THE SINGULAR MEASURE OF A DIRICHLET SPACE

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1. Introduction

We [4], [5] examined some properties of balayaged measures in the theory of a Dirichlet space. In those papers, we showed that the singular measure of a Dirichlet space plays some important roles. In this paper, we shall precisely examine some properties of the singular measure of a Dirichlet space. Let X be a locally compact Hausdorff space in which there exists a positive Radon measure ξ which is everywhere dense in X. First we obtain the following

(1) Let D be a Dirichlet space with respect to X and ξ, and let σ be the singular measure of D. For any couple u and v in D such that $S_u \cap S_v = \phi$,[1] the function $u^*(x)v^*(y)$ in the product space $X \times X$ is σ-integrable and

$$(u, v) = -2 \iint u^*(x)v^*(y)d\sigma(x, y),$$

where u^* and v^* are the refinements of u and v, respectively.

By using this result, we shall obtain more precise results than those in [4]. Moreover we have the following

(2) Let D be the same as the above (1), and let u_μ be a pure potential in D. For an open set ω in X, let μ' be the balayaged measure of μ to ω, and let ν' be the restriction of μ' to ω. For any pure potential u_μ in D and any open set ω contained in the complement CS_μ of the support of μ, ν' is absolutely continuous for ξ if and only if the projection of the singular measure of D to X is absolutely continuous for ξ.

Next we shall examine total masses of balayaged measures. The result in this paper is better than the one in [5].

Finally we shall obtain more precise results in the case of a special Dirichlet space. Especially the following result is important.

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1) For a ξ-measurable function f, S_f means the complement of the largest open set ω such that $f(x) = 0$ in ω.

For any special Dirichlet space D, ν' is always absolutely continuous for ξ.

2. Preliminaries on Dirichlet spaces

Let X be a locally compact Hausdorff space in which there exists a positive Radon measure ξ which is everywhere dense in X (i.e., $\xi(\omega)>0$ for any non-empty open set ω in X). Let C_K be the space of finite continuous functions with compact support provided with the topology usual. According to Beurling & Deny [2], we define a ξ-Dirichlet space on X.

DEFINITION 1. A Hilbert space $D=D(X;\ \xi)$ is called ξ-Dirichlet space (simply, Dirichlet space) on X if each element u in D is locally ξ-summable (simply, summable) real-valued function[2] in X and the following three conditions are satisfied:

(D. 1) For any compact subset K of X, there exists a positive constant $A(K)$ such that

$$\int_K |u(x)|\,d\xi(x)\leq A(K)\,\|u\|$$

for any u in D.

(D. 2) $C_K\cap D$ is dense both in C_K and in D.

(D. 3) For any u in D and any normal contraction T on the real line R, $T\cdot u$ is contained in D and $\|T\cdot u\|\leq\|u\|$.

In the above (D. 3), A transformation T on R into itself is called a normal contraction if it satisfies the following:

$$T(0)=0 \text{ and } |Ta_1-Ta_2|\leq|a_1-a_2|$$

for any couple a_1 and a_2 in R. Two functions which are equal locally almost everywhere (simply, a.e.) for ξ represents the same element in D. The norm of D is denoted by $\|u\|$, the associated scalar product by (u,v). Similarly as Beurling and Deny [2], we define potentials in D.

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2) Beurling & Deny [2] first assumed that each element u in D is a complex-valued function in X. Put $D_r=\{Re\ u\,;\ u\in D\}$. Then D_r is a Dirichlet space in our sense. Conversely, let D be a Dirichlet space in our sense. Put $D'=\{u+iv\,;\ u,\ v\in D\}$. Then D' is a Dirichlet space in Beurling & Deny's sense. In potential theory, it is sufficient to assume that each u in D is real-valued, because important potentials, i.e., balayaged potentials, equilibrium potentials, $\cdots$ are all real-valued.

DEFINITION 2. An element u in D is called a potential in D if there exists a real Radon measure μ in X such that

$$(f, u) = \int f(x) d\mu(x)$$

for any f in $C_K \cap D$. Such an element u is denoted by u_μ. Especially if μ is positive, u_μ is called a pure potential in D. By Definition 1, (D. 1), for each bounded measurable function f with compact support, there exists a unique element u_f in D such that

$$(v, u_f) = \int v(x) f(x) d\xi(x)$$

for any v in D.

Beurling and Deny [2] showed the following important representation theorem.

PROPOSITION 1. *For a Dirichlet space D on X, there exist a positive measure ν in X, a positive Hermitian form $N(f, g)$ on $C_K \cap D$ and a positive symmetric measure σ in $X \times X - \delta$ (δ is the diagonal set of $X \times X$) such that*

$$(f, g) = \int f g d\nu + N(f, g) + \iint (f(x) - f(y))(g(x) - g(y)) d\sigma(x, y)$$

for any couple f and g in $C_K \cap D$. Here $N(f, g)$ has the following local character: if g is constant in some neighborhood of the support S_f of f, then $N(f, g)$ vanishes.

PROPOSITION 2. *For a Dirichlet space D on X, the above representation is unique.*

Proof. Suppose that there exist another positive measure ν' in X, another positive Hermitian form $N'(f, g)$ on $C_K \cap D$ with the above local character and another positive symmetric measure σ' in $X \times X - \delta$ such that

$$(f, g) = \int f g d\nu + N'(f, g) + \iint (f(x) - f(y))(g(x) - g(y)) d\sigma'(x, y)$$

for any couple f and g in $C_K \cap D$. Since $C_K \cap D$ is dense in C_K, the set

$$\{f(x) g(y);\ f, g \in C_K \cap D,\ S_f \cap S_g = \phi\}$$

is dense in $C_K(X \times X - \delta)$.[3] For any couple f and g in $C_K \cap D$ with $S_f \cap S_g = \phi$,

[3] $C_K(X \times X - \delta)$ is the space of finite continuous functions in $X \times X - \delta$ with compact support provided with the topology of uniform convergence.

$$(f, g) = -2\int f(x)g(y)d\sigma(x, y) = -2\int f(x)g(y)d\sigma'(x, y).$$

Hence the equality $\sigma = \sigma'$ holds. Next we shall show the equality $\nu = \nu'$. It is sufficient to prove the equality

$$\int f\ d\nu = \int f\ d\nu'$$

for any f in $C_K \cap D$. Similarly as in the proof of Theorem 1 in [4], there exists a function g in $C_K \cap D$ such that $g(x) = 1$ in some neighborhood of S_f. The Hermitian forms $N(f, g)$ and $N'(f, g)$ having the local character,

$$(f, g) = \int f\ d\nu + \iint (f(x) - f(y))\,(g(x) - g(y))d\sigma(x, y)$$

$$= \int f\ d\nu' + \iint (f(x) - f(y))\,(g(x) - g(y))d\sigma'(x, y).$$

Therefore the equality $\nu = \nu'$ holds, and hence

$$N(f, g) = N'(f, g)$$

on $C_K \cap D$. This completes the proof.

