi_i(j)} \geqslant 1$. The same argument for τ_i^{-1} shows that $m_{\pi_i(j)}/m_j \geqslant 1$. Thus we have $m_j = m_{\pi_i(j)}$.

Since $\xi'(\tau_i') = \tau_i'(0) = \tau_i'P_0 = P_{\pi_i(0)} = (1/m_{\pi_i(0)})M_{\pi_i(0)}^* = (1/m_0)M_{\pi_i(0)}^* = M_{\pi_i(0)}^*$, we have from $(*)$ that $\pi'(\tau_i')M_j^* = M_{\pi_i(j)}^* - m_jM_{\pi_i(0)}^*$. Replacing τ_i by τ_i^{-1} we have

$$\pi'(\tau_i')^{-1}M_j^* = M_{\pi_i^{-1}(j)}^* - m_jM_{\pi_i^{-1}(0)}^* \ (0 \leqslant j \leqslant l').$$

Now it is easy to derive $\pi'(\tau_i')\beta_j = \beta_{\pi_i(j)}$ using $m_j = m_{\pi_i(j)}$ and the above equalities: If $j \neq 0,\ \pi_i^{-1}(0)$, then for $1 \leqslant k \leqslant l'$ we have $(\pi'(\tau_i')\beta_j,\ M_k^*) = (\beta_j,\ \pi'(\tau_i')^{-1}M_k^*) = (\beta_j,\ M_{\pi_i^{-1}(k)}^* - m_kM_{\pi_i^{-1}(0)}^*) = (\beta_j,\ M_{\pi_i^{-1}(k)}^*) = \delta_{\pi_i(j),k} = (\beta_{\pi_i(j)},\ M_k^*)$, so that $\pi'(\tau_i')\beta_j = \beta_{\pi_i(j)}$. We can similarly confirm the same equality for $j = 0$ or $\pi_i^{-1}(0)$.

d) Since the existence and the uniqueness of an element w' of W' such that

$$\{\beta\in\Delta';\ \beta>0,\ w'^{-1}\beta<0\} = \{\beta\in\Delta';\ (\beta, M_i^*)>0\}$$

is known (Kostant [4]), it suffices to show that $\tau' = t'(M_i^*)w'$, with w' as above and $m_i = 1$, leaves S' invariant. We may assume that $i\neq 0$. Take an element h' of S'. Let $1\leqslant j\leqslant l'$, then $(\tau'h', \beta_j) = (w'h' + M_i^*, \beta_j) = (h', w'^{-1}\beta_j) + (M_i^*, \beta_j)$. If $w'^{-1}\beta_j>0$, then $(h', w'^{-1}\beta_j)>0$ since $h'\in S'$. If $w'^{-1}\beta_j<0$, then $(M_i^*, \beta_j)=1$ from the assumption for w' and $(h', w'^{-1}\beta_j)>-1$ since $h'\in S'$. Thus in both cases we have $(\tau'h', \beta_j)>0$. Furthermore we have $(\tau'h', \mu')=(w'h'+M_i^*, \mu')=(h', w'^{-1}\mu')+1$. If $w'^{-1}\mu'<0$, then $(h', w'^{-1}\mu')<0$ since $h'\in S'$, so that $(\tau'h', \mu')<1$. If $w'^{-1}\mu'>0$, then from the assumption for w' we have $(\mu', M_i^*)\leqslant 0$, which is a contradiction. Thus we have $(\tau'h', \mu')<1$. It follows that $\tau'h'$ is also an element of S'. q.e.d.

THEOREM 3. *Let* $\mathscr{F} = \pi'F\subset\mathscr{F}(\mathfrak{k}')$, *that is,* $\mathscr{F}$ *is the image of* $\bar{S}'\cap Z$ *by the injection* $\pi'\xi'^{-1} : \bar{S}'\cap Z'_*\longrightarrow \mathrm{Aut}(\Pi'^*)$, *and let* Z'' *be the free abelian group defined in Lemma* 2. *Then*

$$\pi_1(G)\cong Z''\times\mathscr{F}.$$

If G *has no center, then the rank of* $Z''=Z''_*$ *is the same as the dimension of the center of the maximal compact subgroup* K *of* G. *The set* $2\pi\sqrt{-1}\,(\bar{S}'\cap Z_*)$ *is a complete set of representatives of the torsion part of the center* C *of the universal covering group of* G.

