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LIFTINGS OF SOME TYPES OF TENSOR FIELDS AND CONNECTIONS TO TANGENT BUNDLES OF p^r-VELOCITIES

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§ Introduction.

In the previous paper [6], we studied the liftings of tensor fields to tangent bundles of higher order. The purpose of the present paper is to generalize the results of [6] to the tangent bundles $\overset{r,p}{T}M$ of p^r-velocities in a manifold M—— notions due to C. Ehresmann [1] (see also [2]). In § **1**, we explain the p^r-velocities in a manifold and define the (λ)-lifting of differentiable functions for any multi-index $\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_p)$ of non-negative integers λ_i satisfying $\sum \lambda_i \leq r$. In § **2**, we construct $\langle\lambda\rangle$-lifts of any vector fields and (λ)-lifts of 1-forms. The $\langle\lambda\rangle$-lift is a little bit different from the (λ)-lift of vector fields in [6].

In § **3**, we construct (λ)-lifting of $(0, q)$-tensor fields and then (λ)-lifting of $(1, q)$-tensor fields to $\overset{r,p}{T}M$ for $q \geq 1$. Unfortunately, the author could not construct a natural lifting of (s, q)-tensor fields to $\overset{r,p}{T}M$ for $s \geq 2$.

Nevertheless, our (λ)-liftings of (s, q)-tensor fields for $s = 0$ or 1 are quite sufficient for the geometric applications, because the important tensor fields with which we encounter so far in differential geometry seem to be, fortunately, only of type (s, q) with $s = 0$ or 1.

As an application, we shall consider in § **4**, the prolongations of almost complex structures and prove that if M is a (homogeneous) complex manifold, then $\overset{r,p}{T}M$ is also a (homogeneous) complex manifold.

In § **5**, we consider the liftings of affine connections to $\overset{r,p}{T}M$ and prove that if M is locally affine symmetric then $\overset{r,p}{T}M$ is also locally affine symmetric with respect to the lifted affine connection.

In § **6**, we shall give a proof for the fact that if M is an affine symmetric space then $\overset{r,p}{T}M$ is also an affine symmetric space.

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In this paper, all manifolds and mappings (functions) are assumed to be differentiable of class C^∞, unless otherwise stated.

We shall fix two positive integers r and p throughout the paper.

§1. Tangent bundles of p^r-velocities.

Consider the algebra $C^\infty(R^p)$ of all C^∞-functions on the p-dimensional euclidean space R^p with natural coordinates $(t_1, t_2, \cdots, t_p)$. For any p-tuple $\nu = (\nu_1, \nu_2, \cdots, \nu_p)$ of non-negative integers ν_j we denote as usual by $(\partial/\partial t)^\nu$ the following partial differentiation

$$(1.1) \qquad \left(\frac{\partial}{\partial t}\right)^\nu f = \frac{\partial^{\nu_1+\cdots+\nu_p} f}{\partial t_1^{\nu_1}\cdots\partial t_p^{\nu_p}}$$

for $f \in C^\infty(R^p)$. We define $|\nu|$ and $\nu!$ as follows:

$$|\nu| = \nu_1 + \cdots + \nu_p, \quad \nu! = \nu_1!\, \nu_2! \cdots \nu_p!.$$

We denote by $N(p,r)$ the set of all p-tuples $\nu = (\nu_1, \cdots, \nu_p)$ of non-negative integers ν_i such that $|\nu| \leq r$. The set $N(p,r)$ is a subset of the additive group Z^p of all p-tuples of integers.

Take two elements $f, g \in C^\infty(R^p)$. We say f is r-equivalent to g if $(\partial/\partial t)^\nu f = (\partial/\partial t)^\nu g$ at $t = (t_1, \cdots, t_p) = 0$ for every $\nu \in N(p,r)$ and denote it by $f \underset{r}{\sim} g$. Clearly $\underset{r}{\sim}$ is an equivalence relation in $C^\infty(R^p)$.

Now, let M be an n-dimensional manifold. Consider the set $S_p(M)$ of all maps φ: $R^p \to M$. Take two elements $\varphi, \psi \in S_p(M)$. We say that φ is r-equivalent to ψ if $f \circ \varphi \underset{r}{\sim} f \circ \psi$ for every $f \in C^\infty(M)$ and denote it by $\varphi \underset{r}{\sim} \psi$. The relation $\underset{r}{\sim}$ is also an equivalence relation in $S_p(M)$. We denote by $\overset{r,p}{T}M$ the set of all equivalence classes in $S_p(M)$ with respect to the equivalence relation $\underset{r}{\sim}$. We denote by $[\varphi]_r$ the equivalence class containing $\varphi \in S_p(M)$, and we shall call it a *p^r-velocity* in M at $\varphi(0)$. To introduce the manifold structure in $\overset{r,p}{T}M$, we define local coordinate system on $\overset{r,p}{T}M$ as follows: Take a coordinate neighborhood U in M with coordinate system $\{x_1, x_2, \cdots, x_n\}$. Define the coordinate functions $\{x_i^{(\nu)} \mid i = 1, \cdots, n;\ \nu \in N(p,r)\}$ on $\overset{r,p}{T}U$ by

$$(1.2) \qquad x_i^{(\nu)}([\varphi]_r) = \frac{1}{\nu!}\left[\left(\frac{\partial}{\partial t}\right)^\nu (x_i \circ \varphi)\right]_{t=0}$$

for $[\varphi]_r \in \overset{r,p}{T}U$ (cf. (1.1)). It is straightforward to see that $\overset{r,p}{T}M$ becomes a manifold by the above coordinate systems $\{x_i^{(\nu)}\}$. The projection $\overset{r,p}{\pi}$ defined by $\overset{r,q}{\pi}([\varphi]_r)=\varphi(0)$ for $\varphi\in S_p(M)$ is clearly a differentiable map of $\overset{r,p}{T}M$ onto M.

DEFINITION 1.1. The differentiable manifold $\overset{r,p}{T}M$ with projection $\overset{r,p}{\pi}$ will be called *the tangent bundle of p^r-velocities in M.*

DEFINITION 1.2. For any $f\in C^\infty(M)$, we define *the (λ)-lift $f^{(\lambda)}$ of f*, for every $\lambda\in N(p,r)$, as follows:

$$f^{(\lambda)}([\varphi]_r)=\frac{1}{\lambda!}\left[\left(\frac{\partial}{\partial t}\right)^{\lambda}(f\circ\varphi)\right]_{t=0} \tag{1.3}$$

for $[\varphi]_r\in\overset{r,p}{T}M$. Clearly, $f^{(\lambda)}$ is a well-defined differentiable function on $\overset{r,p}{T}M$. We note also that $(x_i)^{(\nu)}=x_i^{(\nu)}$ holds on $\overset{r,p}{T}U$ for the above coordinate system $\{x_1,\cdots,x_n\}$.

For the sake of convenience we define $f^{(\lambda)}=0$ for any $\lambda\in Z^p$ such that $\lambda\notin N(p,r)$.

LEMMA 1.3. The (λ)-lifting $f\to f^{(\lambda)}$ is a linear map of $C^\infty(M)$ into $C^\infty(\overset{r,p}{T}M)$ and satisfies the following equality

$$(f\cdot g)^{(\lambda)}=\sum_{\mu\in Z^p}f^{(\mu)}\cdot g^{(\lambda-\mu)} \tag{1.4}$$

for every $f,g\in C^\infty(M)$ and $\lambda\in N(p,r)$.

Proof. Straightforward verification similar to the one of Lemma 1.2 [6].

§ 2. Liftings of vector fields and 1-forms.

Let $\mathcal{T}(M)=\sum\mathcal{T}_q^s(M)$ be, as in [6], the tensor algebra of all tensor fields on M.

LEMMA 2.1. *For any $X\in\mathcal{T}_0^1(M)$ and any $\lambda\in N(p,r)$ there exists one and only one $X^{<\lambda>}\in\mathcal{T}_0^1(\overset{r,p}{T}M)$ satisfying the following equality*

$$X^{<\lambda>}f^{(\mu)}=(Xf)^{(\mu-\lambda)} \tag{2.1}$$

for every $f\in C^\infty(M)$ and $\mu\in N(p,r)$.