DEFINITION 3. The above measure ν in X is called the equilibrium measure of X (with respect to D),[4] $N(f, g)$ is called the local form of D and the positive measure σ is called the singular measure of D.

3. Some lemmas

In order to obtain our first main theorem, we need the following lemmas.

LEMMA 1. *Let D be a Dirichlet space on X. For a compact set F_1 and a closed set F_0 in X with $F_1 \cap F_0 = \phi$, let $u_{\mu_1-\mu_0}$ be the condensor potential with respect to F_1 and F_0.*[5] *Then $u_{\mu_1-\mu_0}$ is contained in the closure of the following subset $E_{1,0}$ of D:*

$$E_{1,0} = \{f \in C_K \cap D;\ f(x) = 1 \ \text{on}\ F_1 \ \text{and}\ f(x) = 0 \ \text{on}\ F_0\}.$$

4) Beurling and Deny [2] remarked that for any non-decreasing net $(\omega_\alpha)_{\alpha\in I}$ of relatively compact open sets tending to X, the equilibrium measure of ω_α tends vaguely to ν. Hence we say that ν is the equilibrium measure of X.

5) Beurling and Deny [2] showed that for any couple of open sets ω_1 and ω_0 in X, ω_1 being relatively compact, there exists a potential $u_{\mu_1-\mu_0}$ in D satisfying the following: $0 \leq u_{\mu_1-\mu_0} \leq 1$, $u_{\mu_1-\mu_0}(x) = i$ a.e. in ω_i and μ_i is a positive measure in X supported by $\overline{\omega_i}$. We [6] formed a similar potential in D for a compact set F_1 and a closed set F_0. This potential is called the condensor potential with respect to ω_1 and ω_0 (or F_1 and F_0).

Proof. We put

$$\tilde{E}_{1,0}=\overline{\{f\in C_K\cap D;\ f(x)\geq 1 \text{ on } F_1 \text{ and } f(x)\leq 0 \text{ on } F_0\}}.$$

Then $\tilde{E}_{1,0}$ is a closed convex set and non-empty, because $C_K\cap D$ is dense in C_K. Let $u_{1,0}$ be a unique element in $\tilde{E}_{1,0}$ whose norm is minimal in $\tilde{E}_{1,0}$. Similarly as Beurling and Deny's Condensor Theorem, we obtain that $u_{1,0}$ is equal to a potential u_μ in D and μ^+ (resp. μ^-) is supported by F_1 (resp. F_0). By the condition (D. 3) in Definition 1, $0\leq u_{1,0}\leq 1$ and $u^*_{1,0}(x)=i$ ppp on F_i for $i=1,0,$[6] where $u^*_{1,0}$ is the refinement of $u_{1,0}$.[7] Next we shall show that $u_{\mu_1-\mu_0}=u_{1,0}$. By Beurling and Deny's theorem,[8] there exists a sequence (u_{μ_n}) of linear combinations of pure potentials in D such that (u_{μ_n}) converges strongly to $u_{1,0}$ as $n\longrightarrow+\infty$ and

$$S_{\mu_n}\subset F_1\cup F_0.$$

Then we have

$$\|u_{1,0}\|^2=\lim_{n\to\infty}(u_{1,0},u_{\mu_n})=\lim_{n\to\infty}(u_{\mu_1-\mu_0},\ u_{\mu_n})$$
$$=(u_{\mu_1-\mu_0},u_{1,0})\leq\|u_{\mu_1-\mu_0}\|\cdot\|u_{1,0}\|,$$

because

$$u^*_{\mu_1-\mu_0}(x)=1\ \ ppp \text{ on } F_1 \text{ and } u^*_{\mu_1-\mu_0}(x)=0\ \ ppp \text{ on } F_0.$$

That is,

$$\|u_{1,0}\|\leq\|u_{\mu_1-\mu_0}\|.$$

By the definition of the condensor potential, we obtain that $u_{1,0}=u_{\mu_1-\mu_0}$. Finally we shall prove that $u_{1,0}\in\overline{E_{1,0}}$. By the above assertion, there exists a sequence (f'_n) in $E_{1,0}\cap C_K$ such that (f'_n) converges strongly to $u_{1,0}$ in D. Let T be the unit contraction on R,[9] and put

$$f_n(x)=T\cdot f'_n(x).$$

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[6] A property is said to hold ppp on a subset E in X if the property holds μ-a.e. on E for any pure potential u_μ in D such that $S_\mu\subset E$.

[7] Cf. [2], pp. 209–210.

[8] Cf. [2], p. 214.

[9] We say that the projection on R to the closed interval [0, 1] is the unit contraction on R. Cf. [6].

Then f_n is contained in $E_{1,0}$ and (f_n) converges strongly to $u_{1,0}$ in D as $n \longrightarrow +\infty$, because $(\|f_n\|)$ is bounded and

$$\|u_{1,0}\| = \lim_{n\to\infty} \|f'_n\| \geq \overline{\lim_{n\to\infty}} \|f_n\| .$$

This completes the proof.

