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ON A METHOD OF DESCRIBING FORMAL DEDUCTIONS CONVENIENT FOR THEORETICAL PURPOSES

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Introduction

In my paper [2], I have proposed a method of describing formal deductions which seems to be convenient for practical purposes. In my paper [3], I have employed an index-system to exactly express tree-form configurations of proofs in Gentzen's formalism for sequents. In the present paper, I would like to propose a method of describing formal deductions which seems to be convenient for theoretical purposes. The device employed for this purpose relies mostly on an index-system. Just as in [2] as well as in [3], I propose here also to denote every proposition and every denomination of a variable, or every sequent in Gentzen's formalism, by an index-word.

Although our device, to be illustrated in the present paper, can be applied to a large variety of various formalisms, I will illustrate it in the present paper by taking up two examples. The first example is Gentzen's ***LK*** (notation in the present paper : ***GLK***). It can be applied also to Gentzen's ***LJ*** without any essential modification. The second example is the lower classical predicate logic ***LK*** in the form I have dealt with in my former papers. Our device can be naturally extended to other logics such as the intuitionistic logic, Johansson's minimal logic, the positive logic, etc.

Our index-system introduced in the present paper has the strong point for theoretical purposes, that not only the tree-form configuration of each proof is clearly denoted by the index-system but also the inference rules employed for the deduction of steps are expressed exactly and further their reference steps too can be founded out by index-words only. However, this index-system has the weak point that, for practical purposes, proof-note

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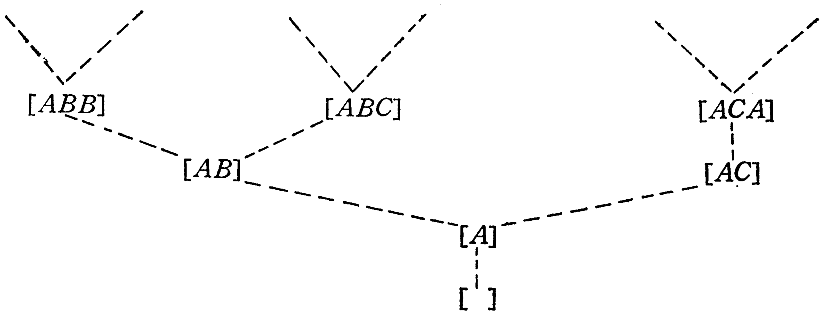
as well as index-words turn out much longer when described by our new method compared with proof-notes or index-words described by the method proposed in [2].

In Section (**1**), I will illustrate our new index-word system with respect to ***GLK*** (Gentzen's ***LK***), and I will illustrate it with respect to the lower classical predicate logic ***LK*** in Section (**2**). Proofs of the logics such as the intuitionistic logic ***LJ***, the minimal logics ***LM*** and ***LN***, the positive logics ***LP*** and ***LQ***, and also the primitive logic ***LO*** in the form I have dealt with in my former papers, can be described along this line without any essential modification. The method of description introduced in Section (**2**) is referred to by ***TD*** (theoretical description) and the method of description introduced in my former paper [2] is referred to by ***PD*** (practical description). When the logic ***LK*** is described in ***TD*** or ***PD***, it is referred to by ***TLK*** or ***PLK***, respectively. More generally, any logic ***LX*** can be referred to by ***TLX*** or ***PLX***, when it is described in ***TD*** or in ***PD***, respectively.

In Section (**3**), I would like to expose the mutual relation between ***TLK*** and ***GLK*** which is formulated in Section (**1**), and also the mutual relation between ***TLK*** and ***PLK***.

(1) Index-system for *GLK* (Gentzen's *LK*)

To denote tree-form configuration of proofs, I have employed in my paper [3] an index-system, in which any index-word is a sequence of letters *A*, *B*, and *C*. Configurations of proofs have been figured out something like



This index-system is enough to show the tree-form configurations of proofs themselves, but it can not show completely, for each step, which inference rule is employed to deduce the step from a step or steps standing above it. In the present section, I will introduce an index-system which enables us to show, for each step of a proof, from which step or steps and by which inference rule the step is deduced.

For this purpose, I will take up the following list of letters

$F,\ V,\ V^*,\ R,\ R^*,\ Q,\ Q^*,\ I',\ I'',\ I^*,\ C',\ C'',\ C^*,\ C^{**},\ D',$
$D'',\ D^*,\ D^{**},\ N,\ N^*,\ U,\ U^*,\ E,\ E^*,\ S',\ S'',$

instead of the list of three letters A, B, C. The letter F does not occur in any index-word except at its tail. It should be occasionally disregarded but occasionally should not be disregarded. When the letter F should be disregarded in an index-word, the index-word is enclosed in "{ }". Namely, both index-word $\underline{s}$ and $\underline{s}F$ is denoted by $\{\underline{s}\}$ as well as by $\{\underline{s}F\}$. Throughout this section, notations of the form $\underline{s}$ (underlined lower case letters) stand for sequences of letters in our list. In any bare index-word having no F explicitly, it is tacitly assumed that no F occurs in the index-word.

For each inference rule, I will give in the following, its notation, its index-word form, and its sequent form. Any sequent indicated by an index-word of the form $\underline{s}F$ should be a fundamental sequent.

Inference rules of *GLK*:

Notation	Index-word form	Sequent form
(F)	$\underline{s}F$	$\mathfrak{A} \vdash \mathfrak{A}$
(V)	$\dfrac{\{\underline{s}V\}}{\underline{s}}$	$\dfrac{\Gamma \vdash \Delta}{\Gamma, \mathfrak{A} \vdash \Delta}$
(V^*)	$\dfrac{\{\underline{s}V^*\}}{\underline{s}}$	$\dfrac{\Gamma \vdash \Delta}{\Gamma \vdash \Delta, \mathfrak{A}}$
(R)	$\dfrac{\{\underline{s}R\}}{\underline{s}}$	$\dfrac{\Gamma, \mathfrak{A}, \mathfrak{B}, \Theta \vdash \Delta}{\Gamma, \mathfrak{B}, \mathfrak{A}, \Theta \vdash \Delta}$
(R^*)	$\dfrac{\{\underline{s}R^*\}}{\underline{s}}$	$\dfrac{\Gamma \vdash \Delta, \mathfrak{A}, \mathfrak{B}, \Lambda}{\Gamma \vdash \Delta, \mathfrak{B}, \mathfrak{A}, \Lambda}$
(Q)	$\dfrac{\{\underline{s}Q\}}{\underline{s}}$	$\dfrac{\Gamma, \mathfrak{A}, \mathfrak{A} \vdash \Delta}{\Gamma \mathfrak{A} \vdash \Delta}$
(Q^*)	$\dfrac{\{\underline{s}Q^*\}}{\underline{s}}$	$\dfrac{\Gamma \vdash \Delta, \mathfrak{A}, \mathfrak{A}}{\Gamma \vdash \Delta, \mathfrak{A}}$
(I', I'')	$\dfrac{\{\underline{s}I'\}\ \{\underline{s}I''\}}{\underline{s}}$	$\dfrac{\Gamma \vdash \Delta, \mathfrak{A} \quad \Gamma, \mathfrak{B} \vdash \Delta}{\Gamma, \mathfrak{A} \to \mathfrak{B} \vdash \Delta}$
(I^*)	$\dfrac{\{\underline{s}I^*\}}{\underline{s}}$	$\dfrac{\Gamma, \mathfrak{A} \vdash \Delta, \mathfrak{B}}{\Gamma \vdash \Delta, \mathfrak{A} \to \mathfrak{B}}$