Proof. $\pi_1(G)$ is isomorphic with $Z''\times F$ by Theorem 1 and Lemma 3, 2) and F is isomorphic with $\mathscr{F}$ by Theorem 2. It follows that $\pi_1(G)$ is isomorphic with $Z''\times\mathscr{F}$. The second statement follows from Lemma 3, 1). The last follows from Lemma 2, 2). q.e.d.

§ 2. Center of a simply connected simple group

Let $\mathfrak{g}_u$ be a compact simple Lie algebra.

(A) Let $\mathfrak{h}_u$ be a Cartan subalgebra of $\mathfrak{g}_u$. Then the complexification $\mathfrak{h}^C$ of $\mathfrak{h}_u$ is a Cartan subalgebra of the complexification $\mathfrak{g}^C$ of $\mathfrak{g}_u$. The real part $\mathfrak{h}_0$ of $\mathfrak{h}^C$ is identified with the dual space of $\mathfrak{h}_0$ as in Section 1 by means of the Killing form (,) of $\mathfrak{g}^C$, so that the root system Δ of $\mathfrak{g}^C$ with respect to $\mathfrak{h}^C$ is a subset of $\mathfrak{h}_0$. Choose a set $\{e_\alpha;\ \alpha\in\Delta\}$ of root vectors of $\mathfrak{g}^{C}$ with respect to $\mathfrak{h}^{C}$ such that $[e_\alpha, e_{-\alpha}] = -\alpha$ $(\alpha \in \Delta)$ and $[e_\alpha, e_\beta] = N_{\alpha,\beta} e_{\alpha+\beta}$ $(\alpha, \beta, \alpha+\beta \in \Delta)$ where $N_{\alpha,\beta} \neq 0$, $N_{\alpha,\beta} \in \boldsymbol{R}$. Let $\Pi = \{\alpha_1, \cdots, \alpha_l\}$ be a fundamental system of Δ and $>$ the lexicographic order of Δ associated with Π. Let $\Lambda_i^* \in \mathfrak{h}_0$ $(1 \leqslant i \leqslant l)$ be the dual basis of Π, that is, $(\Lambda_i^*, \alpha_j) = \delta_{ij}$ $(1 \leqslant i, j \leqslant l)$ and put $\Lambda_0^* = 0$, $\alpha_0 = -\mu$, where μ is the highest root of Δ. Take an involutive trasformation ρ of $\mathfrak{h}_0$ with $\rho\Delta = \Delta$ and $\rho\Pi = \Pi$, and put

$$\mathfrak{h}_+ = \{h \in \mathfrak{h}_0;\ \rho h = h\}.$$

Changing indices of the α_i if necessary, we may assume that $\rho\alpha_i = \alpha_i$ $(1 \leqslant i \leqslant p)$, $\rho\alpha_{p+i} = \alpha_{l_0+i}$ $(1 \leqslant i \leqslant l_0 - p)$ and $\rho\alpha_{l_0+i} = \alpha_{p+i}$ $(1 \leqslant i \leqslant l_0 - p)$. Then we have $\Lambda_i^* \in \mathfrak{h}_+$ if $0 \leqslant i \leqslant p$. Let θ_ρ be the involutive automorphism of $\mathfrak{g}_u$ leaving $\mathfrak{h}_u$ invariant, which is characterized by property that its $\boldsymbol{C}$-linear extension θ_ρ to $\mathfrak{g}^{C}$ satisfies $\theta_\rho = \rho$ on $\mathfrak{h}_0$ and $\theta_\rho e_{\alpha_i} = e_{\rho\alpha_i}$ for any root α_i of Π. Let $\bar{\alpha}$ denote the image of a root α of Δ by the orthogonal projection of $\mathfrak{h}_0$ onto $\mathfrak{h}_+$. Then

$$\Delta_0 = \{\bar{\alpha};\ \alpha \in \Delta\}$$

is the root system of a complex simple Lie algebra of rank l_0 and

$$\Pi_0 = \{\bar{\alpha}_i;\ \alpha_i \in \Pi\} = \{\alpha_1, \cdots, \alpha_p, \bar{\alpha}_{p+1}, \cdots, \bar{\alpha}_{l_0}\}$$

is a fundamental system of Δ_0 (Murakami [6], p. 301, p. 302). The lexicographic order $>$ of Δ_0 associated with Π_0 is nothing but the one induced by the order $>$ of Δ. Let $\mu_0 = n_1\alpha_1 + \cdots + n_p\alpha_p + n_{p+1}\bar{\alpha}_{p+1} + \cdots + n_{l_0}\bar{\alpha}_{l_0}$ be the highest root of Δ_0 and put $n_0 = 1$. Then