Similarly as in the case of a special Dirichlet space, we obtain the following

LEMMA 2. *Let D be a Dirichlet space on X and σ be the singular measure of D. For any compact set K in X and any open neighborhood ω of K,*

$$\iint_{K\times C\omega} d\sigma(x, y) < +\infty .$$

Proof. We take another open neighborhood ω' of K such that $\overline{\omega'} \subset \omega$. Let u_μ be the condensor potential with respect to K and $C\omega'$ and let (f_n) be a sequence in $C_K \cap D$ such that (f_n) converges strongly to u_μ in D as $n \longrightarrow +\infty$ and

$$0 \leq f_n(x) \leq 1,\ f_n(x) = 1 \text{ on } K \text{ and } f_n(x) = 0 \text{ on } C\omega'.$$

Let $(K'_\alpha)_{\alpha\in I}$ be a non-decreasing net of compact subsets in X tending to X and put

$$K_\alpha = K'_\alpha \cap C\omega .$$

Similarly as above, we can take a non-decreasing net (g_α) in $C_K \cap D$ such that

$$S_{g_\alpha} \subset C\overline{\omega'},\ 0 \leq g_\alpha \leq 1 \text{ and } g_\alpha(x) = 1 \text{ on } K_\alpha .$$

Then for any n,

$$\iint_{K\times K_\alpha} d\sigma(x, y) \leq \iint f_n(x) g_\alpha(y) d\sigma(x, y)$$
$$= -\frac{1}{2}(f_n, g_\alpha) .$$

Consequently we have

$$\iint_{K\times K_\alpha} d\sigma(x,y) \leq -\frac{1}{2}(u_\mu, g_\alpha) = \frac{1}{2}\int g_\alpha(x)d\mu^-(x)\,.$$

The total mass of the positive measure μ^- being finite, we obtain that

$$\iint_{K\times C\omega} d\sigma(x,y) \leq \frac{1}{2}\int d\mu^- < +\infty\,.$$

This completes the proof.

3. First main theorem

Now we define the projection of a singular measure of a Dirichlet space.

DEFINITION 4. Let σ be the singular measure of a Dirichlet space D. For a compact set K in X, the projection σ_K of σ to CK is the positive measure in CK defined as follows:

$$\int f d\sigma_K = \int_K\int f(y)d\sigma(x,y)$$

for any f in $C_K(CK)$.

LEMMA 3. *Let σ be the singular measure of a Dirichlet space D. For a compact set K in X and an element u in D such that $K \cap S_u = \phi$, the refinement u^* of u is σ_K-integrable.*

Proof. It is sufficient to prove that there exists a pure potential u_μ in D such that the inequality $\sigma_K \leq \mu$ holds in an open set ω contained with its closure in CK. We take a couple of open sets ω_1 and ω_0 with disjoint closures, ω_1 being relatively compact and holding the following inclusions:

$$\omega_1 \supset K \text{ and } \omega_0 \supset \bar{\omega}\,.$$

Let $u_{\mu_1-\mu_0}$ be the condensor potential with respect to ω_1 and ω_0. Then by the results in the preceding paper,[10] u_{μ_1} and u_{μ_0} are elements in D. Similarly as the above lemmas, there exists a sequence (f_n) in $C_K \cap D$ such that (f_n) converges strongly to $u_{\mu_1-\mu_0}$ as $n \longrightarrow +\infty$,

$$0 \leq f_n \leq 1,\ f_n(x) = 1 \text{ on } K \text{ and } f_n(x) = 0 \text{ on } \bar{\omega}\,.$$

For any f in $C_K^+ \cap D$[11] with support in ω, we have

[10] Cf. Levy-Khinchine's theorem in [2] and [3].

[11] Cf. Lemma 1 and Lemma 3 in [5].

$$\int_K \int f(y) d\sigma(x, y) \leq \iint f_n(x) f(y) d\sigma(x, y) = -\frac{1}{2}(f, f_n)$$

for any n. Making n tend to infinity, we obtain that

$$\int_K \int f(y) d\sigma(x, y) \leq -\frac{1}{2}(f, u_{\mu_1-\mu_0}) = \frac{1}{2}\int f\, d\mu_0 .$$

$C_K \cap D$ being dense in C_K, we obtain that $\sigma_K \leq \frac{1}{2}\mu_0$ in ω. This completes the proof.

By the above lemma, we obtain the following

THEOREM 1. *Let D be a Dirichlet space on X and σ be the singular measure of D. For any potential u_μ in D, let $\mu^{(1)}$ be the restriction of μ to CS_{u_μ}. Then*

$$d\mu^{(1)}(x) = -\frac{1}{2}\int u_\mu^*(y) d\sigma(x, y)$$

in CS_{u_μ}. Furthermore for any couple of elements u_1 and u_2 in D such that $S_{u_1} \cap S_{u_2} = \phi$, we obtain

$$(u_1, u_2) = -2\iint u_1^*(x) u_2^*(y) d\sigma(x, y) .$$

Proof. First we suppose that u_μ is bounded in X. By the conditions (D. 2) and (D. 3) in Definition 1, there exists a sequence (f_n) such that (f_n) converges strongly to u_μ in D as $n \longrightarrow +\infty$, (f_n) is uniformly bounded and S_{f_n} is contained in a fixed neighborhood N of S_{u_μ}. We take any fixed element f in $C_K \cap D$ such that $S_f \subset CS_{u_\mu}$. We may assume that the above function f_n has the support in CS_f. Then

$$(f, f_n) = -2\iint f_n(x) f(y) d\sigma(x, y) .$$

By Lemma 2 and Lebesgue's bounded convergence theorem, making n tend to infinity, we obtain

$$(f, u_\mu) = -2\iint u_\mu^*(x) f(y) d\sigma(x, y) .$$

That is,

$$\int f\, d\mu^{(1)} = -2\iint f(x) u_\mu^*(y) d\sigma(x, y) .$$

Next we shall prove the general case. We may assume that u_μ is non-negative, because in the general case, u_μ^+ and u_μ^- are potentials in D. Put

$$u_{\mu,n}(x) = \inf(u_\mu(x),\ n).$$

Then $u_{\mu,n}$ is contained in D and by the above assertion, we have

$$(u_{\mu,n}, f) = -2\iint f(x)u_{\mu,n}^*(y)d\sigma(x, y).$$ [12]

Since the sequence $(u_{\mu,n})$ converges strongly to u_μ in D[13] and the sequence $(u_{\mu,n}(x))$ is non-decreasing, making n tend to infinity, we have

$$\int f\ d\mu^{(1)} = (u_\mu, f) = -2\iint f(x)u_\mu^*(y)d\sigma(x, y).$$

Let's show the second part of our theorem. First we assume that S_{u_1} is compact and $u_2(x)$ is non-negative. Then we can take a relatively compact open set ω_1 and an open set ω_2 such that

$$\overline{\omega_1} \cap \overline{\omega_2} = \phi,\ S_{u_1} \subset \omega_1 \text{ and } S_{u_2} \subset \omega_2.$$

