Shinji Yamashita
Nagoya Math. J.
Vol. 40 (1970), 33–37

CONTINUATIONS OF ANALYTIC FUNCTIONS OF CLASS S AND CLASS U

SHINJI YAMASHITA

1. Introduction. Let f be of class U in Seidel's sense ([4, p. 32], ="inner function" in [3, p. 62]) in the open unit disk D. Then f has, by definition, the radial limit $f(e^{i\theta})$ of modulus one a.e. on the unit circle K. As a consequence of Smirnov's theorem [5, p. 64] we know that the function

$$\frac{1+f(z)}{1-f(z)}$$

belongs to the Hardy class H_p for any p with $0<p<1$ (and we note that this does not necessarily belong to H_1). This implies that

$$\int_I \left| \frac{1+f(e^{i\theta})}{1-f(e^{i\theta})} \right|^p d\theta < \infty \quad (0<p<1)$$

for any open arc $I=(e^{i\alpha}, e^{i\beta})$ with $\alpha<\beta$, $\beta-\alpha \leqq 2\pi$, of K, where $\int_I$ is an abbreviation of $\int_\alpha^\beta$.

Now, what can we say about f if

$$\int_I \left| \frac{1+f(e^{i\theta})}{1-f(e^{i\theta})} \right| d\theta < \infty \tag{1}$$

for an open arc I of K? One aim of this note is to verify that the condition (1) allows $f(z)$ to possess a meromorphic extension into D_1, the complement of $K-I$ with respect to the extended z-plane (Theorem 2).

For the above purpose we prove a theorem (Theorem 1) concerning analytic extensions of functions of class S ($=N^*$ in [2], D in [5] and N_* in [6], cf. also [7] and [8] for the definition) to a simple rectifiable arc in the z-plane, which gives us an extension of Theorem 1 in [8].

Received June 16, 1969.

2. Polubarinova-Kočina's Theorem.

As a small modification of P.J. Polubarinova-Kočina's theorem [5, p. 80] we prove first

LEMMA. *Let $f(z)$ be in the class $S(G)$ of a Jordan domain G in the z-plane with the rectifiable boundary Γ. Assume that f has the finite asymptotic value $f(\zeta)$ along a simple arc in G terminating at a.e. point $\zeta \in \Gamma$, i.e., except for a set of linear measure zero on Γ. Furthermore assume that*

$$\int_{\Gamma} |f(\zeta)||d\zeta| < \infty. \tag{2}$$

Then f belongs to the class $E_1(G)$ (cf. [5, p. 145 ff.] and [8]).

Proof. Let $z = z(w)$ be a one-to-one conformal map of the disk D onto G. Then the function $F(w) = f(z(w))z'(w)$ is of class $S(D)$ since both the composed function $f(z(w))$ and the derived function $z'(w)$ of $z(w)$ are of class $S(D)$. Let $\zeta = z(e^{i\theta})$ be the natural extension of the map $z = z(w)$ to the circle K. Then we can easily show, by means of Bagemihl's ambiguous point theorem [1], that

$$f^*(e^{i\theta}) = f(\zeta) \text{ for } \zeta = z(e^{i\theta})$$

a.e. on K, where $f^*(e^{i\theta})$ is the radial limit of $f(z(w))$ at $e^{i\theta}$, the existence of which follows from the fact that $f(z(w))$ is of bounded type in D. Thus the condition (2) implies that

$$\int_K |F(e^{i\theta})|d\theta = \int_K |f^*(e^{i\theta})z'(e^{i\theta})|d\theta < \infty,$$

where $F(e^{i\theta})$ ($z'(e^{i\theta})$ resp.) is the radial limit of $F(w)$ ($z'(w)$ resp.) at $e^{i\theta}$. Therefore by Polubarinova-Kočina's theorem [5, p. 80], $F(w)$ is of class H_1, so that $f(z)$ is of class $E_1(G)$.

3. Continuation of functions of class S.

We now prove the following

THEOREM 1. *Let G_1 and G_2 be mutually disjoint Jordan domains in the plane and let I be an open arc lying on the non-empty common boundary of G_1 and G_2 such that the closure of I is rectifiable. Let f_1 and f_2 be analytic functions of class $S(G_1)$ and class $S(G_2)$ respectively. Suppose that for a.e. point ζ of I there exist* *simple arcs* $L_{\zeta,1}$ *and* $L_{\zeta,2}$ *lying in* G_1 *and* G_2 *respectively except for their common terminal point* ζ *such that*

(3) $$\lim_{z\to\zeta} f_1(z) = \lim_{z\to\zeta} f_2(z) = \omega_\zeta \neq \infty,$$

where the limits are taken along $L_{\zeta,1}$ *and* $L_{\zeta,2}$ *respectively. Furthermore suppose that*

(4) *the function* $\varphi(\zeta) = \omega_\zeta$ *defined a.e. on* I *is integrable there.*

Then we can find an analytic function $F(z)$ *in the domain* $G_1 \cup I \cup G_2$ *such that* $F(z) \equiv f_j(z)$ *in* G_j *for* $j = 1, 2$.

Proof. We assert first that

(5) $$f_j(\zeta) = \varphi(\zeta) \text{ a.e. in } I,$$

where $f_j(\zeta)$ is the asymptotic value of $f_j(z)$ at $\zeta \in I$ along the normal $c_{\zeta,j}$ from the interior of G_j to ζ $(j = 1, 2)$. For, corresponding to any point ζ_0 of I we can find a Jordan domain G_0 satisfying the following conditions:

(6) G_0 contains ζ_0 and the domain $G_1 \cup I \cup G_2$ contains the closure of G_0;

(7) $G_{0,j} = G_0 \cap G_j$ is a Jordan domain with the rectifiable boundary $(j = 1, 2)$ (cf. e.g. [8]).