Proof. Take a coordinate neighborhood U in M with coordinate system $\{x_1,\cdots,x_n\}$ and let $X=\sum a_i\cdot\partial/\partial x_i$ $(a_i\in C^\infty(U))$ be the local expression of X in U. Consider the vector field $\tilde{X}=\tilde{X}_U$ on $(\overset{r,p}{\pi})^{-1}(U)$ defined by

$$\tilde{X} = \sum_{\mu \in N(p,r)} \sum_{j=1}^{n} a_j^{(\mu-\lambda)} \frac{\partial}{\partial x_j^{(\mu)}} . \tag{2.2}$$

We see that $\tilde{X}(x_j^{(\mu)}) = a_j^{(\mu-\lambda)} = (Xx_j)^{(\mu-\lambda)}$ for $j = 1, 2, \cdots, n$; $\mu \in N(p, r)$. Now, making use of same arguments as in the proof of Lemma 1.4 [6], we can prove that $\tilde{X}(f^{(\mu)}) = (Xf)^{(\mu-\lambda)}$ for every $f \in C^\infty(U)$ and $\mu \in N(p, r)$. We can also prove that if U' is a coordinate neighborhood in M such that $U \cap U' = U'' \neq \phi$, then $\tilde{X}_U | U'' = \tilde{X}_{U'} | U''$ holds. Thus we obtain a vector field $X^{<\lambda>}$ on $\overset{r,p}{T}M$ such that $X^{<\lambda>} | (\overset{r,p}{\pi})^{-1}(U) = \tilde{X}_U$ for every coordinate neighborhood U in M. This vecotr field $X^{<\lambda>}$ clearly satisfies the condition (2.1) for every $f \in C^\infty(M)$ and $\mu \in N(p, r)$. The uniqueness of $X^{<\lambda>}$ is also easily verified. Q.E.D.

COROLLARY 2.2. *Let $\{x_1, \cdots, x_n\}$ be a local coordinate system on a neighborhood U in M. Then, we have*

$$\left(\frac{\partial}{\partial x_i}\right)^{<\lambda>} = \frac{\partial}{\partial x_i^{(\lambda)}} \tag{2.3}$$

for every $i = 1, \cdots, n$ and $\lambda \in N(p, r)$.

Proof. Clear from the expression (2.2) of $X^{<\lambda>}$ in $(\overset{r,p}{\pi})^{-1}(U)$.

COROLLARY 2.3. *Notations being as in Corollary 2.2, we have*

$$\left(\frac{\partial f}{\partial x_i}\right)^{(\lambda-\mu)} = \frac{\partial f^{(\lambda)}}{\partial x_i^{(\mu)}} \tag{2.4}$$

for every $i = 1, \cdots, n$ and $\lambda, \mu \in N(p, r)$.

Proof. By Corollary 2.2, we have

$$\frac{\partial f^{(\lambda)}}{\partial x_i^{(\mu)}} = \left(\frac{\partial}{\partial x_i}\right)^{<\mu>} f^{(\lambda)} = \left(\frac{\partial f}{\partial x_i}\right)^{(\lambda-\mu)}. \qquad \text{Q.E.D.}$$

DEFINITION 2.4. The vector field $X^{<\lambda>}$ in Lemma 2.1 will be called *the $\langle\lambda\rangle$-lift of X to $\overset{r,p}{T}M$ for $\lambda \in N(p, r)$.* For the sake of convenience, we define $X^{<\lambda>} = 0$ for every $\lambda \in Z^p$ such that $\lambda \notin N(p, r)$. The $\langle\lambda\rangle$-lifting $X \to X^{<\lambda>}$ is a linear map of $\mathcal{T}_0^1(M)$ into $\mathcal{T}_0^1(\overset{r,p}{T}M)$ for every $\lambda \in Z^p$.

LEMMA 2.5. *For $X, Y \in \mathcal{T}_0^1(M)$, we have*

$$[X^{<\lambda>}, Y^{<\mu>}] = [X, Y]^{<\lambda+\mu>} \tag{2.5}$$

for every $\lambda, \mu \in N(p, r)$.

Proof. Assume $\lambda+\mu\in N(p,r)$. Then, for any $g\in C^{\infty}(M)$ and $\nu\in N(p,r)$ we have

$$\begin{aligned}[X^{<\lambda>}, Y^{<\mu>}](g^{(\nu)}) &= X^{<\lambda>}Y^{<\mu>}g^{(\nu)} - Y^{<\mu>}X^{<\lambda>}g^{(\nu)} \\ &= X^{<\lambda>}(Yg)^{(\nu-\mu)} - Y^{<\mu>}(Xg)^{(\nu-\mu)} \\ &= X^{<\lambda>}(Yg)^{(\nu-\mu)} - Y^{<\mu>}(Xg)^{(\nu-\lambda)} \\ &= (XYg - YXg)^{(\nu-\lambda-\mu)} = ([X,Y]g)^{(\nu-\lambda-\mu)} \\ &= [X,Y]^{<\lambda+\mu>}g^{(\nu)}.\end{aligned}$$

Since $g\in C^{\infty}(M)$ and $\nu\in N(p,r)$ are arbitrary we get (2.5) if $\lambda+\mu\in N(p,r)$.

Assume $\lambda+\mu\notin N(p,r)$, then by our convention, we have $[X,Y]^{<\lambda+\mu>}=0$. On the other hand, for any $g\in C^{\infty}(M)$ and $\nu\in N(p,r)$ we have, by the same calculation as above,

$[X^{<\lambda>}, Y^{<\mu>}]g^{(\nu)} = (XYg)^{(\nu-\mu-\lambda)} - (YXg)^{(\nu-\lambda-\mu)} = 0,$ since $\nu-\mu-\lambda\notin N(p,r)$. Thus (2.5) is verified in any case. Q.E.D.

LEMMA 2.6. *For $X\in\mathscr{T}^1_0(M)$ and $f\in\mathscr{T}^0_0(M)$, we have*

$$(f\cdot X)^{<\lambda>} = \sum_{\nu\in N(p,r)} f^{(\nu)}\cdot X^{<\lambda+\nu>} \tag{2.6}$$

for every $\lambda\in N(p,r)$.

Proof. For any $g\in\mathscr{T}^0_0(M)$ and $\mu\in N(p,r)$, we have

$$\begin{aligned}(f\cdot X)^{<\lambda>}g^{(\mu)} &= (fX\cdot g)^{(\mu-\lambda)} = (f\cdot Xg)^{(\mu-\lambda)} \\ &= \sum_{\nu\in Z^p} f^{(\nu)}\cdot(Xg)^{(\mu-\lambda-\nu)} = \sum f^{(\nu)}\cdot X^{<\lambda+\nu>}g^{(\mu)} \\ &= (\sum f^{(\nu)}X^{<\lambda+\nu>})g^{(\mu)}.\end{aligned}$$

Since g and μ are arbitrary, we get (2.6) for every $\lambda\in N(p,r)$. Q.E.D.

Remark 2.7. By our convention (cf. Def. 1.2) we can write (2.6) as follows:

$$(f\cdot X)^{<\lambda>} = \sum_{\nu\in Z^p} f^{(\nu)}X^{<\lambda+\nu>}. \tag{2.7}$$

LEMMA 2.8. *Let $f_i, g_i\in C^{\infty}(M)$ $(i=1,\cdots,k)$ be such that $\sum g_i df_i=0$ on M. Then the following equality*

$$\sum_{i=1}^{k}\sum_{\mu\in Z^p} g_i^{(\mu)}df_i^{(\lambda-\mu)} = 0 \tag{2.8}$$

holds on $\overset{r,p}{T}M$ *for every* $\lambda \in N(p,r)$.

Proof. Similar to the proof of Lemma 2.1 [6]. Q.E.D.

LEMMA 2.9. *There is one and only one lifting* L_λ: $\mathscr{T}_1^0(M) \to \mathscr{T}_1^0(\overset{r,p}{T}M)$ *for every* $\lambda \in N(p,r)$ *satisfying the following condition:*

$$L_\lambda(f \cdot dg) = \sum_{\lambda \in Z^p} f^{(\mu)} dg^{(\lambda-\mu)} \tag{2.9}$$

for every $f, g \in \mathscr{T}_0^0(M)$.

Proof. Similar to the proof of Lemma 2.2 [6].