By Lemma 3, we can define a positive measure $\sigma_{u_2,1}$ in ω_1 such that

$$\int f\ d\sigma_{u_2,1} = \iint f(x)u_2^*(y)d\sigma(x, y)$$

for any f in C_K with support in ω_1. Let's show that the function u_1^* is $\sigma_{u_2,1}$-measurable. By the properties of the refinement, there exists a non-increasing sequence (ω_n) of open sets contained in ω_1 such that u_1^* is continuous on $C\omega_n$ for any n and

$$\lim_{n\to\infty} \operatorname{cap}(\omega_n) = 0.$$ [14]

We take an open set ω_3 such that

$$\overline{\omega_2} \subset \omega_3 \text{ and } \overline{\omega_1} \cap \overline{\omega_3} = \phi.$$

Let u_{μ_n} be the condensor potential with respect to ω_n and ω_3. Then

$$\int_{\omega_n} d\sigma_{u_2,1} \leq -\frac{1}{2}(u_{\mu_n}, u_2) \leq \frac{1}{2}\|u_{\mu_n}\|\cdot\|u_2\|.$$

12) Cf. Proposition 1.

13) Cf. Lemma 4 in [5].

14) For an open set ω, the capacity cap(ω) of ω is defined as follows: cap(ω)=inf$\{\|u\|^2;\ u(x)\geq 1$ a.e. in $\omega\}$, cap(ω)$=+\infty$ if such elements don't exist.

Since the sequence $(\| u_{\mu_n} \|)$ converges to 0 as $n \longrightarrow +\infty$, u_1^* is $\sigma_{u_2,1}$-measurable. If u_1^* is bounded, our conclusion is evident. Put

$$u_{1,n}^+ = \inf(u_1^+, n), \quad u_{1,n}^- = \inf(u_1^-, n).$$

Then the sequences $(u_{1,n}^+)$ and $(u_{1,n}^-)$ are non-decreasing and contained in D. By the above assertion,

$$(u_{1,n}^+, u_2) = -2\iint u_{1,n}^{+*}(x)u_2^*(y)d\sigma(x, y)$$

and

$$(u_{1,n}^-, u_2) = -2\iint u_{1,n}^{-*}(x)u_2^*(y)d\sigma(x, y).$$

Making n tend to infinity, we obtain

$$(u_1^+, u_2) = -2\iint u_1^{+*}(x)u_2^*(y)d\sigma(x, y) \text{ and } (u_1^-, u_2) = -2\iint u_1^{-*}(x)u_2^*(y)d\sigma(x, y).$$

That is, we have

$$(u_1, u_2) = -2\iint u_1^*(x)u_2^*(y)d\sigma(x, y).$$

In the case that u_2 is general, by the above assertion, we have

$$\begin{aligned}(u_1, u_2) &= (u_1, u_2^+) - (u_1, u_2^-) \\ &= -2\iint u_1^*(x)u_2^{+*}(y)d\sigma(x, y) + 2\iint u_1^*(x)u_2^{-*}(y)d\sigma(x, y) \\ &= -2\iint u_1^*(x)u_2^*(y)d\sigma(x, y).\end{aligned}$$

Thus we prove the case that S_{u_1} is compact. We shall prove the case that S_{u_1} is general. Similarly as the above, we may assume that u_1 and u_2 are non-negative. We take a non-decreasing net $(\omega_\alpha)_{\alpha \in I}$ of relatively compact open sets tending to CS_{u_2}. We put $F_\alpha = C\omega_\alpha$. Let $u'_{1,\alpha}$ be the projection of u_1 to $D_{F_\alpha}^{(1)}$, where

$$D_{F_\alpha}^{(1)} = \overline{\{u_\mu \colon \text{a potential in } D,\ S_\mu \subset F_\alpha\}}.$$

Then $u'_{1,\alpha}$ is non-negative.[15] Furthermore we put

[15] Similarly as in [2], p. 214, we obtain the following result: $u^*(x) \geq 0$ ppp on the spectrum of u implies $u \geq 0$. Cf. [5].

$$u_{1,\alpha} = u_1 - u'_{1,\alpha}.$$

By the above assertion,

$$(u_{1,\alpha}, u_2) = -2\iint u^*_{1,\alpha}(x)u^*_2(y)d\sigma(x, y).$$

The net $(u'_{1,\alpha})$ tends to 0, and hence the net $(u_{1,\alpha})$ tends strongly to u_1 in D. Hence we can choose a subsequence (u_{1,α_n}) of $(u_{1,\alpha})$ such that (u_{1,α_n}) converges strongly to u_1. By Fatou's lemma, we have

$$\iint u^*_1(x)u^*_2(y)d\sigma(x, y) \leq \varliminf_{n\to\infty} \iint u^*_{1,\alpha_n}(x)u^*_2(y)d\sigma(x, y)$$

$$= \varliminf_{n\to\infty} -\frac{1}{2}(u_{1,\alpha_n}, u_2) = -\frac{1}{2}(u_1, u_2).$$

On the other hand, since $u^*_1(x) - u^*_{1,\alpha}(x) \geqq 0$ ppp in X for any $\alpha \in I$,

$$\iint u^*_1(x)u^*_2(y)d\sigma(x, y) \geqq \iint u^*_{1,\alpha}(x)u^*_2(y)d\sigma(x, y).$$

Consequently we obtain

$$(u_1, u_2) = -2\iint u^*_1(x)u^*_2(y)d\sigma(x, y)$$

This completes the proof.

Applying this theorem, we obtain the following corollary.

Let F be a closed set in the product space $X \times X$. The x-section F_x of F means the projection $\{x\} \times X \cap F$ to X, and for an arbitrary subset A of X, the A-section F_A means the union $\bigcup_{x\in A} F_x$.