(C')	$\dfrac{\{\underline{s}C'\}}{\underline{s}}$	$\dfrac{\Gamma, \mathfrak{A} \vdash \Delta}{\Gamma, \mathfrak{A} \wedge \mathfrak{B} \vdash \Delta}$
(C'')	$\dfrac{\{\underline{s}C''\}}{\underline{s}}$	$\dfrac{\Gamma, \mathfrak{A} \vdash \Delta}{\Gamma, \mathfrak{B} \wedge \mathfrak{A} \vdash \Delta}$
(C^*, C^{**})	$\dfrac{\{\underline{s}C^*\} \ \{\underline{s}C^{**}\}}{\underline{s}}$	$\dfrac{\Gamma \vdash \Delta, \mathfrak{A} \quad \Gamma \vdash \Delta, \mathfrak{B}}{\Gamma \vdash \Delta, \mathfrak{A} \wedge \mathfrak{B}}$
(D', D'')	$\dfrac{\{\underline{s}D'\} \ \{\underline{s}D''\}}{\underline{s}}$	$\dfrac{\Gamma, \mathfrak{A} \vdash \Delta \quad \Gamma, \mathfrak{B} \vdash \Delta}{\Gamma, \mathfrak{A} \vee \mathfrak{B} \vdash \Delta}$
(D^*)	$\dfrac{\{\underline{s}D^*\}}{\underline{s}}$	$\dfrac{\Gamma \vdash \Delta, \mathfrak{A}}{\Gamma \vdash \Delta, \mathfrak{A} \vee \mathfrak{B}}$
(D^{**})	$\dfrac{\{\underline{s}D^{**}\}}{\underline{s}}$	$\dfrac{\Gamma \vdash \Delta, \mathfrak{A}}{\Gamma \vdash \Delta, \mathfrak{B} \vee \mathfrak{A}}$
(N)	$\dfrac{\{\underline{s}N\}}{\underline{s}}$	$\dfrac{\Gamma \vdash \Delta, \mathfrak{A}}{\Gamma, \sim\mathfrak{A} \vdash \Delta}$
(N^*)	$\dfrac{\{\underline{s}N^*\}}{\underline{s}}$	$\dfrac{\Gamma, \mathfrak{A} \vdash \Delta}{\Gamma \vdash \Delta, \sim\mathfrak{A}}$
(U)	$\dfrac{\{\underline{s}U\}}{\underline{s}}$	$\dfrac{\Gamma, \mathfrak{A}(t) \vdash \Delta}{\Gamma, (x)\mathfrak{A}(x) \vdash \Delta}$
(U^*)	$\dfrac{\{\underline{s}U^*\}}{\underline{s}}$	$\dfrac{\Gamma \vdash \Delta, \mathfrak{A}(t)}{\Gamma \vdash \Delta, (x)\mathfrak{A}(x)}$
(E)	$\dfrac{\{\underline{s}E\}}{\underline{s}}$	$\dfrac{\Gamma, \mathfrak{A}(t) \vdash \Delta}{\Gamma, (\exists x)\mathfrak{A}(x) \vdash \Delta}$
(E^*)	$\dfrac{\{\underline{s}E^*\}}{\underline{s}}$	$\dfrac{\Gamma \vdash \Delta, \mathfrak{A}(t)}{\Gamma \vdash \Delta, (\exists x)\mathfrak{A}(x)}$
(S', S'')	$\dfrac{\{\underline{s}S'\} \ \{\underline{s}S''\}}{\underline{s}}$	$\dfrac{\Gamma \vdash \Delta, \mathfrak{A} \quad \Gamma, \mathfrak{A} \vdash \Delta}{\Gamma \vdash \Delta}$

Any non-empty finite set $\mathfrak{T}$ of index-words is called a *proof-tree* if and only if it satisfies the following conditions:

$(TG\,1)$ *For any index-word of the form* $\underline{s}F$ *in* $\mathfrak{T}$, *there is no index-word beginning with* $\underline{s}$ *other than* $\underline{s}F$ *in* $\mathfrak{T}$.

$(TG\,2)$ *For any index-word of the form* $\underline{s}X$ *in* $\mathfrak{T}$, *where* X *differs from* F, *the index-word* $\underline{s}$ *belongs to* $\mathfrak{T}$.

$(TG\,3)$ *For any index-word* $\underline{s}$ *in* $\mathfrak{T}$ *having no* F *at its tail, there is at least one index-word of the form* $\underline{s}X$ *in* $\mathfrak{T}$.

$(TG\,4)$ *For any index-word of the form* $\{\underline{s}X\}$ *in* $\mathfrak{T}$, *there is no index-word beginning with* $\underline{s}$ *other than* $\underline{s}$ *and those beginning with* $\underline{s}X$ *in* $\mathfrak{T}$, *where* X *stands for any one of the letters* V, V^*, R, R^*, Q, Q^*, I^*, C', C'', D^*, D^{**}, N, N^*, U, U^*, E, E^*.

(TG 5) *For any index-word of the form* $\{\underline{s}X\}$ *in* $\mathfrak{T}$, *there is the index-word* $\{\underline{s}Y\}$ *and no index-word other than* $\underline{s}$ *and those beginning with* $\underline{s}X$ *or* $\underline{s}Y$ *in* $\mathfrak{T}$, *where the pair of letters* X *and* Y *stands for any one of the non-ordered pairs of letters* $\{I', I''\}$, $\{C^*, C^{**}\}$, $\{D', D''\}$, $\{S', S''\}$.

Any function Π which maps a proof-tree into the domain of sequents is called a *proof-note* if and only if Π satisfies the following conditions:

(ΠG 1) $\Pi(\underline{s}F)$ *is a fundamental sequent.*

(ΠG 2) $\Pi(\{\underline{s}X\})$ *and* $\Pi(\underline{s})$ *have the forms of the sequents indicated by* $\{\underline{s}X\}$ *and* $\underline{s}$ *in the inference rule* (X), *respectively, where* X *stands for any one of the letters* V, V^*, R, R^*, Q, Q^*, I^*, C', C'', D^*, D^{**}, N, N^*, U, U^*, E, E^*. *Furthermore, in the case where* X *stands for any one of the letters* U, U^*, E, E^*, *the variable* t *should never occur in the range of any quantifier of the bound variable* x *in* $\mathfrak{A}(t)$, *and in the cases where* X *stands for* U^* *or* E, *the variable* t *should never occur in* $\Pi(\underline{s})$.