$$\theta = \theta_\rho \exp \pi\sqrt{-1}\ \mathrm{ad}\, \Lambda_{i_0}^* \qquad (0 \leqslant i_0 \leqslant p,\ n_{i_0} = 1 \text{ or } 2)$$

is an involutive automorphism of $\mathfrak{g}_u$. We put

$$\mathfrak{k} = \{x \in \mathfrak{g}_u;\ \theta x = x\},\ \mathfrak{p}_u = \{x \in \mathfrak{g}_u;\ \theta x = -x\},$$

$$\mathfrak{g} = \mathfrak{k} + \sqrt{-1}\ \mathfrak{p}_u.$$

Then $\mathfrak{g}$ is a real simple Lie algebra, which is a real form of $\mathfrak{g}^{C}$, and $\mathfrak{k}$ is a maximal compact subalgebra of $\mathfrak{g}$. Let $\mathfrak{h}' = \mathfrak{h}_+ \cap \sqrt{-1}\ \mathfrak{k}'$, where $\mathfrak{k}'$ is the derived algebra of $\mathfrak{k}$, and $\mathfrak{c}$ the orthogonal complement of $\mathfrak{h}'$ in $\mathfrak{h}_+$. Then $\mathfrak{h}_+$, $\mathfrak{h}'$ and $\mathfrak{c}$ play the same roles as those in Section 1. So we shall use the same notation as there.

(B) Let $\mathfrak{g}$ be the scalor restriction to $\boldsymbol{R}$ of the complexification $(\mathfrak{g}_u)^C$ of $\mathfrak{g}_u$. Then $\mathfrak{g}$ is a real simple Lie algebra, whose maximal compact subalgebra is isomorphic with $\mathfrak{g}_u$.

THEOREM. (Murakami [6], p. 295, p. 303) *Any real simple Lie algebra* $\mathfrak{g}$ *is obtained from a compact simple Lie algebra* $\mathfrak{g}_u$ *by the construction* (A) *or* (B). *In case* (A), *a fundamental system* Π' *of the root system* Δ' *of* $\mathfrak{k}'^C$ *with respect to* $\mathfrak{h}'^C$ *and* $\mathfrak{c}$ *are obtained as follows.*

1) $\rho = 1,\ i_0 = 0 \qquad \Pi' = \Pi = \{\alpha_1, \cdots, \alpha_l\},\ \mathfrak{c} = \{0\}.$

2) $\rho = 1,\ 1 \leqslant i_0 \leqslant l,\ n_{i_0} = 2 \qquad \Pi' = (\Pi - \{\alpha_{i_0}\}) \cup \{\alpha_0\},\ \mathfrak{c} = \{0\}.$

3) $\rho = 1,\ 1 \leqslant i_0 \leqslant l,\ n_{i_0} = 1 \qquad \Pi' = \Pi - \{\alpha_{i_0}\},\ \mathfrak{c} = \boldsymbol{R}\Lambda^*_{i_0}.$

4) $\rho \neq 1,\ i_0 = 0 \qquad \Pi' = \Pi_0,\ \mathfrak{c} = \{0\}.$

5) $\rho \neq 1,\ 1 \leqslant i_0 \leqslant p,\ n_{i_0} = 1$ *or* $2 \qquad \Pi' = (\Pi_0 - \{\alpha_{i_0}\{) \cup \{\xi\},\ \mathfrak{c} = \{0\},$

where $\xi = \alpha_{i_0} + \alpha_{i_1} + \alpha_{i_2} + \cdots + \alpha_{i_t} + \alpha_k,$

$1 \leqslant i_1, \cdots, i_t \leqslant p, p+1 \leqslant k \leqslant l_0,$

$(\alpha_{i_0}, \alpha_{i_1}),\ (\alpha_{i_1}, \alpha_{i_2}), \cdots, (\alpha_{i_{t-1}}, \alpha_{i_t}), (\alpha_{i_t}, \alpha_k)$ *are all negative.*