COROLLARY 1. *Let D be a Dirichlet space on X, and let σ be the singular measure of D. Given a symmetric closed set F in $X \times X$ containing the diagonal set δ of $X \times X$, the following two conditions are equivalent.*

(1. 1) *For any pure potential u_μ in D and any open set ω contained in CS_μ, let $u_{\mu'}$, be the balayaged potential of u_μ to ω. Then*

$$S_{\mu'} \subset F_{C\omega} \cap \overline{\omega}.$$

(1. 2) $$S_\sigma \subset F.$$

In the preceding paper [4], we proved this result in the case that F is regular, i.e., F_x is compact for any $x \in X$ and the point-to-set map: $x \longrightarrow F_x$

is continuous. Let's prove this corollary. First we shall prove the implication (1. 1) $\Rightarrow$ (1. 2). Suppose that $S_\sigma \not\subset F$. Then there exist two functions f_1 and f_2 in $C_K^+ \cap D$ such that

$$S_{f_1} \cap F_{S_{f_2}} = \phi, \quad S_{f_2} \cap F_{S_{f_1}} = \phi$$

and

$$\iint f_1(x) f_2(y) d\sigma(x, y) > 0.$$[16]

Hence there exists a pure potential u_μ in D such that $S_\mu \subset S_{f_1}$ and

$$\iint (u_\mu^*(x) - u_{\mu'}^*(x)) f_2(y) d\sigma(x, y) > 0,$$

where $u_{\mu'}$ is the balayaged potential of u_μ to CS_{f_1}. On the other hand, since

$$S_{(u_\mu - u_{\mu'})} \cap S_{f_2} = \phi,$$

we have

$$2\iint (u_\mu^*(x) - u_{\mu'}^*(x)) f_2(y) d\sigma(x, y) = \int f_2(x) d\mu'(x) = 0,$$

because

$$S_{\mu'} \subset F_{S_{f_1}}$$

by our assumption. This is a contradiction. The proof of the implication (1. 2) $\Rightarrow$ (1. 1) is evident by the fact that $u_\mu(x) - u_{\mu'}(x) = 0$ a.e. in ω and Theorem 1. This completes the proof.

In order to characterize the absolute continuity of balayaged measures, first we give the following definition.

DEFINITION 5. Let σ be the singular measure of a Dirichlet space D. We say that the projection of σ to X is absolutely continuous for ξ if for any compact set K in X, the positive measure σ_K in CK is absolutely continuous for ξ.

Remark. If σ is absolutely continuous for $\xi \times \xi$, the projection of σ to

16) Cf. [4], Lemma 6.

X is absolutely continuous for ξ. But the converse is not valid. We can easily construct a counter example.

Another corollary of Theorem 1 is the following

COROLLARY 2. *Let D be a Dirichlet space on X and σ be the singular measure of D. The following two conditions are equivalent.*

(2. 1) *For any pure potential u_μ in D and any open set ω contained in CS_μ, let $u_{\mu'}$ be the balayaged potential of u_μ to ω. Then the restriction of μ' to ω is absolutely continuous for ξ.*

(2. 2) *The projection of σ to X is absolutely continuous for ξ.*

Proof. First we shall prove the implication (2. 1) $\Rightarrow$ (2. 2). For a compact set K in X, it is sufficient to prove that the positive measure σ_K is absolutely continuous for ξ in any open set ω such that $\bar{\omega} \subset CK$. We take another open set ω_1 in X such that

$$K \subset \omega_1, \ \overline{\omega_1} \cap \bar{\omega} = \phi \,.$$

Let $u_{\mu_1-\mu_0}$ be the condensor potential with respect to ω_1 and ω. By Theorem 1, for any f in C_K^+ with support in ω, we have

$$\int f \ d\sigma_K \leq \iint f(x) u^*_{\mu_1-\mu_0}(y) d\sigma(x, y) = \frac{1}{2} \int f \ d\mu_0 \,.$$

That is, the inequality $\sigma_K \leq \frac{1}{2} \mu_0$ holds in ω. Since u_{μ_1} is contained in D and μ_0 is the balayaged measure of μ_1 to ω, we obtain that σ_K is absolutely continuous for ξ in ω.

Next we shall prove the converse. First suppose that $C\omega$ is compact in X. By Theorem 1, the restriction $\mu'^{(1)}$ of μ' to ω satisfies the following:

$$\int f \ d\mu'^{(1)} = 2 \iint f(x) \, (u^*_\mu(y) - u^*_{\mu'}(y)) d\sigma(x, y)$$

for any f in C_K with support in ω. Hence it is evident that the condition (2. 1) is satisfied if $u^*_\mu(x) - u^*_{\mu'}(x)$ is bounded. In the general case, we put

$$u_n(x) = \inf (u_\mu(x) - u_{\mu'}(x), n).$$

Then u_n is in D. By our assumption, for any compact set K in X such that $\xi(K) = 0$ and $K \subset \omega$,

$$\int_K\int u_n^*(x)d\sigma(x, y) = 0.$$

Making n tend to infinity, we obtain

$$\int_K\int (u_\mu^*(x) - \mu_{\mu'}^*(x))d\sigma(x, y) = 0,$$

and hence $\mu'(K) = 0$. That is, $\mu'^{(1)}$ is absolutely continuous for ξ. Next we shall prove the case that ω is general. We take a decreasing net $(\omega_\alpha)_{\alpha \in I}$ of open sets such that $C\omega_\alpha$ is compact in X for any $\alpha \in I$ and it tend to ω. Let $u_{\mu'_\alpha}$ be the balayaged potential of u_μ to ω_α. Then the positive measure $\mu_\alpha'^{(1)}$ is absolutely continuous for ξ. Since the net $(u_{\mu'_\alpha})$ is non-decreasing and converges strongly to $u_{\mu'}$, there exists a subsequence $(u_{\mu'_{\alpha_n}})$ of $(u_{\mu'_\alpha})$ which is non-decreasing and converges strongly to $u_{\mu'}$ as $n \longrightarrow +\infty$. Similarly as the above calculation, we obtain that $\mu'^{(1)}$ is absolutely continuous for ξ.
This completes the proof.

4. Second main theorems

In this section, first we shall examine some properties of equilibrium measures and equilibrium potentials in a Dirichlet space.[17] We shall prove the following lemmas.