(ΠG 3) $\Pi(\{\underline{s}X\})$, $\Pi(\{\underline{s}Y\})$, *and* $\Pi(\underline{s})$ *have forms of the sequents indicated by* $\{\underline{s}X\}$, $\{\underline{s}Y\}$, *and* $\underline{s}$ *in the inference rule of* (X, Y), *respectively, where the pair of* X *and* Y *stands for any one of the ordered pairs of letters* $\langle I', I''\rangle$, $\langle C^*, C^{**}\rangle$, $\langle D', D''\rangle$, $\langle S', S''\rangle$.

Any proof-note Π is regarded as a proof of $\Pi(\emptyset)$, where $\emptyset$ denotes the null-sequence index-word. Notice that $\emptyset$ belongs to any proof-tree according to the condition (TG 2). It can be seen without difficulty that this system is equivalent to Gentzen's ***LK***. Also, it would be easily seen that our device can be extended agreeably to describe Gentzen's ***LJ***.

(2) Index-system for the classical predicate logic *LK*.

The logic I am going to formulate here is the classical predicate logic ***LK*** of my former papers. The logic ***LK*** has the negation notion, but $\sim\mathfrak{F}$ can be agreeably replaced by $\mathfrak{F} \to \curlywedge$ by adopting the proposition constant $\curlywedge$. I do not give the inference rule for negation with respect to the logical constant "$\sim$", but I will give here the inference rules with respect to the proposition constant $\curlywedge$.

The index-system for ***LK*** is similar to the index-system I have given in the preceeding section. Namely, I take up the following list of letters

F, I', I'', i^*, C', C'', C^*, C^{**}, D, d', d'', D^*, D^{**}, U, U^*, E, e, E^*, $\forall$, $\curlywedge$, P

together with an auxiliary symbol "$-$". Any index-word contains at most one auxiliary symbol "$-$". Hence, starting from a sequence $\underline{s}$ of letters of the length n, we can make $n+1$ index-words by either inserting "$-$" just after k letters of the sequence ($k=0, \cdots, n-1$) or not inserting "$-$". The index-word obtained by inserting "$-$" just after k letters of $\underline{s}$ is denoted by $\underline{s}^k$. In the following, I would like to establish a convention that any symbol of the form $\underline{s}$ denotes an index-word containing no "$-$", any symbol of the form $\underline{s}^-$ denotes an index-word having "$-$" somewhere, and any symbol of the form $\underline{s}^m$ for $m \geqq n$ denotes $\underline{s}$. Hence, symbols of the form $\underline{s}^k$ in general can stand both for $\underline{s}^-$ and for $\underline{s}$.

The inference rules of ***LK*** read:

(F) $\ll\underline{s})\ \mathfrak{A}\gg$ *is deducible from* $\ll\underline{s}^jF)\ \mathfrak{A}\gg$ *for any* j.

(I', I'') $\ll\underline{s})\ \mathfrak{A}\gg$ *is deducible from* $\ll\underline{s}^jI')\ \mathfrak{B}\gg$ *and* $\ll\underline{s}^kI'')\ \mathfrak{B}\rightarrow\mathfrak{A}\gg$ *for any* j *and* k.

(i^*) $\ll\underline{s})\ \mathfrak{A}\rightarrow\mathfrak{B}\gg$ *is deducible from the fact that* $\ll\underline{s}i^*)\ \mathfrak{B}\gg$ *is deducible from steps of the form* $\ll\underline{s}-i^*\underline{g})\ \mathfrak{A}\gg$.

(C') $\ll\underline{s})\ \mathfrak{A}\gg$ *is deducible from* $\ll\underline{s}^jC')\ \mathfrak{A}\wedge\mathfrak{B}\gg$ *for any* j.

(C'') $\ll\underline{s})\ \mathfrak{A}\gg$ *is deducible from* $\ll\underline{s}^jC'')\ \mathfrak{B}\wedge\mathfrak{A}\gg$ *for any* j.

(C^*, C^{**}) $\ll\underline{s})\ \mathfrak{A}\wedge\mathfrak{B}\gg$ *is deducible from* $\ll\underline{s}^jC^*)\ \mathfrak{A}\gg$ *and* $\ll\underline{s}^kC^{**})\ \mathfrak{B}\gg$ *for any* j *and* k.

(D, d', d'') $\ll\underline{s})\ \mathfrak{A}\gg$ *is deducible from* $\ll\underline{s}^jD)\ \mathfrak{B}\vee\mathfrak{C}\gg$ *and the facts that* $\ll\underline{s}d')\ \mathfrak{A}\gg$ *is deducible from steps of the form* $\ll\underline{s}-d'\underline{g})\ \mathfrak{B}\gg$ *and that* $\ll\underline{s}d'')\ \mathfrak{A}\gg$ *is deducible from steps of the form* $\ll\underline{s}-d''\underline{h})\ \mathfrak{C}\gg$.

(D^*) $\ll\underline{s})\ \mathfrak{A}\vee\mathfrak{B}\gg$ *is deducible from* $\ll\underline{s}^jD^*)\ \mathfrak{A}\gg$ *for any* j.

(D^{**}) $\ll\underline{s})\ \mathfrak{A}\vee\mathfrak{B}\gg$ *is deducible from* $\ll\underline{s}^jD^{**})\ \mathfrak{B}\gg$ *for any* j.

(U) $\ll\underline{s})\ \mathfrak{A}(t)\gg$ *is deducible from* $\ll\underline{s}^jU)\ (x)\mathfrak{A}(x)\gg$ *for any* j.

$(U^*, \forall)$ $\ll\underline{s})\ (x)\mathfrak{A}(x)\gg$ *is deducible from the fact that* $\ll\underline{s}U^*)\ \mathfrak{A}(t)\gg$ *is deducible for* $\ll\underline{s}\forall)\ \forall t:\gg$

$(E, e, \forall)$ $\ll\underline{s})\ \mathfrak{A}\gg$ *is deducible from* $\ll\underline{s}^jE)\ (\exists x)\mathfrak{B}(x)\gg$ *and the fact that* $\ll\underline{s}e)\ \mathfrak{A}\gg$ *is deducible from steps of the form* $\ll\underline{s}-e\underline{g})\ \mathfrak{B}(t)\gg$ *for* $\ll\underline{s}\forall)\ \forall t:\gg$.

(E^*) $\ll\underline{s})\ (\exists x)\mathfrak{A}(x)\gg$ *is deducible from* $\ll\underline{s}^jE^*)\ \mathfrak{A}(t)\gg$ *for any* j.

$(\curlywedge)$ $\ll\underline{s})\ \mathfrak{A}\gg$ *is deducible from* $\ll\underline{s}^j\curlywedge)\ \curlywedge\gg$ *for any* j.