Now we want to calculate the center C of the simply connected Lie group with the Lie algebra $\mathfrak{g}$ constructed in (A) or (B). The center C is isomorphic with the fundamental group $\pi_1(G)$ of the adjoint group G of $\mathfrak{g}$. In case (B), the problem is reduced to the one in case (A), 1), since $\pi_1(G) \cong \pi_1(G_u)$ where G_u is the adjoint group of $\mathfrak{g}_u$. In case (A), 1), we have $\pi_1(G) = \pi_1(G_u) \cong \mathscr{F}(\mathfrak{g}_u)$, which can be calculated by Theorem 2. So we shall restrict ourselves to find $\bar{S}' \cap Z_*$ in cases (A), 2) $\sim$ 5) and a generator of the free part of $\pi_1(G)$ in case (A), 3). Let $\mathfrak{k}' = \sum_{k=1}^{r} \mathfrak{k}'_k$ be the decomposition of $\mathfrak{k}'$ into simple factors and $\Pi'^* = \bigcup_{k=1}^{r} \Pi'^*_k$ and $\bar{S}' \cap Z'_* = \prod_{k=1}^{r} \bar{S}'_k \cap (Z'_*)_k$ be the corresponding decompositions. We can associate to any element γ of Π'^* a positive integer m_γ and an element M^*_γ of $\mathfrak{h}'$ as in Theorem 2: If $\gamma \in \Pi'_k$, then m_γ is the coefficient of γ in the expression of the highest root of Π'_k as the linear combination of fundamental roots. $\{M^*_\gamma;\ \gamma \in \Pi'\} \subset \mathfrak{h}'$ is the dual basis of Π'. If $\gamma \in \Pi'^* - \Pi'$, then $m_\gamma = 1$ and $M^*_\gamma = 0$. Then by Theorem 2 any element z' of $\bar{S}'\cap Z'_*$ is of the form $z' = \sum_{k=1}^{r} M^*_{\gamma_k}$, where $M^*_{\gamma_k}$ is an element of $\bar{S}'_k\cap(Z'_*)_k$, that is, $\gamma_k\in\Pi'^*_k$ and $m_{\gamma_k}=1$.

Case (A), 2). We have $\mu=\sum_{i=1}^{l} n_i\alpha_i$ since $\rho=1$. We associate to any element γ of Π'^* a non-negative integer n'_γ as follows: $n'_\gamma=n_i$ for $\gamma=\alpha_i\in\Pi'$ and $n'_\gamma=0$ for $\gamma\in\Pi'^*-\Pi'$. Let $z'=\sum_{k=1}^{r} M^*_{\gamma_k}$ be an element of $\bar{S}'\cap Z'_*$. Then for $i\neq i_0$, $1\leqslant i\leqslant l$, we have $(\alpha_i, z')\in(\Pi', Z'_*)\subset \boldsymbol{Z}$ and $(\alpha_{i_0}, z') = (-(1/2)(\alpha_0+\sum_{\substack{i\neq i_0\\ 1\leqslant i\leqslant l}} n_i\alpha_i), z') = -(1/2)(\sum_{\gamma\in\Pi'} n'_\gamma\gamma, \sum_k M^*_{\gamma_k}) = -(1/2)\sum_k n'_{\gamma_k}$. It follows that

$$\bar{S}'\cap Z_* = \{\sum_k M^*_{\gamma_k};\ m_{\gamma_k}=1 \cdot \textit{for all } k,\ \sum_k n'_{\gamma_k}\in 2\boldsymbol{Z}\}.$$