LEMMA 2.10. *For* $f \in \mathscr{T}_0^0(M)$ *and* $\theta \in \mathscr{T}_1^0(M)$, *we have*

$$(f \cdot \theta)^{(\lambda)} = \sum_{\mu \in Z^p} f^{(\mu)} \cdot \theta^{(\lambda-\mu)} \tag{2.10}$$

for every $\lambda \in N(p,r)$.

Proof. Similar to the proof of Corollary 2.4 [6].

LEMMA 2.11. *For* $\theta \in \mathscr{T}_1^0(M)$ *and* $X \in \mathscr{T}_0^1(M)$, *we have*

$$\theta^{(\lambda)}(X^{<\mu>}) = (\theta(X))^{(\lambda-\mu)} \tag{2.11}$$

for every $\lambda, \mu \in N(p,r)$.

Proof. Let $\theta = \sum f_i dx_i$ be the local expression of θ. Making use of Lemma 2.1, we calculate as follows:

$$\begin{aligned}
\theta^{(\lambda)}(X^{<\mu>}) &= (\textstyle\sum f_i dx_i)^{(\lambda)}(X^{<\mu>}) \\
&= \sum_i \sum_{\nu \in Z^p} f_i^{(\nu)} dx_i^{(\lambda-\nu)}(X^{<\mu>}) \\
&= \sum_i \sum_\nu f_i^{(\nu)}(X^{<\mu>} x_i^{(\lambda-\nu)}) = \sum_i \sum_\nu f_i^{(\nu)}(Xx_i)^{(\lambda-\nu-\mu)} \\
&= \sum_i \sum_\nu f_i^{(\nu)}(dx_i(X))^{(\lambda-\nu-\mu)} \\
&= \textstyle\sum (f_i \cdot dx_i(X))^{(\lambda-\mu)} = (\theta(X))^{(\lambda-\mu)}.
\end{aligned}$$

Q.E.D.

§ 3. Lifting of $(1, q)$-tensor fields.

Let $\mathscr{T}_*(M)$ be the subalgebra of $\mathscr{T}(M)$ consisting of all covariant tensor fields on M. We denote by $\overset{r,p}{\mathscr{T}_*}M$ the $m(r,p)$ times direct sum of $\mathscr{T}_*(\overset{r,p}{T}M)$, where $m(r,p)$ denotes the number of elements in $N(p,r)$. i.e.

$$\overset{r,p}{\mathscr{T}}_*(M) = \sum_{q=0}^{\infty} \sum_{\lambda\in N(p,r)} (\mathscr{T}^0_q(\overset{r,p}{T}M))_\lambda,$$

where $(\mathscr{T}^0_q(\overset{r,p}{T}M))_\lambda = \mathscr{T}^0_q(\overset{r,p}{T}M)$ for all $\lambda\in N(p,r)$.

Take two elements $\theta = (\theta^\lambda)$ and $\eta = (\eta^\lambda)$ in $\overset{r,p}{\mathscr{T}}_*(M)$. We define the multiplication $\theta\otimes\eta$ of θ and η by the following:

$$(\theta\otimes\eta)^\lambda = \sum_{\mu,\, \lambda-\mu\in N(p,r)} \theta^\mu \otimes \eta^{\lambda-\mu} \tag{3.1}$$

for $\lambda\in N(p,r)$. We can readily see that $\overset{r,p}{\mathscr{T}}_*(M)$ is an associative graded algebra over $C^\infty(\overset{r,p}{T}M)$ by this multiplication $\otimes$.

We have defined, in Lemma 2.9, the lifting L_λ of $\mathscr{T}^0_1(M)$ into $\mathscr{T}^0_1(\overset{r,p}{T}M)$ for $\lambda\in N(p,r)$. Define L: $\mathscr{T}^0_1(M)\to \overset{r,p}{\mathscr{T}}_*(TM)$ by $L(\theta) = (L_\lambda(\theta))_{\lambda\in N(p,r)}$ for $\theta\in T^0_1(M)$.

Lemma 3.1. *There exists one and only one homomorphism* $\tilde{L}$: $\mathscr{T}_*(M)\to \overset{r,p}{\mathscr{T}}_*(M)$ *such that* $\tilde{L}|\mathscr{T}^0_1(M) = L$.

Proof. Define L^q: $(\mathscr{T}^0_1(M))^q\to \overset{r,p}{\mathscr{T}}_*(M)$ by

$$L^q(\theta_1, \cdots, \theta_q) = L(\theta_1)\otimes\cdots\otimes L(\theta_q)$$

for $\theta_i\in\mathscr{T}^0_1(M)$ $i = 1, 2, \cdots, q$. Then, L is a multilinear map satisfying the following condition:

$$L^q(f_1\theta_1, \cdots, f_q\theta_q) = L(f_1\cdots f_q)\otimes L^q(\theta_1, \cdots, \theta_q)$$

for $\theta_i\in\mathscr{T}^0_1(M)$ and $f_i\in\mathscr{T}^0_0(M)$ $i = 1, \cdots, q$, from which we conclude that there is a linear map $\tilde{L}^q$ of $\mathscr{T}^0_q(M)$ into $\overset{r,p}{\mathscr{T}}_*(M)$ such that

$$\tilde{L}^q(\theta_1\otimes\cdots\otimes\theta_q) = L(\theta_1)\otimes\cdots\otimes L(\theta_q)$$

for $\theta_i\in\mathscr{T}^0_1(M)$, $i = 1, \cdots, q$. Thus $\tilde{L}^q(q\geqq 0)$ define a homomorphism $\tilde{L}: \overset{r,p}{\mathscr{T}}_*(M)\to \mathscr{T}_*(M)$ such that $\tilde{L}(\theta) = L(\theta)$ for $\theta\in\mathscr{T}^0_1(M)$. Q.E.D.

Definition 3.2. For $K\in\mathscr{T}^0_q(M)$ we denote by $K^{(\lambda)}$ the λ-component of $\tilde{L}(K)$ for $\lambda\in N(p,r)$, i.e.

$$\tilde{L}(K) = (K^{(\lambda)}).$$

We shall call $K^{(\lambda)}$ *the* (λ)*-lift of* K. For the sake of convenience we put $K^{(\lambda)} = 0$ for $\lambda\in Z^p$ such that $\lambda\notin N(p,r)$.

LEMMA 3.3. *The notation* α_X^k *being as in Lemma* 3.7 [6], *for any* $K \in \mathscr{T}_q^0(M)$ *and* $X \in \mathscr{T}_0^1(M)$, *we have*

$$\alpha_{X^{<\lambda>}}^k K^{(\mu)} = (\alpha_X^k K)^{(\mu-\lambda)} \tag{3.2}$$

for $\lambda, \mu \in N(p, r)$.

Proof. Using Lemma 2.11, we can prove the lemma in the same way as the one of Lemma 3.7 [6].

COROLLARY 3.4. *For* $K \in \mathscr{T}_q^0(M)$ *and* $X_i \in \mathscr{T}_0^1(M)$ $i = 1, \cdots, q$, *we have*

$$K^{(\lambda)}(X_1^{<\mu_1>}, \cdots, X_q^{<\mu_q>}) = (K(X_1, \cdots, X_q))^{(\lambda-\Sigma\mu_i)}$$

for every $\lambda, \mu_i \in N(p, r)$, $i = 1, \cdots, q$.

Proof. We use Lemma 3.3 q-times. Q.E.D.

LEMMA 3.5. *For any* $K \in \mathscr{T}_q^1(M)$ *and* $\nu \in N(p, r)$, *there is a unique* $\tilde{K} = K^{(\nu)} \in \mathscr{T}_q^1(\overset{r,p}{T}M)$ *such that*

$$\tilde{K}(X_1^{<\lambda_1>}, \cdots, X_q^{<\lambda_q>}) = (K(X_1, \cdots, X_q))^{<\lambda+\nu>} \tag{3.3}$$

for every $X_i \in \mathscr{T}_0^1(M)$ *and* $\lambda_i \in N(p, r)$, *where* $\lambda = \sum_i \lambda_i$.