(P) $\ll\underline{s})\ \mathfrak{A}\gg$ *is deducible from* $\ll\underline{s}^jP)\ (\mathfrak{A}\rightarrow\mathfrak{B})\rightarrow\mathfrak{A}\gg$ *for any* j.

Now, I will define proof-trees in ***LK***. Namely, any non-empty finite set $\mathfrak{T}$ of index-words is called a *proof-tree* if and only if it satisfies the following conditions:

$(T\,1)$ *Any index-word in* $\mathfrak{T}$ *contains at most one* "$-$", *and any* "$-$" *in an* *index-word stands just before* i^*, d', d'', *or* e. *The letter* $\forall$ *can stand at the end of an index-word, if any. Any index-word ending with* $\forall$ *can not contain* "$-$". *For any index-word of the form* $\underline{s}-g$ *in* $\mathfrak{T}$, $\underline{s}-g$ *is the only index-word in* $\mathfrak{T}$ *which begins with an index-word of the form* $(\underline{s}g)^j$.

(T 2) *For any index-word* $\underline{s}$ *in* $\mathfrak{T}$, *either there is at least one index-word of the form* $\underline{s}^j X$, *or* $\underline{s}$ *ends with* $\forall$.

(T 3) *For any index-word* $\underline{s}^j X$ *in* $\mathfrak{T}$, $\underline{s}$ *belongs to* $\mathfrak{T}$.

(T 4) *For any index-word of the form* $\underline{s}^j X$ *in* $\mathfrak{T}$, *any index-word beginning with* $\underline{s}^k$ *for any* k *is either* $\underline{s}$ *or an index-word beginning with* $(\underline{s}X)^h$ *for some* h, *where* X *stands for any one of the letters* F, i^*, C', C'', D^*, D^{**}, U, E^*, $\curlywedge$, P.

(T 5) *For any index-word of the form* $\underline{s}^j X$ *in* $\mathfrak{T}$, *there is an index-word of the form* $\underline{s}^k Y$ *for some* k *in* $\mathfrak{T}$, *but there is no index-word beginning with an index-word of the form* $\underline{s}^h$ *in* $\mathfrak{T}$ *other than* $\underline{s}$ *and those index-words beginning with index-words of the forms* $(\underline{s}X)^p$ *or* $(\underline{s}Y)^q$ *for some* p *and* q, *where the pair of* X *and* Y *stands either for any one of the non-ordered pairs* $\{I', I''\}$, $\{C^*, C^{**}\}$ *or for the ordered pair* $\langle U^*, \forall \rangle$.

(T 6) *For any index-word of the form* $\underline{s}^j X$ *in* $\mathfrak{T}$, *there are index-words of the forms* $\underline{s}^h Y$ *and* $\underline{s}^k Z$ *for some* h *and* k *in* $\mathfrak{T}$, *but there can not be any index-word beginning with an index-word of the form* $\underline{s}^g$ *in* $\mathfrak{T}$ *other than* $\underline{s}$ *and those index-words beginning with index-words of the forms* $(\underline{s}X)^p$, $(\underline{s}Y)^q$, *or* $(\underline{s}Z)^r$ *for some* p, q, r, *where the triple of* X, Y, *and* Z, *stands either for the non-ordered triple* $\{D, d', d''\}$ *or any one of the ordered triples* $\langle E, e, \forall \rangle$ *and* $\langle e, E, \forall \rangle$.

(T 7) *For any index-word of the form* $\underline{s}\forall$ *in* $\mathfrak{T}$, *there is either the index-word* $\underline{s}U^*$ *or the index-word of the form* $\underline{s}e$ *in* $\mathfrak{T}$.

Any function Π which maps a proof-tree into the domain of propositions and denominations of variables is called a *proof-note* if and only if Π satisfies the following conditions:

(Π 1) $\Pi(\underline{s}\forall)$ *is a denomination of the form* $\langle\!\langle \forall t : \rangle\!\rangle$. $\Pi(\underline{s}^j)$ *is a proposition.*

(Π 2) $\Pi(\underline{s}^j X)$ *and* $\Pi(\underline{s})$ *have the forms of the propositions indicated by* $\underline{s}^j X$ *and* $\underline{s}$ *in the inference rule* (X), *respectively, where* X *stands for any one of the letters* F, C', C'', D^*, D^{**}, U, E^*, $\curlywedge$, P.

(Π 3) $\Pi(\underline{s}i^*)$ *and* $\Pi(\underline{s})$ *have the forms of the propositions indicated by the index-words* $\underline{s}i^*$ *and* $\underline{s}$ *in the inference rule* (i^*), *respectively, and all the values*

$\Pi(\underline{s}-i^*\underline{g})$ for various $\underline{g}$ are identical to the same proposition having the form of the proposition indicated by the index-word $\underline{s}-i^*\underline{g}$ in the same inference rule (i^*).

(Π 4) *$\Pi(\underline{s}U^*)$ and $\Pi(\underline{s})$ have the forms of the propositions indicated by the index-words $\underline{s}U^*$ and $\underline{s}$ in the inference rule $(U^*, \forall)$, respectively. The denominated variable t in $\ll\underline{s}\forall)\ \forall t:\gg$ should occur neither in the range of any quantifier of the bound variable x of the proposition $\Pi(\underline{s}U^*)$, i.e. $\mathfrak{A}(t)$, nor in any proposition of the form $\Pi(\underline{s}^{-}\underline{g})$.*

(Π 5) *$\Pi(\underline{s}^{k}X)$, $\Pi(\underline{s}^{j}Y)$, and $\Pi(\underline{s})$ have the forms of the propositions indicated by the index-words $\underline{s}^{j}X$, $\underline{s}^{k}Y$, and $\underline{s}$ in the inference rule (X, Y), respectively, where the pair of X and Y stands for either of the ordered pairs $<I', I''>$ or $<C^*, C^{**}>$.*

(Π 6) *$\Pi(\underline{s}^{j}E)$, $\Pi(\underline{s}e)$, and $\Pi(\underline{s})$ have the forms of propositions indicated by $\underline{s}^{j}E$, $\underline{s}e$, and $\underline{s}$ in the inference rule $(E, e, \forall)$, respectively, and all $\Pi(\underline{s}-e\underline{g})$ of Π for various $\underline{g}$ are identical to the same proposition. The denominated variable t of $\ll\underline{s}\forall)\ \forall t:\gg$ should occur neither in the range of any quantifier of the bound variable x of the proposition $\Pi(\underline{s}-e\underline{g})$, i.e. $\mathfrak{B}(t)$, nor in the proposition $\Pi(\underline{s}^{j}E)$, i.e. $(\exists x)\mathfrak{B}(x)$, nor in $\Pi(\underline{s}e)$, i.e. $\mathfrak{A}$, nor in any proposition of the form $\Pi(\underline{s}^{-}\underline{h})$.*

(Π 7) *$\Pi(\underline{s}^{j}D)$, $\Pi(\underline{s}d')$, $\Pi(\underline{s}d'')$, and $\Pi(\underline{s})$ have the forms of the propositions indicated by $\underline{s}^{j}D$, $\underline{s}d'$, $\underline{s}d''$, and $\underline{s}$ in the inference rule (D, d', d''), respectively, and all $\Pi(\underline{s}-d'\underline{g})$ for various $\underline{g}$ are identical to the same proposition. Likewise, all $\Pi(\underline{s}-d''\underline{h})$ for various $\underline{h}$ are identical to the same proposition. $\Pi(\underline{s}-d'\underline{g})$ and $\Pi(\underline{s}-d''\underline{h})$ are propositions of the forms of the propositions indicated by the index-words $\underline{s}-d'\underline{g}$ and $\underline{s}-d''\underline{h}$ in the inference rule (D, d', d''), respectively.*

Any proposition $\mathfrak{P}$ is called *provable by a proof-note* Π if and only if Π is a correct proof-note and $\Pi(\emptyset)$ is the proposition $\mathfrak{P}$.