LEMMA 4. *Let D be a Dirichlet space on X. For an open set ω in X, the equilibrium potential u_ν of ω exists in D if cap$(\omega) < +\infty$.*

Proof. By the definition of the capacity, the set

$$E_\omega = \{u \in D;\ u(x) \geq 1 \text{ a.e. in } \omega\}$$

is non-empty and closed convex subset of D. Similarly as Beurling & Deny [2], a unique element whose norm is minimum in E is the equilibrium potential of ω.

LEMMA 5. *Let D be a Dirichlet space on X. For two open sets ω_1 and ω_2*

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17) Let D be a Dirichlet space on X. Beurling and Deny [2] showed that for any relatively compact open set ω, there exists a pure potential u_ν in D such that $0 \leq u_\nu \leq 1$, $u_\nu = 1$ a.e. in ω and $S_\nu \subset \overline{\omega}$. This potential u_ν is called the equilibrium potential of ω and this positive measure ν is called the equilibrium measure of ω.

in X such that $\omega_1 \subset \omega_2$ and $cap(\omega_2) < +\infty$, let u_{ν_1} and u_{ν_2} be the equilibrium potentials of ω_1 and ω_2, respectively. Then, for any Borel set A contained in ω_1,

$$\nu_1(A) \geqq \nu_2(A).$$

Proof. It is sufficient to prove that for any f in $C_K^+ \cap D$ with support in ω_1,

$$\int f \ d\nu_1 \geqq \int f \ d\nu_2,$$

because $C_K^+(\omega_1) \cap D$ is dense in $C_K^+(\omega_1)$.[18] Using the domination theorem, we obtain that

$$u_{\mu_2} \geqq u_{\mu_1} \text{ and } S_{(u_{\mu_2} - u_{\mu_1})} \subset C\omega_1.$$

Then by Theorem 1, we have

$$\int f \ d\mu_1 - \int f \ d\mu_2 = 2 \iint f(x)(u_{\mu_2}^*(y) - u_{\mu_1}^*(y)) d\sigma(x, y) \geqq 0.$$

This completes the proof.

By Lemma 4, for any open set ω in X, there exists a positive measure ν supported by $\bar{\omega}$ such that for any net (ω_α) of relatively compact open sets contained in ω tending to ω, the equilibrium measure ν_α of ω_α converges vaguely to ν. We say that this positive measure ν is the equilibrium measure of ω. Similarly as the above, we obtain the following

LEMMA 5′. *Let D be a Dirichlet space on X. For two open sets ω_1 and ω_2 such that $\omega_1 \subset \omega_2$ ($cap(\omega_2)$ is finite or not), let ν_i be the equilibrium measure of ω_i for $i = 1, 2$. Then for any Borel set A contained in ω_1,*

$$\nu_1(A) \geqq \nu_2(A).$$

This follows immediately from the above lemma. By the above two lemmas, we obtain the following corollary.

COROLLARY 3. *Let D be a Dirichlet space on X. Suppose that for any relatively compact open set ω in X, the equilibrium measure ν of ω is absolutely continuous for ξ. Then, for any open set ω in X, the equilibrium measure ν of ω*

18) Because the closure of $C_K(\omega_1) \cap D$ by the norm of D is a Dirichlet space on ω_1. Cf. [5].

is absolutely continuous for ξ. Especially the equilibrium measure of X is absolutely continuous for ξ.

Similarly as in Theorem 1, we obtain the following theorem.

THEOREM 2. *Let D be a Dirichlet space on X, and let ν, σ be the equilibrium measure of X, the singular measure of D, respectively. For an open set ω in X with $cap(\omega) < +\infty$, let μ be the equilibrium measure of ω and $\mu^{(1)}$ be the restriction of μ to ω. Then*

$$\int f\ d\mu^{(1)} = 2\iint f(x)(u_\mu^*(x) - u_\mu^*(y))\,d\sigma(x,y) + \int f\ d\nu$$

for any f in C_K with support in ω. Furthermore, for any couple u_1 and u_2 in D such that $u_2(x) = c$ a.e. in some neighborhood of S_{u_1},

$$(u_1, u_2) = c\int u_1^*(x)d\nu(x) + 2\iint u_1^*(x)(u_2^*(x) - u_2^*(y))d\sigma(x,y)\,,$$

where c is constant.

In order to prove this theorem, we need the following lemma.

LEMMA 6. *Let D be a Dirichlet space on X. Given a relatively compact open set ω in X, let u_μ be the equilibrium potential of ω. Then there exist unrefinement u_μ^* of u_μ such that the equality $u_\mu^*(x) = 1$ holds everywhere in ω.*

Proof. It is sufficient to prove that for any open set ω_1 such that $\overline{\omega_1} \subset \omega$, the equality $u_\mu^*(x) = 1$ holds everywhere in ω_1. By Lemma 1, there exists a sequence (f_n) in $C_K \cap D$ such that (f_n) converges strongly to u_μ as $n \longrightarrow +\infty$, $0 \leq f_n \leq 1$ and $f_n(x) = 1$ in ω_1 for any n. We may assume that

$$\sum_{n=1}^{\infty} 4^n \| f_{n+1} - f_n \|^2 < +\infty\,.$$

By the definition of the refinement, the sequence (f_n) is uniformly convergent to u_μ^* in CE_k, where

$$E_k = \bigcup_{n=k}^{\infty} E_n' = \bigcup_{n=k}^{\infty} \{x \in X;\ |f_{n+1}(x) - f_n(x)| > 1/2^n\}$$

for any integer n. The inclusion $\omega_1 \subset CE_k$ exists for any integer n, and hence we obtain that u_μ^* is continuous in ω_1 and the equality $u_\mu^*(x) = 1$ holds everywhere in ω_1. This completes the proof.

Remark. The above lemma is valid for any open set ω with finite capacity.