Proof. Define $\tilde{L}_\nu : \mathscr{T}_0^1(M) \times \mathscr{T}_q^0(M) \to \mathscr{T}_q^1(\overset{r,p}{T}M)$ by the following

$$\tilde{L}_\nu(X, T) = \sum_{\mu \in Z^p} X^{<\mu+\nu>} \otimes T^{(\mu)} \tag{3.4}$$

for $X \in \mathscr{T}_0^1(M)$ and $T \in \mathscr{T}_q^0(M)$. It is clear that $\tilde{L}$ is a bilinear map over R. We now assert that the following

$$\tilde{L}_\nu(fX, T) = \tilde{L}_\nu(X, fT) \tag{3.5}$$

holds for every $X \in \mathscr{T}_0^1(M)$, $T \in \mathscr{T}_q^0(M)$ and $f \in \mathscr{T}_0^0(M)$. For, making use of Remark 2.7 and Lemma 3.1, we calculate as follows:

$$\begin{aligned}
\tilde{L}_\nu(fX, T) &= \sum_\mu (fX)^{<\mu+\nu>} \otimes T^{(\mu)} \\
&= \sum_\mu \sum_\lambda f^{(\lambda)} X^{<\lambda+\mu+\nu>} \otimes T^{(\mu)} \\
&= \sum_\mu \sum_{\lambda'} f^{(\lambda'-\mu-\nu)} X^{<\lambda'>} \otimes T^{(\mu)} \\
&= \sum_{\lambda'} \sum_\mu X^{<\lambda'>} \otimes f^{(\lambda'-\mu-\nu)} T^{(\mu)}
\end{aligned}$$

$$= \sum_{\lambda'} X^{<\lambda'>} \otimes (fT)^{(\lambda'-\nu)}$$

$$= \sum_{\lambda} X^{<\lambda+\nu>} \otimes (fT)^{(\lambda)} = \tilde{L}_\nu(X, fT),$$

which proves our assertion. Thus, we obtain a linear map L_ν of $\mathscr{T}^1_q(M)$ into $\mathscr{T}^1_q(\overset{r,p}{T}M)$ such that

$$L_\nu(X \otimes T) = \sum_{\mu \in Z^p} X^{<\mu+\nu>} \otimes T^{(\mu)}$$

for $X \in \mathscr{T}^1_0(M)$ and $T \in \mathscr{T}^0_q(M)$. Put $\tilde{K} = L_\nu(K)$. It is now sufficient to prove (3.3) for $K = X \otimes T$ with $X \in \mathscr{T}^0_q(M)$ and $T \in \mathscr{T}^0_q(M)$. Using Corollary 3.4 and Lemma 2.6, we can calculate as follows:

$$\tilde{K}(X_1^{<\lambda_1>}, \cdots, X_q^{<\lambda_q>}) = \sum_{\mu} T^{(\mu)}(X_1^{<\lambda_1>}, \cdots, X_q^{<\lambda_q>}) X^{<\mu+\nu>}$$

$$= \sum_{\mu} (T(X_1, \cdots, X_q))^{(\mu-\lambda)} X^{<\mu+\nu>}$$

$$= \sum_{\mu'} (T(X_1, \cdots, X_q))^{(\mu')} X^{<\mu'+\nu+\lambda>}$$

$$= (T(X_1, \cdots, X_q) \cdot X)^{<\nu+\lambda>} = (K(X_1, \cdots, X_q))^{<\nu+\lambda>}.$$

The uniqueness of $\tilde{K}$ is clear, since (3.3) holds for every $X_i \in \mathscr{T}^1_0(M)$ and $\lambda_i \in N(p, r)$. Q.E.D.

DEFINITION 3.6. For $K \in \mathscr{T}^1_q(M)$ and $\nu \in N(p, r)$, we denote $\tilde{K}$ in Lemma 3.5 by $\tilde{K} = K^{(\nu)}$ and call it *the* (ν)-*lift of* K, i.e.

(3.6) $$K^{(\nu)}(X_1^{<\lambda_1>}, \cdots, X_q^{<\lambda_q>}) = (K(X_1, \cdots, X_q))^{<\lambda+\nu>}$$

for $X_i \in \mathscr{T}^1_0(M)$, $\lambda_i \in N(p, r)$, where $\lambda = \sum \lambda_i$. We call $K^{(0)}$ the complete lift of K to $\overset{r,p}{T}M$.

LEMMA 3.7. *For* $K \in \mathscr{T}^1_q(M)$ $(q \geqq 1)$ *and* $X \in \mathscr{T}^1_0(M)$, *we have*

(3.7) $$\alpha^k_{X^{<\lambda>}} K^{(\mu)} = (\alpha^k_X K)^{(\mu+\lambda)}$$

for $k \leqq q$ *and* $\lambda, \mu \in N(p, r)$.

Proof. It suffices to prove (3.7) for $K = Y \otimes T$ with $Y \in \mathscr{T}^1_0(M)$, $T \in \mathscr{T}^0_q(M)$. Using Lemma 3.3, we calculate as follows:

$$\alpha^k_{X^{<\lambda>}} K^{(\mu)} = \alpha^k_{X^{<\lambda>}} \sum_{\nu} Y^{<\nu+\mu>} \otimes T^{(\nu)}$$

$$= \sum_{\nu} Y^{<\nu+\mu>} \otimes \alpha^k_{X^{<\lambda>}} T^{(\nu)}$$

$$= \sum_{\nu} Y^{<\nu+\mu>} \otimes (\alpha_X^k T)^{(\nu-\lambda)}$$

$$= \sum Y^{<\nu'+\lambda+\mu>} \otimes (\alpha_X^k T)^{(\nu')}$$

$$= (Y \otimes \alpha_X^k T)^{(\lambda+\mu)} = (\alpha_X^k K)^{(\lambda+\mu)} \qquad \text{Q.E.D.}$$

COROLLARY 3.8. *We have*

$$\alpha_X^k{}_{<0>} K^{(\mu)} = (\alpha_X^k K)^{(\mu)}$$

for every $X \in \mathscr{T}_0^1(M)$, $K \in \mathscr{T}_q^1(M)$ *and* $\mu \in N(p, r)$.

§4. Prolongations of almost complex structures.

LEMMA 4.1. *For any* $A, B \in \mathscr{T}_1^1(M)$, *we have*

$$(A \circ B)^{(0)} = A^{(0)} \circ B^{(0)}. \tag{4.1}$$

Let $I_M \in \mathscr{T}_1^1(M)$ *be the* (1,1)*-tensor field of identity transformations of tangent spaces to* M. *Then, we have*

$$(I_M)^{(0)} = I_{\underset{TM}{p,r}} \tag{4.2}$$

Proof. Making use of (3.6), we have, for any $X \in \mathscr{T}_0^1(M)$,

$$A^{(0)} \circ B^{(0)}(X^{<\lambda>}) = A^{(0)}(B^{(0)}(X^{<\lambda>}))$$

$$= A^{(0)}((B(X))^{<\lambda>}) = (ABX)^{<\lambda>}$$

$$= ((A \circ B)X)^{<\lambda>} = (A \circ B)^{(0)}(X^{<\lambda>})$$

for every $\lambda \in N(p, r)$. Therefore we get (4.1).

To prove (4.2), let $I_M = \sum (\partial/\partial x_i) \otimes dx_i$ be the local expression of I_M, where $\{x_i, \cdots, x_n\}$ is a local coordinate system. Then, we have

$$(I_M)^{(0)} = \sum_{i,\mu} \left(\frac{\partial}{\partial x_i}\right)^{<\mu>} \otimes (dx_i)^{(\mu)}$$

$$= \sum_{i,\mu} \frac{\partial}{\partial x_i^{(\mu)}} \otimes dx_i^{(\mu)} = I_{\underset{TM}{p,r}},$$

which proves (4.2).

COROLLARY 4.2. *For any polynomial* $P(x)$ *of one variable* x *with real coefficients and for any* $A \in \mathscr{T}_1^1(M)$, *we have*

$$(P(A))^{(0)} = P(A^{(0)}). \tag{4.3}$$

Proof. Use (4.1) and (4.2) repeatedly. Q.E.D.

THEOREM 4.3. *Let J be an almost complex structure on M with its Nijenhuis tensor N_J. Then, the bundle $\overset{r,p}{T}M$ of p^r-velocities in M has an almost complex structure $J^{(0)}$ with its Nijenhuis tensor $(N_J)^{(0)}$.*

THEOREM 4.4. *If a manifold M is a complex manifold with almost complex structure J, so is the bundle $\overset{r,p}{T}M$ of p^r-velocities in M with almost complex structure $J^{(0)}$.*

§ 5. Lifting of affine connections.