Although I have illustrated our method of description with respect to ***LK*** only, it can be easily seen that our index-system device can be extended agreeably to descriptions of other logics such as the intuitionistic predicate logic ***LJ***, the minimal predicate logics ***LM*** and ***LN*** as well as the positive predicate logics ***LP*** and ***LQ***, the primitive logic ***LO***, etc.

In the logic ***LO***, the inference rules (F), (I', I''), (i^*), (U), and $(U^*, \forall)$ are admitted. In all the other logics ***LJ***, ***LK***, ***LM***, ***LN***, ***LP***, and ***LQ***, the inference rules (C'), (C''), (C^*, C^{**}), (D, d', d''), (D^*), (D^{**}), $(E, e, \forall)$, and (E^*) are further admitted. In the logics ***LJ***, ***LK***, ***LM***, and ***LN***, the proposition constant $\curlywedge$ is assumed. In the logics ***LJ*** and ***LK***, the inference rule $(\curlywedge)$ is admitted. In the logics ***LK***, ***LN***, and ***LQ***, the inference rule (P) is further admitted.

When any logic ***LX*** is described by making use of the index-system illustrated in the present paper, I will refer to it by ***TLX***.

(3) Mutual relations

The main purpose of the present paper is to expose the mutual relation between ***TLK*** and ***GLK*** as has been formulated in Section (**1**), which I will denote hereafter by ***TGLK***. I will also expose the mutual relation between ***TLK*** and ***PLK***.

To show the mutual relation between ***TLK*** and ***TGLK*** very clearly, it seems better to modulate either ***TGLK*** or ***TLK*** so that it matches better with the other. It was really a splendid idea of Gentzen to deal with any finite number of cases (Δ of any sequent of the form $\Gamma \vdash \Delta$) simultaneously. To match with ***TGLK***, we might modulate ***TLK*** so that we can deal with any finite number of cases simultaneously. Indeed, this must be an interesting task. In the present paper, however, I will modulate ***TGLK*** so as to match with ***TLK***. Namely, by ***TG*LK***, I will denote the system dealing with only sequents of the form $\Gamma \vdash \mathfrak{A}$ and having the inference rules of (F), (V), (R), (Q), (I^*), (C'), (C''), (C^*, C^{**}), (D', D''), (D^*), (D^{**}), (U), (U^*), (E), (E^*) with respect to sequents of the form $\Gamma \vdash \mathfrak{A}$, together with the following inference rules:

$$(I', I'') \qquad \frac{\{\underline{s}I'\}\ \{\underline{s}I''\}}{\underline{s}} \qquad \frac{\Gamma \vdash \mathfrak{A} \quad \Gamma, \mathfrak{B} \vdash \mathfrak{C}}{\Gamma, \mathfrak{A} \to \mathfrak{B} \vdash \mathfrak{C}}$$

$$(S', S'') \qquad \frac{\{\underline{s}S'\}\ \{\underline{s}S''\}}{\underline{s}} \qquad \frac{\Gamma \vdash \mathfrak{A} \quad \Gamma, \mathfrak{A} \vdash \mathfrak{B}}{\Gamma \vdash \mathfrak{B}}$$

and the two kinds of new fundamental sequents

$$(\curlywedge) \qquad \underline{s}\curlywedge \qquad \curlywedge \vdash \mathfrak{A},$$

$$(P) \qquad \underline{s}P \qquad (\mathfrak{A} \to \mathfrak{B}) \to \mathfrak{A} \vdash \mathfrak{A}.$$

The negation "$\sim\mathfrak{F}$" is defined by "$\mathfrak{F} \to \curlywedge$". Accordingly, we need the following list of letters

F, V, R, Q, I', I'', I^*, C', C'', C^*, C^{**}, D', D'', D^*, D^{**}, U, U^*, E, E^*, S', S'', $\curlywedge$, P.

If any index-word $\underline{s}$ is enclosed in "{ }", the letters F, $\curlywedge$, or P in $\underline{s}$ should be disregarded.

Any non-empty finite set of index-words is called a *proof-tree* in ***TG*LK***, if and only if it satisfies the following conditions:

(TG^* 1) *For any index-word of the forms, $\underline{s}F$, $\underline{s}\curlywedge$, or $\underline{s}P$ in $\mathfrak{T}$, there is no index-word beginning with $\underline{s}$ in $\mathfrak{T}$ other than $\underline{s}F$, $\underline{s}\curlywedge$, or $\underline{s}P$.*

(TG^* 2) *For any index-word of the form $\underline{s}X$, where X differs from F, $\curlywedge$, and P, in $\mathfrak{T}$, the index-word $\underline{s}$ belongs to $\mathfrak{T}$.*

(TG^* 3) *For any index-word $\underline{s}$ in $\mathfrak{T}$ having neither F nor $\curlywedge$ nor P at its tail, there is at least one index-word of the form $\underline{s}X$ in $\mathfrak{T}$.*

(TG^* 4) *The same as* (TG 4), *the letters V^*, R^*, Q^*, N, N^* being deleted.*

(TG^* 5) *The same as* (TG 5).

Just as in ***TGLK***, any function Π which maps a proof-tree into the domain of sequents of the form $\Gamma \vdash \mathfrak{A}$ is called a *proof-note in* ***TG*LK*** if and only if Π satisfies the following conditions:

(ΠG^* 1) *Any sequent of the forms* $\Pi(\underline{s}F)$, $\Pi(\underline{s}\curlywedge)$, *and* $\Pi(\underline{s}P)$ *is a fundamental sequent.*

(ΠG^* 2) *The same as* (ΠG 2), *the letters V^*, R^*, Q^*, N, N^* being deleted.*

(ΠG^* 3) *The same as* (ΠG 3).