Case (A), 3). Let $a_{ij}=2(\alpha_i,\alpha_j)/(\alpha_j,\alpha_j)$ $(1\leqslant i,\ j\leqslant l)$ be Cartan integers of Π and (b_{ij}) the inverse of the Cartan matrix (a_{ij}). We associate to any element γ of Π'^* a non-negative real number λ_γ as follows: $\lambda_\gamma = b_{i_0,i}/b_{i_0,i_0}$ for $\gamma=\alpha_i\in\Pi'$ and $\lambda_\gamma=0$ for $\gamma\in\Pi'^*-\Pi'$. We shall show first that $\mathfrak{h}'$-component of α_{i_0} is $-\sum_{\gamma\in\Pi'}\lambda_\gamma\gamma$. Let $\alpha_{i_0}=\lambda\Lambda^*_{i_0}+\sum_{\alpha_i\in\Pi'}\lambda'_i\alpha_i$ $(\lambda,\lambda'_i\in\boldsymbol{R})$. From $1=(\alpha_{i_0},\Lambda^*_{i_0})=\lambda(\Lambda^*_{i_0},\Lambda^*_{i_0})$, we have $\lambda=1/(\Lambda^*_{i_0},\Lambda^*_{i_0})$. For $i\neq i_0$, $1\leqslant i\leqslant l$, from $0=(\alpha_{i_0},\Lambda^*_i)=\lambda(\Lambda^*_{i_0},\Lambda^*_i)+\lambda'_i$, we have $\lambda'_i=-\lambda(\Lambda^*_{i_0},\Lambda^*_i)=-(\Lambda^*_{i_0},\Lambda^*_i)/(\Lambda^*_{i_0},\Lambda^*_{i_0})$. If we put $c_{ij}=c_{ji}=(\Lambda^*_i,\Lambda^*_j)$ $(1\leqslant i,\ j\leqslant l)$, we have $\Lambda^*_i=\sum_{j=1}^{l}c_{ij}\alpha_j$ and $\delta_{ki}=\delta_{ik}=(\Lambda^*_i,\alpha_k)=\sum_j c_{ij}(\alpha_j,\alpha_k)=\sum_j (c_{ij}(\alpha_j,\alpha_j)/2)\,(2(\alpha_j,\alpha_k)/(\alpha_j,\alpha_j))=\sum_j a_{kj}((\alpha_j,\alpha_j)c_{ji}/2)$ $(1\leqslant i,\ k\leqslant l)$. It follows that $b_{ji}=(\alpha_j,\alpha_j)c_{ji}/2$ and $c_{ij}=(2/(\alpha_i,\alpha_i))b_{ij}$. Hence $\lambda'_i=-c_{i_0,i}/c_{i_0,i_0}=-b_{i_0,i}/b_{i_0,i_0}=-\lambda_{\alpha_i}$ $(1\leqslant i\leqslant l,\ i\neq i_0)$, as is desired.

Let $z'=\sum_k M^*_{\gamma_k}$ be an element of $\bar{S}'\cap Z'_*$. For $i\neq i_0$, $1\leqslant i\leqslant l$, we have $(\alpha_i,z')\in(\Pi',Z'_*)\subset\boldsymbol{Z}$ and $(\alpha_{i_0},z')=(-\sum_{\gamma\in\Pi'}\lambda_\gamma\gamma,\ \sum_k M^*_{\gamma_k})=-\sum_k\lambda_{\gamma_k}$. It follows that

$$\bar{S}'\cap Z_* = \{\sum_k M^*_{\gamma_k};\ m_{\gamma_k}=1\ \textit{for all } k,\ \sum_k \lambda_{\gamma_k}\in\boldsymbol{Z}\}.$$

Let again $z'=\sum_k M^*_{\gamma_k}$ be an element of $\bar{S}'\cap Z'_*$. If we put $z=\lambda''\Lambda^*_{i_0}+z'$ $(\lambda''\in\boldsymbol{R})$, then for $i\neq i_0$, $1\leqslant i\leqslant l$, we have $(z,\alpha_i)=(z',\alpha_i)\in(Z'_*,\Pi')\subset\boldsymbol{Z}$ and $(z,\alpha_{i_0})=\lambda''-\sum_k\lambda_{\gamma_k}$. It follows that $z\in\bar{S}\cap Z_*$ if and only if $\lambda''-\sum_k\lambda_{\gamma_k}\in\boldsymbol{Z}$. Let

$$\lambda_{z'}=\mathrm{Min}\,\{|\sum_k\lambda_{\gamma_k}+m|;\ m\in\boldsymbol{Z},\ \sum_k\lambda_{\gamma_k}+m\neq 0\},$$

$$\lambda_0=\mathop{\mathrm{Min}}_{z'\in\bar{S}'\cap Z'_*}\lambda_{z'}.$$

Let λ_0 be attained by $z'_0 = \sum_k M^*_{\gamma^0_k} \in \bar{S}' \cap Z'_*$, that is, $\lambda_0 = \sum_k \lambda_{\gamma^0_k} + m_0$ for some integer m_0. Let $w'_0 = \pi' \xi'^{-1}(z'_0)$ and w_0 the trivial extension of w'_0 to $\mathfrak{h}_+$. Then by Lemma 3 $z_0 = \lambda_0 \Lambda^*_{i_0} + z'_0$ gives a representative of a generator of the free part of C by multiplying $2\pi\sqrt{-1}$ and $\tau_0 = t(z_0)w_0$ is a generator of the free part of $\tilde{W}_*(S) \cong \pi_1(G)$.