Let ∇ be the covariant differentiation defined by an affine connection of M.

THEOREM 5.1. *There exists one and only one affine connection of $\overset{r,p}{T}M$ whose covariant differentiation $\tilde{\nabla}$ satisfies the following condition:*

$$\tilde{\nabla}_{X^{<\lambda>}} Y^{<\mu>} = (\nabla_X Y)^{<\lambda+\mu>} \tag{5.1}$$

for every $X, Y \in \mathscr{T}^1_0(M)$ and $\lambda, \mu \in N(p,r)$.

Proof. Take a coordinate neighborhood U with coordinate system $\{x_1, \cdots, x_n\}$ and let Γ^k_{ij} be the connection components of ∇ with respect to $\{x_1, \cdots, x_n\}$, i.e.

$$\nabla_{\frac{\partial}{\partial x_i}} \frac{\partial}{\partial x_j} = \sum_k \Gamma^k_{ij} \frac{\partial}{\partial x_k} \tag{5.2}$$

for $i, j = 1, \cdots, n$. Let Γ'^k_{ij} be the connection components of ∇ with respect to another coordinate system $\{y_1, \cdots, y_n\}$ on U. Then, we have the following equalities:

$$\Gamma'^k_{ij} = \sum_{a,b,c} \frac{\partial x_b}{\partial y_i} \frac{\partial x_c}{\partial x_j} \frac{\partial y_k}{\partial x_a} \Gamma^a_{bc} + \sum \frac{\partial^2 x_a}{\partial y_i \partial y_j} \frac{\partial y_k}{\partial x_a} \tag{5.3}$$

for $i, j, k = 1, 2, \cdots, n$. (cf. for instance [3] p. 27). Let $\{x_i^{(\nu)} \mid i = 1, \cdots, n;\ \nu \in N(p,r)\}$ (resp. $\{y_i^{(\nu)}\}$) be the induced coordinate system on $(\overset{r,p}{\pi})^{-1}(U)$. Define

$$\tilde{\Gamma}^{(k,\lambda)}_{(i,\nu)(j,\mu)} = (\Gamma^k_{ij})^{(\lambda-\nu-\mu)} \tag{5.4}$$

for $i, j, k = 1, 2, \cdots, n$; $\lambda, \mu, \nu \in N(p,r)$. We can now prove that there exists a connection $\tilde{\nabla}$ whose connection components with respect to $\{x_i^{(\nu)}\}$ are given by (5.4). For, we can verify (5.5) [6] for $\lambda, \mu, \nu \in N(p, r)$ in the same way as the proof of (5.5) [6], since we can use the equalities

$$\frac{\partial f^{(\lambda)}}{\partial x_i^{(\mu)}} = \left(\frac{\partial f}{\partial x_i}\right)^{(\lambda-\mu)}$$

for every $\lambda, \mu \in N(p, r)$ and $f \in C^\infty(U)$ (cf. Cor. 2.3).

Next, we shall verify the following

$$(5.5) \qquad \tilde{\nabla}_{X_i^{<\lambda>}} X_j^{<\mu>} = (\nabla_{X_i} X_j)^{<\lambda+\mu>}$$

for every $i, j = 1, \cdots n$ and $\lambda, \mu \in N(p, r)$, where we have put $X_i = \dfrac{\partial}{\partial x_i}$. Making use of Lemma 2.6 we calculate as follows:

$$\tilde{\nabla}_{X_i^{<\lambda>}} X_j^{<\mu>} = \tilde{\nabla}_{\frac{\partial}{\partial x_i^{<\lambda>}}}\left(\frac{\partial}{\partial x_j^{(\mu)}}\right) = \sum_{\nu, k} \tilde{\Gamma}^{(k,\nu)}_{(i,\lambda),(j,\mu)} \frac{\partial}{\partial x_k^{(\nu)}}$$

$$= \sum_{\nu, k} (\Gamma^k_{ij})^{(\nu-\lambda-\mu)} \frac{\partial}{\partial x_k^{(\nu)}} = \sum_{\nu, k} (\Gamma^k_{ij})^{(\nu\prime)}\left(\frac{\partial}{\partial x_k}\right)^{<\lambda+\mu+\nu>}$$

$$= \left(\sum_k \Gamma^k_{ij} \frac{\partial}{\partial x_k}\right)^{<\lambda+\mu>} = (\nabla_{X_i} X_j)^{<\lambda+\mu>}.$$

Now, we shall verify

$$(5.6) \qquad \tilde{\nabla}_{(f \cdot X_i)^{<\lambda>}} X_j^{<\mu>} = (\nabla_{f X_i} X_j)^{<\lambda+\mu>}$$

for $f \in C^\infty(U)$, $i, j = 1, \cdots, n$ and $\lambda, \mu \in N(p, r)$.

For, the left hand side of (5.6) is equal to

$$\tilde{\nabla}_{\Sigma f^{(\nu)} X_i^{<\lambda+\nu>}} X_j^{<\mu>} = \sum f^{(\nu)} \tilde{\nabla}_{X_i^{<\lambda+\nu>}} X_j^{<\mu>}$$

$$= \sum f^{(\nu)} (\nabla_{X_i} X_j)^{<\lambda+\nu+\mu>} = (f \cdot \nabla_{X_i} X_j)^{<\lambda+\mu>} = (\nabla_{f X_i} X_j)^{<\lambda+\mu>},$$

which proves (5.6). Thus (5.1) is proved for $Y = \dfrac{\partial}{\partial x_j}$ and for every $X \in \mathscr{T}^1_0(M)$.

Finally, we shall verify (5.1) for $Y = \sum f_i X_j \in \mathscr{T}^1_0(M)$ as follows:

$$\tilde{\nabla}_{X^{<\lambda>}} (\textstyle\sum f_i X_j)^{<\mu>} = \tilde{\nabla}_{X^{<\lambda>}} \displaystyle\sum_{j,\nu} f_j^{(\nu)} X_j^{<\nu+\mu>}$$

$$= \sum_{j,\nu} \{ f_j^{(\nu)} \tilde{\nabla}_{X^{<\lambda>}} X_j^{<\nu+\mu>} + X^{<\lambda>} f_j^{(\nu)} \cdot X_j^{<\nu+\mu>} \}$$

$$= \sum_{j,\nu} \{ f_j^{(\nu)} (\nabla_X X_j)^{<\nu+\lambda+\mu>} + (X f_j)^{(\nu-\lambda)} X_j^{<\nu+\mu>} \}$$

$$= \sum_j \{ (f_j \cdot \nabla_X X_j)^{<\lambda+\mu>} + (X f_j \cdot X_j)^{<\lambda+\mu>} \}$$

$$= (\nabla_X (\textstyle\sum f_j X_j))^{<\lambda+\mu>}$$

The uniqueness of $\tilde{\nabla}$ is clear, since (5.1) holds for every $X, Y \in \mathcal{T}^1_0(M)$ and $\lambda, \mu \in N(p, r)$. Q.E.D.

DEFINITION 5.2. We denote $\tilde{\nabla}$ in Theorem 5.1 by $\tilde{\nabla} = \overset{r,p}{\nabla}$ and call it *the complete lift of* ∇ *to* $\overset{r,p}{T}M$.

PROPOSITION 5.3. *Let* $\tilde{T}$, $\tilde{R}$ *be the torsion and the curvature tensor field of* $\nabla = \overset{r,p}{\nabla}$. *Then we have*

$$\tilde{T} = T^{(0)} \text{ and } R = \tilde{R}, \tag{5.7}$$

where $T^{(0)}$ *and* $R^{(0)}$ *are the complete lift of* T *and* R (*cf. Def.* 3.6).

Proof. Using the relation (3.6), we calculate as follows:

$$\begin{aligned} T^{(0)}(X^{<\lambda>}, Y^{<\mu>}) &= (T(X, Y))^{<\lambda+\mu>} \\ &= (\nabla_X Y - \nabla_Y X - [X, Y])^{<\lambda+\mu>} \\ &= \tilde{\nabla}_{X^{<\lambda>}} Y^{<\mu>} - \tilde{\nabla}_{Y^{<\mu>}} X^{<\lambda>} - [X^{<\lambda>}, Y^{<\mu>}] = \tilde{T}(X^{<\lambda>}, Y^{<\mu>}) \end{aligned}$$

for every $X, Y \in \mathcal{T}^1_0(M)$ and $\lambda, \mu \in N(p, r)$, which proves $T^{(0)} = \tilde{T}$.