Any sequent of the form $\Gamma \vdash \mathfrak{A}$ is called provable in ***TG*LK*** if and only if there is a proof-note Π of ***TG*LK*** such that $\Pi(\emptyset)$ is the sequent $\Gamma \vdash \mathfrak{A}$. Now, let $\Gamma \vdash \mathfrak{A}_1, \cdots, \mathfrak{A}_s$ be any sequent, and let $\mathfrak{A}$ stand for $\mathfrak{A}_1 \vee \cdots \vee \mathfrak{A}_s$. For the case $s=0$, let $\mathfrak{A}$ stand for $\curlywedge$. Then, we can see without difficulty that ***TGLK*** and ***TG*LK*** are mutually equivalent in the sense that any sequent $\Gamma \vdash \mathfrak{A}_1, \cdots, \mathfrak{A}_s$ is provable in ***TGLK*** if and only if the corresponding sequent $\Gamma \vdash \mathfrak{A}$ is provable in ***TG*LK***.

The next step is to show the mutual relation between ***TG*LK*** and ***TLK***. This becomes clear when we associate with every sequent of ***TG*LK*** not a step of ***TLK*** but a whole proof-part to the step in ***TLK***. To meet this situation, I will at first introduce the notions "*proof-branch*" and "*semi-proof-note*" in ***TLK***. Namely, let Π be any proof-note in ***TLK***, and $\mathfrak{T}$ be the proof-tree associated with Π. For any index-word $\underline{s}$ in $\mathfrak{T}$, the set $\mathfrak{T}(\underline{s})$ of index-words beginning with $\underline{s}$ satisfies the conditions (T 1), (T 2), (T 3) (which should be read "For any index-word $\underline{s}\underline{r}^{j}X$ in $\mathfrak{T}$, $\underline{s}\underline{r}$ belongs to $\mathfrak{T}$"), (T 4), (T 5), and (T 6). The set $\mathfrak{T}(\underline{s})$ is called the *proof-branch* of $\mathfrak{T}$ with respect to $\underline{s}$. The restricted function $\Pi[$——$\underline{s}]$ of Π to the domain $\mathfrak{T}(\underline{s})$ is called naturally a *semi-proof-note* of Π. With every semi-proof-note $\Pi[$——$\underline{s}]$ of Π, we can associate a sequent $\dot{\sum}(\underline{s})$. If $\Pi(\underline{s})$ is a denomination of the form "$\forall t$:", we disregard it. If $\Pi(\underline{s})$ is a proposition $\mathfrak{A}$, and Γ is the sequence of all the propositions of the form $\Pi(\underline{s}^{-}\underline{g})$, we take the sequent $\Gamma \vdash \mathfrak{A}$ as $\sum(\underset{\sim}{s})$. Any sequent of the form $\Gamma \vdash \mathfrak{A}$ is called *provable in* ***TLK***, if and only if there is such semi-proof-note $\Pi[\text{——}\underset{\sim}{s}]$ for which $\sum(\underset{\sim}{s})$ is the sequent $\Gamma \vdash \mathfrak{A}$.

To show that ***TLK*** and ***TG*LK*** are equivalent, let us assume at first that a sequent $\Gamma \vdash \mathfrak{A}$ is provable in ***TLK***. Then, we can associate with each step $\underset{\sim}{s}$ of the semi-proof-note of the sequent a sequent $\sum(\underset{\sim}{s})$. We can prove without difficulty that the totality of the sequents $\sum(\underset{\sim}{s})$ form a framework of a proof-note of $\Gamma \vdash \mathfrak{A}$ in ***TG*LK***, although a pretty long series of verifications are necessary for that. To illustrate this by an example only, let us take up a deduction by the inference rule $(E, e, \forall)$. In this case, the deduction in ***TLK*** must have the form

$$\textstyle\sum(\underset{\sim}{s})\colon\ \Gamma \vdash \mathfrak{A}, \quad \sum(\underset{\sim}{s}E)\colon\ \Gamma \vdash (\exists x)\mathfrak{B}(x), \quad \sum(\underset{\sim}{s}e)\colon\ \Gamma, \mathfrak{B}(t) \vdash \mathfrak{A},$$

where t occurs neither in Γ, nor in $(\exists x)\mathfrak{B}(x)$, nor in $\mathfrak{A}$. By making use of the inference rules (E) and (S', S'') of ***TG*LK***, we can deduce $\Gamma \vdash \mathfrak{A}$ from $\Gamma \vdash (\exists x)\mathfrak{B}(x)$ and $\Gamma, \mathfrak{B}(t) \vdash \mathfrak{A}$ also in ***TG*LK***.

Next, let us assume that we have a proof-note Π of a sequent $\Gamma \vdash \mathfrak{A}$ in ***TG*LK***. Then, I can show that there is a semi-proof-note Φ of the sequent $\vdash \Gamma \rightarrow \mathfrak{A}$, where $\Gamma \rightarrow \mathfrak{A}$ stands for the proposition $\mathfrak{C}_1 \rightarrow (\mathfrak{C}_2 \rightarrow (\cdots (\mathfrak{C}_n \rightarrow \mathfrak{A}) \cdots))$ assuming that Γ is the sequence $\mathfrak{C}_1, \cdots, \mathfrak{C}_n$. Accordingly, the sequent $\Gamma \vdash \mathfrak{A}$ corresponds to the index-word $\underset{\sim}{j}$ of the length n consisting exclusively of $i^{*\prime}s$. Any proposition of the form $\Phi(\underset{\sim}{j}^{k-1}\underset{\sim}{g})$ for $\underset{\sim}{j}^{k-1}\underset{\sim}{g}$ in $\mathfrak{T}(\underset{\sim}{j})$ is $\mathfrak{C}_k$ $(k \leq n)$. Evidently, any fundamental sequent of ***TG*LK*** can be transformed into a semi-proof-note of this kind by making use of the inference rules (F), $(\wedge)$, and (P). Accordingly, we have only to confirm that, for every deduced sequent (a sequent indicated by an index-word ending with neither F, nor $\wedge$, nor P), we can produce a semi-proof-note of this kind for the sequent by assuming that there is a semi-proof-note of the same kind for each sequent from which the above sequent is deduced. In reality, we need a long series of easy verifications. It would be enough to illustrate the verification by two examples.

The first example is a deduction by the inference rule (I', I''). Namely, let

$$(\underset{\sim}{s})\ \Gamma, \mathfrak{A} \rightarrow \mathfrak{B} \vdash \mathfrak{C}, \quad (\underset{\sim}{s}I')\ \Gamma \vdash \mathfrak{A}, \quad (\underset{\sim}{s}I'')\ \Gamma, \mathfrak{B} \vdash \mathfrak{C}$$

be the deduction in question in ***TG*LK***, and Φ' and Φ'' be semi-proof-note of the same kind for the sequents $(\underset{\sim}{s}I')$ and $(\underset{\sim}{s}I'')$, respectively, and

$\mathfrak{T}'(\underline{j})$ and $\mathfrak{T}''(\underline{j}i^*)$ be the proof-branches associated to the semi-proof-note Φ' and Φ'', respectively. Here, we assume that Γ is a sequence of n propositions, and $\underline{j}$ is the sequence of n i^*'s.