Proof of Theorem 2. Let ω be the open set in our theorem. For any f in C_K^+ supported in ω, let σ_f be a positive measure in CS_f similarly as in the proof of Theorem 1. By Lemma 1 and Theorem 1, the function $1-u_\mu^*(x)$ is σ_f-integrable. Let (f_n) be a sequence in $C_K \cap D$ such that (f_n) converges strongly to u_μ in D as $n \longrightarrow +\infty$, $0 \leq f_n(x) \leq 1$ and $f_n(x)=1$ in some neighborhood of S_f for any n. Then by Beurling-Deny's representation theorem, we have

$$(f_n, f) = \int f(x) d\nu(x) + \iint (f(x)-f(y))(f_n(x)-f_n(y)) d\sigma(x,y)$$
$$= \int f(x)\, d\nu(x) + 2\int f(x)(1-f_n(y)) d\sigma(x,y)\,.$$

By Lebesgue's bounded convergence theorem, we obtain that

$$\int f(x) d\mu(x) = (u_\mu, f)$$
$$= \int f(x) d\nu(x) + 2\iint f(x)(1-u_\mu^*(y)) d\sigma(x,y)$$
$$= \int f(x) d\nu(x) + 2\iint f(x)(u_\mu^*(x)-u_\mu^*(y)) d\sigma(x,y)\,.$$

From this equality, we obtain the first required equality. Let's prove the second part of our theorem. We may assume that u_2^* is equal to c everywhere in some neighborhood ω of S_{u_1}. Similarly as the proof of Theorem 1 and the proof of the first part of our theorem, we obtain

$$(u_1, u_2) = c\int u_1^*(x)\, d\nu(x) + 2\iint u_1^*(x)(u_2^*(x)-u_2^*(y)) d\sigma(x,y)\,.$$

In the above equality, the ν-measurablity of u_1^* is followed from Lemma 5. This completes the proof.

As an application of the above theorem, we obtain the following theorem. This result is more precise than in [5].

THEOREM 3. *Let D be a Dirichlet space on X and ν be the equilibrium measure of X. For a pure potential u_μ in D such that $\int d\mu < +\infty$ and an open set ω in X such that cap $(C\omega) < +\infty$, let $u_{\mu'}$ be the balayaged potential of u_μ to ω. Then*

$$\int (u_\mu^*(x) - u_{\mu'}^*(x))\,d\nu(x) = \int d\mu - \int d\mu'.$$

Furthermore, for a non-decreasing net $(K_\alpha)_{\alpha\in I}$ of compact sets in X tending to X, let $u_{\mu'_\alpha}$ be the balayaged potential of u_μ to $\omega_\alpha = CK_\alpha$. Then the net $\left(\int d\mu'_\alpha\right)_{\alpha\in I}$ is non-increasing and

$$\int u_\mu^*(x)\,d\nu(x) = \int d\mu - a_\mu,$$

where

$$a_\mu = \lim_{\alpha\in I} \int d\mu'_\alpha.$$

Before we give the proof of this theorem, we remark the following

COROLLARY 3. *Let the notations be the same as in the above theorem. For any pure potential u_μ in D with $\int d\mu < +\infty$ and any open set ω in X such that cap$(C\omega) < +\infty$, $\int d\mu = \int d\mu'$ (resp. $\int d\mu > \int d\mu'$) if and only if $\nu = 0$ (resp. ν is everywhere dense in X).*

The proof of this corollary is immediate from the above theorem. This corollary was partially proved in [5].

Proof of Theorem 3. First we shall prove the case that $C\omega$ is compact in X. We take a non-decreasing net $(\omega_\alpha)_{\alpha\in I}$ of relatively compact open sets in X such that $\omega_\alpha \supset C\omega$ for any $\alpha \in I$ and the net (ω_α) tends to X. Then, for any $\alpha \in I$, we have

$$(u_\mu - u_{\mu'}, u_{\mu_\alpha}) = \int (u_\mu^*(x) - u_{\mu'}^*(x))\,d\mu_\alpha(x)$$

$$= \int (u_\mu^*(x) - u_{\mu'}^*(x))\,d\nu(x) + 2\iint (u_\mu^*(x) - u_{\mu'}^*(x))(u_{\mu_\alpha}^*(x) - u_{\mu_\alpha}^*(y))\,d\sigma(x, y)$$

$$= \int (u_\mu^*(x) - u_{\mu'}^*(x))\,d\nu(x) + 2\iint (u_\mu^*(x) - u_{\mu'}^*(x))(1 - u_{\mu_\alpha}^*(y))\,d\sigma(x, y).$$

Since the net $(1 - u_{\mu_\alpha}^*)_{\alpha\in I}$ is non-increasing and tends to 0 in X, the second part of the last hand converges non-increasingly to 0. Hence we have

$$\lim_{\alpha\in I} (u_\mu - u_{\mu'}, u_{\mu_\alpha}) = \int (u_\mu^*(x) - u_{\mu'}^*(x))\,d\nu(x).$$

On the other hand, the net $(u_{\mu_\alpha})_{\alpha\in I}$ tending non-decreasingly to 1 in X, we obtain that

$$\lim_{\alpha\in I}(u_\mu, u_{\mu_\alpha}) = \int d\mu, \quad \lim_{\alpha\in I}(u_{\mu'}, u_{\mu_\alpha}) = \int d\mu'.$$

That is,

$$\int (u_\mu^*(x) - u_{\mu'}^*(x))d\nu(x) = \int d\mu - \int d\mu'.$$

Next we shall show the case that $C\omega$ is general. We take a decreasing net $(\omega_\alpha)_{\alpha\in I}$ of open sets such that $C\omega_\alpha$ is compact and $\omega_\alpha \supset \omega$ for any $\alpha \in I$, and that the net tends to ω. Let ν' be the restriction of ν to some fixed open set containing $C\omega$ with finite capacity. By Lemma 5, a potential $u_{\nu'}$ exists in D. Hence

$$\begin{aligned}\int (u_\mu^*(x) - u_{\mu'_\alpha}^*(x))d\nu(x) &= \int (u_\mu^*(x) - u_{\mu'_\alpha}^*(x))d\nu'(x)\\ &= (u_\mu - u_{\mu'_\alpha}, u_{\nu'}) \longrightarrow (u_\mu - u_{\mu'}, u_{\nu'})\\ &= (u_\mu^*(x) - u_{\mu'}^*(x))d\nu(x),\end{aligned}$$

because the net $(u_\mu - u_{\mu'_\alpha})_{\alpha\in I}$ converges strongly to $u_\mu - u_{\mu'}$, in D, where μ'_α is the balayaged measure of μ to ω_α. On the other hand, similarly as the proof of theorem 1 in [5],

$$\lim_{\alpha\in I}\int d\mu'_\alpha = \int d\mu'.$$

Thus the first part of our theorem is proved and the second part can be obtained by the usual limiting process. This completes the proof.