Then the restriction of f_j to $G_{0,j}$ belongs to the class $S(G_{0,j})$, so that f_j can be represented as the quotient of two bounded analytic functions in $G_{0,j}$ $(j = 1, 2)$. It now follows from the generalized Fatou's theorem [5, p. 129] that f_j has the finite asymptotic value along $c_{\zeta,j}$ at a.e. point ζ of $I_0 = G_0 \cap I$ $(j = 1, 2)$. Since ζ_0 is arbitrary, our assertion (5) follows from Bagemihl's ambiguous point theorem [1] and hypothesis (3).

We take an arbitrary point ζ_0 of I again and construct a Jordan domain G_0 satisfying the properties (6) and (7) with the additional property that

(8) the Jordan arc $\gamma_j = G_j \cap$ (the boundary of G_0) is normal to I at both its terminal points ζ_1 and ζ_2 and that f_j has the finite asymptotic values $\varphi(\zeta_1)$ and $\varphi(\zeta_2)$ at ζ_1 and ζ_2 respectively along γ_j $(j = 1, 2)$.

Then (8) implies that the restriction of f_j to γ_j is integrable there $(j = 1, 2)$, so that combining (4) with (5) we obtain (2) in Lemma with $G = G_{0,j}$ $(j = 1, 2)$. On the other hand, we know that f_j is in $S(G_{0,j})$ $(j = 1, 2)$. Therefore by Lemma we obtain $f_j \in E_1(G_{0,j})$ $(j = 1, 2)$. Applying now the lemma in [8] to $G_{0,1}$, $G_{0,2}$, I_0, f_1 and f_2 we can find an analytic function in G_0 which is identical with f_j in $G_{0,j}$ $(j = 1, 2)$. Since ζ_0 is chosen at will, the proof is complete.

Remark. As a consequence of the present theorem we may replace the word "bounded" in the condition (b) of Theorem 1 in [8] by the word "integrable".

4. Continuation of functions of class U.

Returning to section 1 we now consider the function $f(z)$ of class U satisfying the condition (1). We set

$$f_1(z) = i\left\{\frac{1+f(z)}{1-f(z)}\right\} \text{ in } D$$

and

$$f_2(z) = \overline{f_1(1/\bar{z})} \text{ in } D^* : 1 < |z| \leqq \infty.$$

Then f_1 and f_2 are of class $S(D)$ and class $S(D^*)$ respectively because H_p $(p > 0)$ is a subclass of S (cf. e.g. [5] and [7]). For any point $e^{i\theta}$ of the arc I we may construct a sufficiently small disk d with the centre $e^{i\theta}$ and we set $V_1 = D \cap d$ and $V_2 = D^* \cap d$, so that we can apply Theorem 1 to V_1, V_2, $I \cap d$, f_1 and f_2. Since $e^{i\theta}$ is arbitrary, it follows that $f_1(z)$ has an analytic continuation into D_1. We have thus established

THEOREM 2. *Let $f(z)$ be of class U in the open unit disk D. Assume that* (1) *in section* 1 *holds for an open arc I of the unit circle K, where $f(e^{i\theta})$ is the radial limit of f at $e^{i\theta}$ of K. Then there exists a meromorphic function $F(z)$ in the complement of the closed arc $K - I$ with respect to the extended z-plane, such that $F(z) \equiv f(z)$ in D.*

Remark 1. The converse is not true as the following simple example shows: $f(z) = z$ and I contains 1.

Remark 2. If

$$\int_K \left|\frac{1+f(e^{i\theta})}{1-f(e^{i\theta})}\right| d\theta < \infty,$$

then f must be a constant. For, in this case, the function $f_1(z)$ must be a constant by Liouville's theorem.

References

[1] F. Bagemihl, Curvilinear cluster sets of arbitrary functions, Proc. Nat. Acad. Sci. U.S.A., **41**(1955), 379–382.

[2] F.W. Gehring, The asymptotic values for analytic functions with bounded characteristic, Quart. J. Math. Oxford, **9**(1958), 282–289.

[3] K. Hoffman, Banach spaces of analytic functions, Englewood Cliffs, N.J., 1962.

[4] K. Noshiro, Cluster sets, Berlin-Göttingen-Heidelberg, 1960.

[5] I.I. Privalov, Randeigenschaften analytischer Funktionen, Berlin, 1956.

[6] W. Rudin, A generalization of a theorem of Frostman, Math. Scand., **21**(1967), 136–143.

[7] S. Yamashita, On some families of analytic functions on Riemann surfaces, Nagoya Math. J., **31**(1968), 57–68.

[8] ———————, Some remarks on analytic continuations, Tôhoku Math. J., **21**(1969), 328–335.

Mathematical Institute
Tôhoku University
Sendai 980, *Japan*

Added in proof. A modified form of the lemma in section 2 is stated without proof in p. 70 of G. C. Tumarkin and S. Ya. Havinson's monograph: "Classes of analytic functions in multiply-connected domains", Researches on contemporary problems of the theory of functions of a complex variable, edited by A.I. Markusevic, Moscow, 1960, pp. 45–77, in Russian.