Now, we produce a new semi-proof-note Φ of the sequent $\Gamma, \mathfrak{A} \to \mathfrak{B} \vdash \mathfrak{C}$ and the proof-branch $\mathfrak{T}(\underline{j}i^*)$ associated with it in ***TLK*** as follows:

1) *All the index-words* $\underline{s}$ *of* $\mathfrak{T}''(\underline{j}i^*)$ *belongs to* $\mathfrak{T}(\underline{j}i^*)$ *except such* $\underline{s}$ *that begins with* "$\underline{j}-i^*$". *For any index-word* $\underline{s}$ *of the form* $\underline{j}-i^*\underline{g}$ *in* $\mathfrak{T}''(\underline{j}i^*)$ *which surely indicates the proposition* $\mathfrak{B}$, *the index-word* $\underline{j}i^*\underline{g}$ *belongs to* $\mathfrak{T}(\underline{j}i^*)$. *In both cases,* $\Phi(\underline{s})$ *is the same as* $\Phi''(\underline{s})$.

2) *All the index-words of the form* $\underline{j}^p i^*\underline{g}I'\underline{h}$ *belong to* $\mathfrak{T}(\underline{j}i^*)$ *for any index-word of the form* $\underline{j}^p\underline{h}$ *in* $\mathfrak{T}'(\underline{j})$ *and for any index-word of the form* $\underline{j}-i^*\underline{g}$ *in* $\mathfrak{T}''(\underline{j}i^*)$. $\Phi(\underline{j}^p i^*\underline{g}I'\underline{h})$ *is the same as* $\Phi'(\underline{j}^p\underline{h})$.

3) *All the index-words of the form* $\underline{j}-i^*\underline{g}I''$ *belong to* $\mathfrak{T}(\underline{j}i^*)$ *for any index-word of the form* $\underline{j}-i^*\underline{g}$ *in* $\mathfrak{T}''(\underline{j}i^*)$. $\Phi(\underline{j}-i^*\underline{g}I'')$ *is the proposition* $\mathfrak{A} \to \mathfrak{B}$.

We can easily confirm that Φ is a semi-proof-note of the requested kind and $\mathfrak{T}(\underline{j}i^*)$ is a proof-branch associated with it.

Another example is a deduction by the inference rule (U^*). Namely, let

$$(\underline{s})\ \Gamma \vdash (x)\mathfrak{A}(x), \qquad (\underline{s}U^*)\ \Gamma \vdash \mathfrak{A}(t)$$

be the deduction in question in ***TG*LK***. Then, we can assume that there is a semi-proof-note Φ' of $\Gamma \vdash \mathfrak{A}(t)$ and a proof-branch $\mathfrak{T}'(\underline{j})$ associated with it. The variable t does not occur in any $\Phi'(\underline{j}^-\underline{g})$ for any $\underline{j}^-\underline{g}$ in $\mathfrak{T}'(\underline{j})$. Then, we can produce a semi-proof-note Φ of $\Gamma \vdash (x)\mathfrak{A}(x)$ and a proof-branch $\mathfrak{T}(\underline{j})$ associated with it as follows:

1) $\Phi(\underline{j})$ *is the proposition* $(x)\mathfrak{A}(x)$.

2) *All the index-words of the form* $\underline{j}^pU^*\underline{g}$ *belong to* $\mathfrak{T}(\underline{j})$ *for any index-word* $\underline{j}^p\underline{g}$ *in* $\mathfrak{T}'(\underline{j})$. $\Phi(\underline{j}^pU^*\underline{g})$ *is the same as* $\Phi'(\underline{j}^p\underline{g})$.

3) *The index-word* $\underline{j}\forall$ *belongs to* $\mathfrak{T}(\underline{j})$. $\Phi(\underline{j}\forall)$ *is naturally the denomination* "$\forall t$:".

We can easily confirm that Φ is a semi-proof-note of the requested kind and $\mathfrak{T}(\underline{j})$ is a proof-branch associated with it, respectively.

Lastly, I will show the equivalence of ***TLK*** and ***PLK***.

Namely, let us assume at first that we have a proof-note Π of a proposition $\mathfrak{A}$ in ***TLK***, and let $\mathfrak{T}$ be the proof-tree associated with Π. Let us define the rank of any index-word by

Rank ($\underset{\sim}{s}$) = the number of letters i^*, d', d'', e, U^*, and $\vee$ in $\underset{\sim}{s}$,

Rank ($\underset{\sim}{s} - \underset{\sim}{g}$) = (the number of letters i^*, d', d'', e, and U^* in $\underset{\sim}{s}$) + 1.

Let $\mathfrak{T}_r$ be the set of index-words of the rank r in $\mathfrak{T}$. With any index-word $\underset{\sim}{s}^p$ in $\mathfrak{T}_{r+1}$, we can associate an index-word $\underset{\sim}{g}$ in $\mathfrak{T}_r$, which is obtained from $\underset{\sim}{s}^p$ by deleting the letters after "—" (including "—" itself), if any, or by deleting the letters after the last letter i^*, d', d'', e, U^*, or $\vee$ (including the letter itself) in $\underset{\sim}{s}^p$. The operation from $\underset{\sim}{s}^p$ to $\underset{\sim}{g}$ is denoted by f ($\underset{\sim}{g} = f(\underset{\sim}{s}^p)$).

Now, for any index-word of $\underset{\sim}{g}$, let $\mathfrak{B}(\underset{\sim}{g})$ be the set of inverse images of $\underset{\sim}{g}$ with respect to the function f, *i.e.*, the set of all such $\underset{\sim}{s}^p$ in $\mathfrak{T}$ that satisfy $\underset{\sim}{g} = f(\underset{\sim}{s}^p)$. With any index-word of $\mathfrak{T}$, I will associate an index-word in ***PLK*** recursively by the following rules:

1) Arrange all the index-words of $\mathfrak{T}_0$ in a series, placing longer ones before shorter ones. Let $\underset{\sim}{s}_1, \cdots, \underset{\sim}{s}_n, \underset{\sim}{s}$ be the series thus arranged, where $\underset{\sim}{s}$ is surely the null sequence. Then, we associate the letters $b, c, \cdots$ in the lexicographic order of letters up to the n-th letter with $\underset{\sim}{s}_1, \cdots, \underset{\sim}{s}_n$, respectively, and associate $\in$ to the null sequence $\underset{\sim}{s}$. Propositions indicated by the index-words remain unchanged.