Similarly, we have:

$$\begin{aligned} & R^{(0)}(X^{<\lambda>}, Y^{<\mu>})Z^{(\nu)} = (R(X, Y)Z)^{<\lambda+\mu+\nu>} \\ &= (\nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X, Y]} Z)^{<\lambda+\mu+\nu>} \\ &= \tilde{\nabla}_{X^{<\lambda>}} \tilde{\nabla}_{Y^{<\mu>}} Z^{<\nu>} - \tilde{\nabla}_{Y^{<\mu>}} \tilde{\nabla}_{X^{<\lambda>}} Z^{<\nu>} - \tilde{\nabla}_{[X^{<\lambda>}, Y^{<\mu>}]} Z^{<\nu>} \\ &= \tilde{R}(X^{<\lambda>}, Y^{<\mu>}) Z^{<\nu>} \end{aligned}$$

for every X, Y, $Z \in \mathcal{T}^1_0(M)$ and λ, μ, $\nu \in N(p, r)$, which proves $R^{(0)} = \tilde{R}$.

Q.E.D.

PROPOSITION 5.4. *For any* $K \in \mathcal{T}^s_q(M)$ ($s = 0$ *or* 1) *and* $X \in \mathcal{T}^1_0(M)$, *we have*

$$\tilde{\nabla}_{X^{<0>}} K^{(\mu)} = (\nabla_X K)^{(\mu)}, \tag{5.8}$$

$$\tilde{\nabla} K^{(\mu)} = (\nabla K)^{(\mu)} \tag{5.9}$$

for every $\mu \in N(p, r)$.

Proof. It is sufficient to prove (5.8) for $K = Y \otimes T$, where $Y \in \mathcal{T}^1_0(M)$, $T \in \mathcal{T}^0_q(M)$. Now, since $K^{(\mu)} = \sum Y^{<\nu+\mu>} \otimes T^{(\nu)}$, and since $\tilde{\nabla}_{X^{<0>}}$ is a derivation of $\overset{r,p}{\mathcal{T}}(TM)$, it suffices to verify (5.8) in the special cases, where

$K=f\in\mathcal{T}^0_0(M)$ and $K=Y\in\mathcal{T}^1_0(M)$ and $K=\theta\in\mathcal{T}^0_1(M)$.

If $K=f$, then we have

$$\tilde{\nabla}_{X^{<0>}}f^{(\mu)}=X^{<0>}f^{(\mu)}=(Xf)^{(\mu)}=(\nabla_X f)^{(\mu)}.$$

If $K=Y$, then we have

$$\tilde{\nabla}_{X^{<0>}}Y^{<\mu>}=(\nabla_X Y)^{<\mu>}=(\nabla_X Y)^{(\mu)}.$$

If $K=\theta$, then we have, for $\mu,\nu\in N(p,r)$ and $Y\in\mathcal{T}^1_0(M)$

$$\begin{aligned}(\tilde{\nabla}_{X^{<0>}}\theta^{(\mu)})Y^{<\nu>}&=\tilde{\nabla}_{X^{<0>}}(\theta^{(\mu)}Y^{<\nu>})-\theta^{(\mu)}(\tilde{\nabla}_{X^{<0>}}Y^{<\nu>})\\&=\tilde{\nabla}_{X^{<0>}}(\theta(Y))^{(\mu-\nu)}-\theta^{(\mu)}((\nabla_X Y))^{<\nu>})\\&=(\nabla_X\theta(Y))^{(\mu-\nu)}-(\theta(\nabla_X Y))^{(\mu-\nu)}\\&=((\nabla_X\theta Y)^{(\mu-\nu)}=(\nabla_X\theta)^{(\mu)}(Y^{<\nu>}),\end{aligned}$$

and hence we get $\tilde{\nabla}_{X^{<0>}}\theta^{(\mu)}=(\nabla_X\theta)^{(\mu)}$.

To prove (5.9), using Corollary 3.8, we calculate as follows

$$\alpha_{X^{<0>}}\tilde{\nabla}K^{(\mu)}=\tilde{\nabla}_{X^{<0>}}K^{(\mu)}=(\nabla_X K)^{(\mu)}=(\alpha_X(\nabla K))^{(\mu)}=\alpha_{X^{<0>}}(\nabla K)^{(\mu)}.$$

Since $(X^{<0>})_{[\varphi]_r}(X\in\mathcal{T}^1_0(M))$ spans the tangent space to $\overset{r,p}{T}M$ at $[\varphi]_r\in\overset{r,p}{T}M$, we conclude that (5.9) holds. Q.E.D.

Combining Proposition 5.3 and 5.4 we have proved the following

THEOREM 5.5. *Let T and R be the torsion and the curavture tensor field of an affine connection ∇ of M. According as $T=0$, $T=0$, $R=0$ or $\nabla R=0$, we have $T^{(0)}=0$, $\overset{r,p}{\nabla}T^{(0)}=0$, $R^{(0)}=0$ or $\overset{r,p}{\nabla}R^{(0)}=0$. In particular, if M is affine locally symmetric with respect to ∇, so is $\overset{r,p}{T}M$ with respect to $\overset{r,p}{\nabla}$.*

§6. Affine symmetric spaces.

Let $\Phi: M\to N$ be a map of a manifold M into another manifold N. Then, the map Φ induces a map $\overset{r,p}{T}\Phi$ of $\overset{r,p}{T}M$ into $\overset{r,p}{T}N$ as follows:

$$(\overset{r,p}{T}\Phi)([\varphi]_r)=[\Phi\circ\varphi]_r \tag{6.1}$$

for $[\varphi]_r\in\overset{r,p}{T}M$. The map $\overset{r,p}{T}\Phi$ is a well-defined differentiable map, which will be called the (r,p)-tangent to Φ. It is clear that if Φ is a diffeomorphism then $\overset{r,p}{T}\Phi$ is also a diffeomorphism.

LEMMA 6.1. *For any* $f\in C^\infty(N)$, *we have*

$$f^{(\mu)}\circ \overset{r,p}{T}\Phi = (f\circ\Phi)^{(\mu)} \tag{6.2}$$

for every $\mu\in N(p,r)$.

Proof. Take a point $[\varphi]_r\in \overset{r,p}{T}M$. Then we have

$$(f^{(\mu)}\circ \overset{r,p}{T}\Phi)([\varphi]_r) = f^{(\mu)}([\Phi\circ\varphi]_r) = \frac{1}{\mu!}\left[\left(\frac{\partial}{\partial t}\right)^{\mu}(f\circ|(\Phi\circ\varphi))\right]_{t=0}$$
$$= \frac{1}{\mu!}\left[\left(\frac{\partial}{\partial t}\right)^{\mu}((f\circ\Phi)\circ\varphi)\right]_{t=0} = (f\circ\Phi)^{(\mu)}([\varphi]_r). \qquad \text{Q.E.D.}$$

LEMMA 6.2. *Let* $\Phi: M\to N$ *be a diffeomorphism of* M *onto* N. *Then for any* $X\in\mathscr{T}^1_0(M)$ *we have*

$$T\overset{r,p}{T}\Phi(X^{<\lambda>}) = (T\Phi X)^{<\lambda>} \tag{6.3}$$

for every $\lambda\in N(p,r)$.

Proof. Take a function $f\in C^\infty(N)$. Then, by making use of Lemma 6.1 and 2.1, we have, for any $\mu\in N(p,r)$:

$$T\overset{r,p}{T}\Phi(X^{<\lambda>})f^{(\mu)} = X^{<\lambda>}(f^{(\mu)}\circ\overset{r,p}{T}\Phi) = X^{<\lambda>}(f\circ\Phi)^{(\mu)}$$
$$= (X(f\circ\Phi))^{(\mu-\lambda)} = ((T\Phi X)f)^{(\mu-\lambda)} = (T\Phi X)^{<\lambda>}f^{(\mu)}.$$

Since $f\in C^\infty(N)$ and $\mu\in N(p,r)$ are arbitrary, we get (6.3). Q.E.D.