Case (A), 4). Since $\Pi' = \Pi_0$, we have

$$\bar{S}' \cap Z_* = \bar{S}' \cap Z'_* \quad \textit{and} \quad \mathscr{F} = \mathscr{F}(\mathfrak{k}).$$

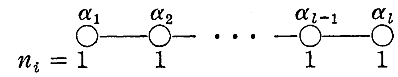
Case (A), 5). Let z' be an element of $\bar{S}' \cap Z'_*$. For $i \neq i_0$, $1 \leqslant i \leqslant l$, we have $(\alpha_i, z') = (\bar{\alpha}_i, z') \in (\Pi', Z'_*) \subset \boldsymbol{Z}$ and $(\alpha_{i_0}, z') = (\xi - \alpha_{i_1} - \cdots - \alpha_{i_t} - \alpha_k, z') = (\bar{\xi} - \alpha_{i_1} - \cdots - \alpha_{i_t} - \bar{\alpha}_k, z')$ is contained in the subgroup of $\boldsymbol{Z}$ generated by (Π', Z'_*). It follows again that

$$\bar{S}' \cap Z_* = \bar{S}' \cap Z'_* \quad \textit{and} \quad \mathscr{F} = \mathscr{F}(\mathfrak{k}).$$

Example of Case (A), 3).

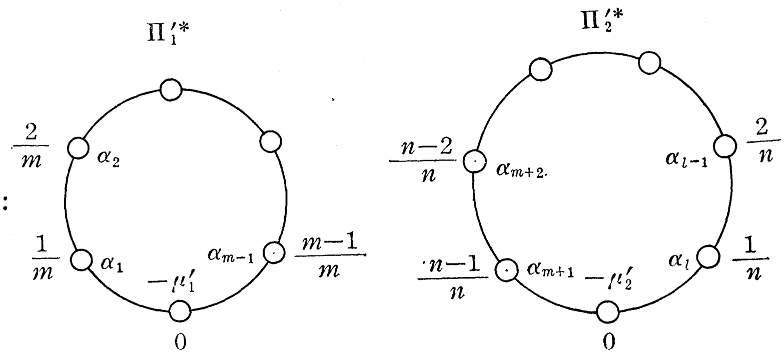
$\mathfrak{g}_u = A_l \quad (l \geqslant 1)$.

Π:



Let $i_0 = m$, $1 \leqslant m \leqslant (l+1)/2$ and put $n = l + 1 - m$. Then $b_{m,i} = in/(m+n)$ $(1 \leqslant i \leqslant m)$ and $b_{m,m+i} = (n-i)m/(m+n)$ $(1 \leqslant i \leqslant n-1)$.

Π'^*:



We wrote the number λ_γ at the vertex γ. $m_\gamma = 1$ for all root γ of Π'^*. Let $\{M^*_i;\ 1 \leqslant i \leqslant l,\ i \neq m\} \subset \mathfrak{h}'$ be the dual basis of $\{\alpha_i;\ 1 \leqslant i \leqslant l,\ i \neq m\}$ and put $M^*_0 = M^*_m = 0$. Then $\bar{S}' \cap Z'_* = \{M^*_i + M^*_{m+j}\ ;\ 0 \leqslant i \leqslant m-1, 0 \leqslant j \leqslant n-1\}$. It follows that $\mathscr{F}(\mathfrak{k}')$ is the direct product of the groups of "rotations" of

$\Pi_1'^*$ and $\Pi_2'^*$ so that $\mathscr{F}(\mathfrak{k}') \cong \boldsymbol{Z}_m \times \boldsymbol{Z}_n$. Let $d = (m, n)$ and a and b the integers such that $0 \leqslant a \leqslant m-1$, $0 \leqslant b \leqslant n-1$ and $an + bm \equiv d \pmod{mn}$. Put $p = m/d$ and $q = n/d$. Then we have

$$\bar{S}' \cap Z_* = \{M^*_{pk} + M^*_{m+qk};\ 0 \leqslant k \leqslant d-1\}$$

so that $\mathscr{F} \cong \boldsymbol{Z}_d$. We have $\lambda_0 = d/mn$ so that

$$z_0 = (d/mn)\, \Lambda^*_m + M^*_a + M^*_{m+n-b}$$

gives a representative of a generator of the free part of C by multiplying $2\pi\sqrt{-1}$.

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