2) Assume that we have already associated an index-word of ***PLK*** with any one of letters belonging to any one of $\mathfrak{T}_0, \cdots, \mathfrak{T}_r$. I will give a rule of association of an index-word to any index-word of ***PLK*** by giving a rule of association of an index-word in ***PLK*** to every index-word of $\mathfrak{B}(\underset{\sim}{g})$ for any index-word $\underset{\sim}{g}$ in $\mathfrak{T}_r$. Let $\underset{\sim}{s}$ be the index-word in ***PLK*** associated with $\underset{\sim}{g}$. If $\mathfrak{B}(\underset{\sim}{g})$ is not empty, only the following four cases are possible:

2.1) $\mathfrak{B}(\underset{\sim}{g})$ contains $\underset{\sim}{g}i^*$. In this case, associate $\underset{\sim}{s}A$ with all the index-words of the form $\underset{\sim}{g} - i^*\underset{\sim}{v}$. This is possible because all the index-words of this form indicate the same proposition by assumption. All the other index-words of $\mathfrak{B}(\underset{\sim}{g})$ begin with $\underset{\sim}{g}i^*$. Arrange these index-words in a series placing longer ones before shorter ones. Associate with the series of the index-words of this kind, the series $\underset{\sim}{s}A, \cdots, \underset{\sim}{s}\in$ of index-words of ***PLK*** respectively. Propositions indicated by the index-words remain unchanged.

2.2) $\mathfrak{B}(\underset{\sim}{g})$ contains $\underset{\sim}{g}d'$ and $\underset{\sim}{g}d''$. In this case, associate $\underset{\sim}{s}A'$ with all the index-words of the form $\underset{\sim}{g}-d'\underset{\sim}{v}$ and $\underset{\sim}{s}A''$ with all the index-words of the form $\underset{\sim}{g}-d''\underset{\sim}{w}$. This is possible because all the index-words of the form $\underset{\sim}{g}-d'\underset{\sim}{v}$ indicate the same proposition, and all the index-words of the form $\underline{g}-d''\underline{w}$ indicate the same proposition by assumption. $\mathfrak{B}(\underline{g})$ can be arranged in two series of index-words, the one is a series of index-words beginning with $\underline{g}d'$, and the other beginning with $\underline{g}d''$, in each series placing longer ones before shorter ones. Then, associate the index-words $\underline{s}b', \cdots, s\in'$ and $\underline{s}b'', \cdots, \underline{s}\in''$ of ***PLK*** with these series of index-words, respectively. Propositions indicated by these index-words remain unchanged.

2. 3) $\mathfrak{B}(\underline{g})$ contains $\underline{g}e$ and $\underline{g}\forall$. In this case, delete $\underline{g}\forall$, which indicates a denomination of the form "$\forall t$:", and associate $\underline{s}A$ with all the index-words of the form $\underline{g}-e\underline{v}$. This is possible because all the index-words of this form indicate the same proposition, say $\mathfrak{A}(t)$, by assumption. Arrange the index-words of $\mathfrak{B}(\underline{g})$ beginning with $\underline{g}e$ in a series placing longer ones before shorter ones. Associate with the series of index-words, the series $\underline{s}b, \cdots, \underline{s}\in$ of index-words of ***PLK***, respectively. The proposition indicated by $\underline{s}A$ becomes the denomination "$\forall t$: $\mathfrak{A}(t)$" (originally, the proposition $\mathfrak{A}(t)$ of $\underline{g}-e\underline{v}$). Propositions indicated by the other index-words remain unchanged.

2. 4) $\mathfrak{B}(\underline{g})$ contains $\underline{g}U^*$. In this case, associate $\underline{s}A$ with $\underline{g}\forall$, which indicates a denomination of the form "$\forall t$:". Arrange all the index-words of $\mathfrak{B}(\underline{g})$ beginning with $\underline{g}U^*$ in a series placing longer ones before shorter ones. Then, associate with the series of index-words the series of index-words $\underline{s}b, \cdots, \underline{s}\in$ of ***PLK***, respectively. Propositions and denominations indicated by the index-words remain unchanged.

It can be confirmed without difficulty that we have a correct proof-note Φ of the proposition $\mathfrak{A}$ in ***PLK***.

Now, conversely, let us assume that we have a proof-note Π of a proposition $\mathfrak{A}$ in ***PLK***. Then, let us arrange it in the fundamental order of steps. Let the number of steps be n. Starting from the step $\in$ of Π we can make many threads of references. Namely, any sequence of steps $\underline{s}^1, \cdots, \underline{s}^m$ is called a *thread* if and only if $\underline{s}^1$ is $\in$, $\underline{s}^m$ is an assumption step of the forms $\underline{t}A, \underline{t}A'$, or $\underline{t}A''$, and $\underline{s}^{i+1}$ is referred by $\underline{s}^i$ for any $i=1, \cdots, m-1$. Here we also say that any step of the forms $\underline{t}\in$, $\underline{t}\in'$, or $\underline{t}\in''$ is referred by $\underline{t}$. We can see evidently that we can associate a definite index-word of ***TLK*** with each terms of any thread except for denominations of the form "$\forall t$:". Especially, the last term $\underline{s}^m$ of any thread must be any one of the forms $\underline{t}A$, $\underline{t}A'$, or $\underline{t}A''$, and there must be a step $\underline{s}^k$ of the forms $\underline{t}\in$, $\underline{t}\in'$, or $\underline{t}\in''$ in the thread, respectively. If the index-word $\underline{g}$ of ***TLK*** is associated with the step $\underline{s}^{k-1}$ of the thread, then an index-word of the form $\underline{g}X$ must be associated with $\underline{s}^k$ for a letter X standing for any one of the letters i^*, d', d'', e, or U^*. If X stands for any one of the letters i^*, d', or d'', an index-word of the form $\underline{g}-X\underline{h}$ should be associated with $\underline{s}^m$. If X stands for the letter e, an index-word of the form $\underline{g}-e\underline{h}$ should be associated with $\underline{s}^m$ and at the same time the index-word $\underline{g}\forall$ should be attached to the thread. If X stands for the letter U^*, we delete $\underline{s}^m$ from the thread after attaching $\underline{g}\forall$ to the thread.

For every index-word $\underline{g}^p$ associated with a step $\underline{s}$, $\Phi(\underline{g}^p)$ denotes the proposition indicated by the step $\underline{s}$ in the original proof-note Π. If $\underline{g}^p$ is an index-word of the form $\underline{h}-e\underline{k}$, the step $\underline{s}$ indicates a denomination of the form "$\forall t: \mathfrak{A}(t)$". In this case, $\Phi(\underline{g}^p)$ denotes the proposition $\mathfrak{A}(t)$. For any index-word $\underline{g}$, $\Phi(\underline{g}\forall)$ denotes naturally a denomination of the form "$\forall t$:", the variable t being the variable of the denomination "$\forall t: \mathfrak{A}(t)$" of the step in Π with which an index-word of the form $\underline{g}-e\underline{h}$ is associated.

In every thread, steps proceed from the bottom upward in the original proof-note arranged in the fundamental order of steps. Therefore, the length of any thread does not exceed n. Accordingly, the number of all the threads with respect to the proof-note Π must be finite. So, we can thus define a proof-note Φ of the proposition $\mathfrak{A}$ in ***TLK***. It would not be necessary to go into detailed examination of the fact that Φ is really a correct proof-note in ***TLK***.

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