H. Morikawa
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ON COMMUTATIVE COMPOSITIONS DETERMINED BY THEIR ORIGINS

HISASI MORIKAWA

1. Let K be the universal domain. Let G be a finite additive group of odd order $|G|$ and $X_a (a \in G)$ be indeterminates indexed by the elements in G. We mean by P_G the projective space $Proj_k(K[(X_a)_{a \in G}])$. Denote by δ_{-1} and $\tau_b (b \in G)$ the automorphisms of P_G of which duals δ^*_{-1} and τ^*_b are the ring-automorphisms of $Z[(X_a)_G]$ such that

$$\delta^*_{-1}(X_a) = X_{-a}, \quad \tau^*_b(X_a) = X_{b+a} \quad (a, b \in G).$$

For the sake of simplicity we denote briefly

$$x^{-1} = \delta^{-1}(x), \quad x(b) = \tau_b(x) \quad (x \in P_G, \quad b \in G).$$

DEFINITION 1. 1 Let $e = (e_a)_G$ be a point on P_G satisfying

$$(1) \qquad e_{-a} = e_a \quad (a \in G).$$

Then two points $x = (x_a)_G$ and $y = (y_a)_G$ are called to be composable with respect to e, if there exist two vectors $u = (u_a)_G$ and $v = (v_a)_G$ such that

$$(2) \qquad \text{rank}\begin{pmatrix} (e_{-a+b}e_{a+b})_{G,G} & (y_{-a+d}\, y_{a+d})_{G,G} \\ {}^t(x_{-c+b}x_{c+b})_{G,G} & (u_{-c+d}v_{c+d})_{G,G} \end{pmatrix} = \text{rank}\,(e_{-a+b}e_{a+b})_{G,G},$$

where $(e_{-a+b}e_{a+b})_{G,G}$, $(x_{-a+b}x_{a+b})_{G,G}$, $(y_{-a+b}y_{a+b})_{G,G}$ and $(u_{-a+b}v_{a+b})_{G,G}$ are $|G| + |G|$-matrices of which (a, b)-components are $e_{-a+b}e_{a+b}$, $x_{-a+b}x_{a+b}$, $y_{-a+b}y_{a+b}$ and $u_{-a+b}v_{a+b}$, respectively, $(a, b \in G)$.

Since the order $|G|$ is odd, the pair $(-a+b,\ a+b)$ runs over all the elements in $G \times G$. Therefore the system of equations

$$u_{-a+b}v_{a+b} = u'_{-a+b}v'_{a+b} \quad (a, b \in G)$$

implies $u_a/u'_a = u_b/u'_b$, $v_a/v'_a = v_b/v'_b$ $(a, b \in G)$. Namely the point $u = (u_a)_G$ and $v = (v_a)_G$ in (2) are uniquely determined by x and y as points in P_G.

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Definition 1. 2. If $x=(x_a)_G$ and $y=(y_a)_G$ are composable with respect to e, we denote by $x\circ y$ the unique point $v=(v_a)_G$ given in (2) and call it the composition of x and y with respect to e.

Proposition 1. 3. *If $x=(x_a)_G$ and $y=(y_a)_G$ are composable with respect to e, them it follow*

$$(3)\quad \operatorname{rank}\begin{pmatrix} (e_{-a+b}e_{a+b})_{G,G} & (y_{-a+d}y_{a+d})_{G,G} \\ {}^t(x_{-c+b}x_{c+b})_{G,G} & ((\lambda(x^{-1}\circ y)_{-c+d}(x\circ y)_{c+d})_{G,G}(x^{-1}\circ y)_{-c+d}(x\circ y)_{c+d})_{G,G} \end{pmatrix}$$
$$= \mathit{rank}\,(e_{-a+b}e_{a+b})_{G,G}$$

with non-zero λ, where λ depends on the homogeneous coordinates.

Proof. Replacing x by x^{-1} in (2), we know that the unique point $u=(u_a)_G$ in (2) is $x^{-1}\circ y$.

Proposition 1. 4 *If $x\circ y$ is well-defined, then $y\circ x$ and $x\circ e(a)$ ($a\in G$) are also well-defined and they satisfy*

$$(4)\quad x\circ y = y\circ x,$$

$$(5)\quad x\circ e(a) = e(a)\circ x = x(a) \quad (a\in G),$$

$$(6)\quad e(a)\circ e(b) = e(a+b) \qquad (a, b\in G).$$

This is an immediate consequence from the relation (3).

2. Since $(e_{-a+b}e_{a+b})_{G,G}$ is symmetric, there exists a subset H in G such that the cardinal $|H|$ equals to the rank of $(e_{-a+b}e_{a+b})_{G,G}$ and $\det(e_{-a'+b'}e_{a'+b'})_{H,H} \neq 0$, where $(e_{-a'+b'}e_{a'+b'})_{H,H}$ is an $|H|\times|H|$-matrix of which (a', b')-component is $e_{-a'+b'}e_{a'+b'}$ $(a', b'\in H)$.

Using the inverse matrix

$$(7)\quad (\alpha_{a',b'})_{H,H} = (e_{-a'+b'}e_{a'+b'})^{-1}_{H,H},$$

we can express the relation (3) by the following explicite polynomial relations:

$$(8)\quad x_{-a+b}x_{a+b} = \sum_{c',d'\in H} \alpha_{c',d'}e_{-c'+a}e_{c'+a}x_{-d'+b}x_{d'+b}$$

$$(8')\quad y_{-a+b}y_{a+b} = \sum_{c',d'\in H} \alpha_{c',d'}e_{-c'+a}e_{c'+a}y_{-d'+b}y_{d'+b}$$

$$(8'')\quad \lambda(x^{-1}\circ y)_{-a+b}(x\circ y)_{a+b} = \sum_{c',d'\in H} \alpha_{c',d'}x_{-c'+a}x_{c'+a}y_{-d'+b}y_{d'+b} \quad (a, b\in G)$$

with non-zero λ.

DEFINITION 2. 1. We denote by V_e the closed subscheme in P_G which is the Zariski-closure of all the point x such that $x^{-1} \circ x$ is well-defined and $x^{-1} \circ x = e$. We call V_e the projective scheme associating with e.

Using $(\alpha_{a',b'})_{H,H}$ we can define V_e as the closed subscheme defined by the relations

$$(9) \quad X_{-a+b}X_{a+b} = \sum_{c',d' \in H} \alpha_{c',d'} e_{-c'+a} e_{c'+a} X_{-d'+b} X_{d'+b} = 0 \quad (a, b \in G)$$

and

$$(10) \quad \sum \alpha_{c',d'} \{ e_c X_{-c'-a+b} X_{c'-a+b} X_{-d'+a+b} X_{d'+a+b} - e_a X_{-c'-c+b} X_{c'-c+b} X_{-d'+c+b} X_{d'+c+b} \} = 0. \quad (a, b, c \in G).$$

Under what condition on $e = (e_a)_G$ the projective scheme V_e is an abelian variety? This is very difficult problem, which is equivalent to giving the reasonable explicite generators of the relations between theta-constants. We shall be concerved with this problem in the next paper.

Institute of Mathematics Nagoya University.