LEMMA 6.3. *Let* ∇ (*resp.* ∇') *be an affine connection on* M (*resp.* N) *and let* $\Phi: M\to N$ *be a diffeomorphism transforming* ∇ *onto* ∇', *i.e. we have*

$$T\Phi(\nabla_X Y) = \nabla'_{T\Phi X}T\Phi Y$$

for $X, Y\in\mathscr{T}^1_0(M)$. *Then the map* $\overset{r,p}{T}\Phi$ *transforms* $\overset{r,p}{\nabla}$ *onto* $\overset{r,p}{\nabla}{}'$.

Proof. Put $\tilde\Phi = T\overset{r,p}{T}\Phi$. It suffices to verify

$$\tilde\Phi\nabla_{X^{<\lambda>}}Y^{<\mu>} = \nabla_{\tilde\Phi X^{<\lambda>}}\tilde\Phi Y^{<\mu>} \tag{6.4}$$

for every $X, Y\in\mathscr{T}^1_0(M)$ and $\lambda,\mu\in N(p,r)$. Now, by making use of Theorem 5.1 and Lemma 6.2, we see the left hand side of (6.7) is equal to

$$T\overset{r,p}{T}\Phi(\nabla_X Y)^{<\lambda+\mu>} = (T\Phi(\nabla_X Y))^{<\lambda+\mu>}$$

$$= (\nabla'_{T\Phi X} T\Phi Y)^{<\lambda+\mu>} = \overset{r,p}{\nabla}'_{(T\Phi X^{<\lambda>}}(T\Phi Y)^{<\mu>} = \overset{r,p}{\nabla}'_{\tilde{\Phi}X^{<\lambda>}}\tilde{\Phi}Y^{<\mu>}.$$

Q.E.D.

LEMMA 6.4. *Take a point $x_0 \in M$ and let Φ be a diffeomorphism of M onto itself such that $\Phi(x_0) = x_0$ and that $T_{x_0}\Phi = -1_{T_{x_0}M}$. Consider the constant map γ_{x_0} of R^p into M defined by $\gamma_{x_0}(u) = x_0$ for $u \in R^p$. Put $\tilde{x}_0 = [\gamma_{x_0}]_r$. Then, we have $(\overset{r,p}{T}\Phi)(\tilde{x}_0) = \tilde{x}_0$ and that*

$$T_{\tilde{x}} \overset{}{T}\Phi = -1_{T_{\tilde{x}_0}(\overset{r,p}{T}M)}. \tag{6·5}$$

Proof. Take an element $[\varphi]_1 \in T_{\tilde{x}_0}(\overset{r,d}{T}M)$, where $\varphi : R \to \overset{r,p}{T}M$ with $\varphi(0) = \tilde{x}_0$. Making use of the same arguments as in the proof of Lemma 1.1 [5], we can find a differentiable map $\psi : R^{p+1} \to M$ such that $\varphi(t) = [\psi_t]_r$ for small t, where we have put $\psi_t(u) = \psi(t,u)$ for $t \in R$ and $u \in R^p$. Put $\psi^u(t) = \psi(t,u)$. Then, since $\varphi(0) = [\psi_0]_r = \tilde{x}_0 = [\gamma_{x_0}]$, we can assume that $\psi(0,u) = x_0$ for small $u \in R^p$ (cf. the expression of $(\tilde{\psi})$ in the proof of Lemma 1.1 [5]). Take a coordinate neighborhood U of x_0 with coordinate system $\{x_1, \cdots, x_n\}$. Put $x_{i,\nu} = x_i^{(\nu)}$ for $i = 1, \cdots, n$ and $\nu \in N(p,r)$. Then $\{x_{i,\nu}\}$ is a coordinate system around $\tilde{x}_0$. We have to prove $T\overset{r,p}{T}\Phi([\varphi]_1) = -[\varphi]_1$, i.e. to prove $[\overset{r,p}{T}\Phi \circ \varphi]_1 = -[\varphi]_1$. To prove this, it suffices to prove the following

$$(x_{i,\nu})^{(1)}([\overset{r,p}{T}\Phi \circ \varphi]_1) = -(x_{i,\nu})^{(1)}([\varphi]_1) \tag{6.6}$$

for $i = 1, 2, \cdots, n$ and $\nu \in N(p,r)$.

Since $(\overset{r,p}{T}\Phi \circ \varphi)(t) = \overset{r,p}{T}\Phi(\varphi(t)) = \overset{r,p}{T}\Phi([\psi_t]_r) = [\Phi \circ \psi_t]_r$, we calculate as follows:

$$(x_{i,\nu})^{(1)}([\overset{r,p}{T}\Phi \circ \varphi]_1) = \left[\frac{\partial}{\partial t}(x_{i,\nu} \circ \overset{r,p}{T}\Phi \circ \varphi)\right]_{t=0} = \left[\frac{\partial}{\partial t}(x_{i,\nu}([\Phi \circ \psi_t]_r))\right]_{t=0}$$

$$= \frac{1}{\nu!}\left[\frac{\partial}{\partial t}\left(\left[\left(\frac{\partial}{\partial u}\right)^{\nu}(x_i \circ \Phi \circ \psi_t)\right]_{t=0}\right.\right.$$

$$= \frac{1}{\nu!}\left[\frac{\partial}{\partial t}\left(\left[\left(\frac{\partial}{\partial u}\right)^{\nu} x_i(\Phi(\psi(t,u)))\right]_{u=0}\right)\right]_{t=0}$$

$$= \frac{1}{\nu!}\left[\left(\frac{\partial}{\partial u}\right)^{\nu}\left(\left[\frac{\partial}{\partial t} x_i(\Phi(\psi^u(t)))\right]_{t=0}\right)\right]_{u=0}$$

Now, making use of our assumption $T_{x_0}\Phi = -1_{T_{x_0}M}$ and the fact that $\psi^u(0) =$

$\psi(0,u)=x_0$ for small $u\in R^p$, we have

$$\left[\frac{\partial}{\partial t}x_i(\Phi(\psi^u(t)))\right]_{t=0}=x_i^{(1)}([\Phi\circ\psi^u]_1)=x_i^{(1)}(T\Phi[\psi^u]_1)$$
$$=-x_i^{(1)}([\psi^u]_1)=-\left[\frac{\partial}{\partial t}x_i(\psi^u(t))\right]_{t=0}.$$

Therefore, we can continue the above calculation as follows:

$$(x_{i,\nu})^{(1)}([\overset{r,p}{T}\Phi\circ\varphi]_1)=-\frac{1}{\nu!}\left[\left(\frac{\partial}{\partial u}\right)^\nu\left(\left[\frac{\partial}{\partial t}x_i\psi^u(t)\right]_{t=0}\right)\right]_{u=0}$$
$$=-\frac{1}{\nu!}\left[\frac{\partial}{\partial t}\left(\left[\left(\frac{\partial}{\partial u}\right)^\nu(x_i\circ\psi_t)\right]_{u=0}=-(x_{i,\nu})^{(1)}([\varphi]_1),\right.\right.$$

which proves (6.6). Q.E.D.

COROLLARY 6.5. *Let M be an affine symmetric space with affine connection ∇. Let $\Phi: M\to M$ be the affine symmetry at a point $x_0\in M$. Then the (r,p)-tangent $\overset{r,p}{T}\Phi$ to Φ is also the affine symmetry of $\overset{r,p}{T}M$ with affiine connection $\overset{r,p}{\nabla}$ at the point $\tilde{x}_0$.*

Proof. Since Φ leaves ∇ invariant, $\overset{r,p}{T}\Phi$ also leaves $\overset{r,p}{\nabla}$ invariant by Lemma 6.3. Next, since Φ is an affine symmetry we see that $T_{x_0}\Phi=-1_{T_{x_0}M}$. Thus, by Lemma 6.4, we get (6.5), which means that $\overset{r,p}{T}\Phi$ is the affine symmetry at $\tilde{x}_0$. Q.E.D.

LEMMA 6.6. *Let ∇ be an affine connection on a manifold M, and let $X\in\mathscr{T}^1_0(M)$ be an infinitesimal affine transofrmation of ∇. Then, the $\langle\lambda\rangle$-lift $X^{<\lambda>}$ of X is also an infinitesimal affine transformation of $\tilde{\nabla}=\overset{r,p}{\nabla}$ on $\overset{r,p}{T}M$ for every $\lambda\in N(p,r)$.*

Proof. A necessary and sufficient condition for X to be an infinitesimal affine transformation of M is that

$$\mathscr{L}_X\circ\nabla_X-\nabla_Y\circ\mathscr{L}_X=\nabla_{[X,Y]}$$

for every $Y\in\mathscr{T}^1_0(M)$, where $\mathscr{L}_X$ denotes the Lie derivation with respect to X. Therefore, we have to prove the following

(6.7) $$\mathscr{L}_{X^{<\lambda>}}\circ\tilde{\nabla}_{\tilde{Y}}K-\nabla_{\tilde{Y}}\circ\mathscr{L}_{X^{<\lambda>}}K=\tilde{\nabla}_{[X^{<\lambda>},\tilde{Y}]}K$$

for every $K\in\mathscr{T}(\overset{r,p}{T}M)$ and $\tilde{Y}\in\mathscr{T}^1_0(\overset{r,p}{T}M)$. To prove (6.7) it suffices to prove (6.7) for the special cases, where $\tilde{Y}=Y^{<\mu>}$ with $Y\in\mathscr{T}^1_0(M)$, $\mu\in N(p,r)$ and $K=Z^{<\nu>}$ or $\theta^{(\nu)}$ with $Z\in\mathscr{T}^1_0(M)$, $\theta\in\mathscr{T}^0_1(M)$ and $\nu\in N(p,r)$. Moreover, to prove (6.7) for the case $K=\theta^{(\nu)}$ with $\theta\in\mathscr{T}^0_1(M)$, it suffices to prove (6.7) for $\theta = df$ with $f \in \mathcal{T}^0_0(M)$.

If $K = Z^{<\nu>}$, then we calculate as follows:

$$\begin{aligned}
&\mathcal{L}_{X^{<\lambda>}}\tilde{\nabla}_{Y^{<\mu>}}Z^{<\nu>} - \tilde{\nabla}_{Y^{<\mu>}}\mathcal{L}_{X^{<\lambda>}}Z^{<\nu>} \\
&= [X^{<\lambda>}, (\nabla_Y Z)^{<\mu+\nu>}] - \tilde{\nabla}_{Y^{<\mu>}}[X^{<\lambda>}, Z^{<\nu>}] \\
&= [X, \nabla_Y Z]^{<\lambda+\mu+\nu>} - (\nabla_Y[X, Z])^{<\lambda+\mu+\nu>} \\
&= ([X, \nabla_Y Z] - \nabla_Y[X, Z])^{<\lambda+\mu+\nu>} = (\mathcal{L}_X \circ \nabla_Y)Z - (\nabla_Y \circ \mathcal{L}_X)Z)^{<\lambda+\mu+\nu>} \\
&= (\nabla_{[X,Y]}Z)^{<\lambda+\mu+\nu>} = \nabla_{[X^{<\lambda>}, Y^{<\mu>}]}Z^{<\nu>},
\end{aligned}$$

which proves (6.7) for $K = Z^{<\nu>}$.

To prove (6.7) for the case $K = df^{(\nu)}$ with $f \in \mathcal{T}^0_0(M)$, we first note that the following equalities hold:

$$(\mathcal{L}_X\theta)(Y) = X(\theta(Y)) - \theta([X, Y]) \tag{6.8}$$

$$(\nabla_X(df))(Y) = XYf - (\nabla_X Y)f \tag{6.9}$$

for $X, Y \in \mathcal{T}^1_0(M)$, $f \in \mathcal{T}^0_0(M)$ and $\theta \in \mathcal{T}^0_1(M)$.

Take a vector field $Z \in \mathcal{T}^1_0(M)$ and $\rho \in N(p, r)$. Making use of (6.8), (6.9), Lemma 2.5 and (5.1), we calculate as follows:

$$\begin{aligned}
&\{\mathcal{L}_{X^{<\lambda>}}(\tilde{\nabla}_{Y^{<\mu>}}(df^{(\nu)})) - \tilde{\nabla}_{Y^{<\mu>}}\mathcal{L}_{X^{<\lambda>}}(df^{(\nu)}))\}(Z^{<\rho>}) \\
&= X^{<\lambda>}((\tilde{\nabla}_{Y^{<\mu>}}(df^{(\nu)}))(Z^{<\rho>})) - \tilde{\nabla}_{Y^{<\mu>}}df^{(\nu)})([X^{<\lambda>}, Z^{<\rho>}]) \\
&\quad - (\tilde{\nabla}_{Y^{<\mu>}}d(X^{<\lambda>}f^{(\nu)}))(Z^{<\rho>}) \\
&= X^{<\lambda>}(Y^{<\mu>}Z^{<\rho>}f^{(\nu)} - (\tilde{\nabla}_{Y^{<\mu>}}Z^{<\rho>})f^{(\nu)}) \\
&\quad - \{Y^{<\mu>}[X^{<\lambda>}, Z^{<\rho>}]f^{(\nu)} - (\tilde{\nabla}_{Y^{<\mu>}}[X^{<\lambda>}, Z^{<\rho>}])f^{(\nu)}\} \\
&\quad - \{Y^{<\mu>}Z^{<\rho>}X^{<\lambda>}f^{(\nu)} - (\tilde{\nabla}_{Y^{<\mu>}}Z^{<\rho>})X^{<\lambda>}f^{(\nu)}\} \\
&= [X\{YZf - (\nabla_Y Z)f\} - \{Y[X, Z]f - (\nabla_Y[X, Z]f - (\nabla_Y[X, Z])f\} \\
&\quad - \{YZXf - (\nabla_Y Z)Xf\}]^{(\nu-\mu-\rho-\lambda)} \\
&= [\{\mathcal{L}_X(\nabla_Y df) - \nabla_Y\mathcal{L}_X df)\}(Z)]^{(\nu-\mu-\rho-\lambda)} \\
&= ((\nabla_{[X,Y]}(df)(Z))^{(\nu-\mu-\rho-\lambda)} = ([X, Y]Zf - (\nabla_{[Y,Y]}Z)f)^{(\nu-\mu-\rho-\lambda)} \\
&= [X^{<\lambda>}, Y^{<\mu>}]Z^{<\rho>}f^{(\nu)} - (\nabla_{[X^{<\lambda>}, Y^{<\mu>}]}Z^{<\rho>})f^{(\nu)} \\
&= (\nabla_{[X^{<\lambda>}, Y^{<\mu>}]}df^{(\nu)})(Z^{<\rho>}),
\end{aligned}$$

which proves (6.7) for $K = df^{(\nu)}$, since $Z \in \mathcal{T}^1_0(M)$ and $\rho \in N(p, r)$ are arbitrary. Thus (6.7) holds for any K and $\tilde{Y}$. Q.E.D.

From Lemma 6.6 we obtain

PROPOSITION 6.7. *If the group of affine transformations of M with ∇ is transitive on M, then the group of affine transformations of $\overset{r,p}{T}M$ with respect to $\overset{r,p}{\nabla}$ is transitive on $\overset{r,p}{T}M$.*

From Proposition 6.7 and Corollary 6.5 we obtain the following

THEOREM 6.8. *If M is an affine symmetric space with connection ∇, then $\overset{r,p}{T}M$ is also an affine symmetric space with connection $\overset{r,p}{\nabla}$.*

§ 7. Remarks.

Let $P(M, \pi, G)$ be a principal fibre bundle with base M, projection π and structure group G. We shall be able to prove that $\overset{r,p}{T}P$ ($\overset{r,p}{T}M$, $\overset{r,p}{T}\pi$, $\overset{r,p}{T}G$) becomes canonically a principal fibre bundle with structure group $\overset{r,p}{T}G$, which is a Lie group by the natural group multiplication. Let ω be a connection form on P. Then by the same methods as in [5], we can construct the prolongation $\omega^{(r,p)}$ of ω to $\overset{r,p}{T}P$. If $P = F(M)$ is the frame bundle of M then a linear connection on M will induce a linear connection on $\overset{r,p}{T}M$ by the above procedure. We shall investigate the relationships between this procedure and the liftings of affine connections in § 5 in a forthcoming paper, where we shall also study the prolongations of G-structures to the tangent bundles of p^r-velocities, which will generalize the results in [4].

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