Evidently we know that a_μ vanishes for any pure potential u_μ in D when X is of finite capacity. But we don't know the condition which a_μ vanishes. Finally we remark that similar theorems as Theorem 1 and Theorem 3 hold for a condensor measure.

6. Special Dirichlet spaces

First, according to Beurling and Deny [2], we define a special Dirichlet space.

DEFINITION 4. A Dirichlet space $D = D(X; \xi)$ is said to be special if X

is a locally compact abelian group, ξ is the Haar measure of X and the following condition is satisfied:

(D. 4) For any u in D and any x in X, the function $U_x u$ is in D and $\| U_x u \| = \| u \|$, where $U_x u$ is the function obtained from u by the translation x (i.e., $U_x u(y) = u(y-x)$).

In the case that D is a special Dirichlet space on X, Proposition 1 reads as follows:

PROPOSITION 3. *Let D be a special Dirichlet space on X. Then there exists a positive constant c, a local form $N(\cdot,\cdot)$ on $C_K \cap D$ and a positive symmetric measure σ' in $X-\{0\}$ such that*

$$(f,g) = c\int fg\, d\xi + N(f,g) + \iint (f(x+y)-f(x))(g(x+y)-g(x))d\sigma'(y)d\xi(x)$$

for any pair f and g in $C_K \cap D$. The above representation is unique.

Proof. By Proposition 1, there exist a positive measure ν in X and a positive symmetric measure σ in $X \times X - \delta$ such that

$$(f,g) = \int fg\, d\nu + N(f,g) + \iint (f(x)-f(y))(g(x)-g(y))d\sigma(x,y)$$

for any pair f and g in $C_K \cap D$. We take an increasing net (K_α) of compact sets in X which tends to X and an increasing net (g_α) of $C_K \cap D$ such that $0 \leq g_\alpha(x) \leq 1$, $g_\alpha(x)=1$ on K_α for any $\alpha \in I$ and the net (g_α) tends to 1 in X. We know the existence of this function g_α by the condition (D. 2) and (D. 3). For any f in $C_K \cap D$ and any x in X,

$$\lim_{\alpha\in I}(f,g_\alpha) = \int f\, d\nu, \quad \lim_{\alpha\in I}(U_x f, U_x g_\alpha) = \int U_x f\, d\nu,$$

and hence

$$\int f\, d\nu = \int U_x f\, d\nu.$$

Consequently $d\nu = c d\xi$, where c is a non-negative constant. Next we shall examine the singular measure σ of D. For any f and g in C_K^+ such that the support S_{f*g} of the convolution $f*g$ doesn't contain the origin 0 of X, the transformation

$$f*g \longrightarrow \iint f(x)g(y)d\sigma(x,y)$$

is positive linear. In fact, suppose that $f_1*g_1 \leq f_2*g_2$. For any h in C_K^+ such that $S_h \cap S_{f_2*g_2} = \phi$,

$$\iint f_1(x)g_1*h(y)\,d\sigma(x,y) = \iint f_1*g_1(x)h(y)d\sigma(x,y)$$
$$\leq \iint f_2*g_2(x)h(y)d\sigma(x,y) = \iint f_2(x)g_2*h(y)d\sigma(x,y).$$

Making h vaguely tend to the unit measure ε at 0, we obtain

$$\iint f_1(x)g_1(y)d\sigma(x,y) \leq \iint f_2(x)g_2(y)d\sigma(x,y).$$

The well-definedness of the above transformation is evidently followed by our assumption, i.e.,

$$\iint f(x)g(y)d\sigma(x,y) = \iint f(x+x_0)g(y+x_0)d\sigma(x,y)$$

for any x_0 in X. Since the totality of such functions $f*g$ is dense in $C_K^+(X-\{0\})$, there exists a positive measure σ' in $X-\{0\}$ such that

$$\int f*g(x)\,d\sigma'(x) = \iint f(x)g(y)\,d\sigma(x,y)$$

for any pair f and g in C_K^+ such that $S_f \cap S_g = \phi$. The symmetricity of σ' follows from the simmetricity of σ. Consequently

$$\iint f(x+y)g(x)d\sigma'(y)d\xi(x) = \iint f(x)g(y)d\sigma(x,y).$$

The uniqueness of the singular measure of D follows from the equality

$$\iint (f(x+y)-f(x))(g(x+y)-g(x))d\sigma'(y)d\xi(x)$$
$$= \iint (f(x)-f(y))(g(x)-g(y))d\sigma(x,y)$$

for any pair f and g in $C_K \cap D$, and hence the proof is completed.

In this case, we call the above positive measure σ' the singular measure of D. Furthermore the local form $N(\cdot,\cdot)$ satisfies the following condition: $N(f,g) = N(U_x f, U_x g)$ for any pair f, g in $C_K \cap D$ and any x in X. Hence the above proof is one of Levy-Khinchine's theorem.[19] Then we obtain the following corollary.

[19] Cf. [2], [3], and [4].

COROLLARY 4. *Let D be a special Dirichlet space on X. The above positive constant c doesn't vanish if and only if $D \subset L^2$ and the mapping: $f \longrightarrow f$ on D into L^2 is continuous.*

The proof is evident by the above proposition. As another application of the above proposition, we obtain the following

THEOREM 4. *Let D be a special Dirichlet space on X, and let σ be the singular measure of D. For any pure potential u_μ in D and any open set ω contained in CS_μ, let $u_{\mu'}$ be the balayaged potential of u_μ to ω, and let $\mu'^{(1)}$ be the restriction of μ' to ω. Then $\mu'^{(1)}$ is absolutely continuous for ξ.*

Proof. By Theorem 1,

$$\int f\, d\mu' = -(u_\mu - u_{\mu'}, f)$$

$$= 2\iint (u^*_\mu(x+y) - u^*_{\mu'}(x+y))f(x)d\sigma(y)d\xi(x)$$

for any f in $C_K \cap D$ with support in ω. Now the function

$$f_{\mu,\omega}(x) = 2\int (u^*_\mu(x+y) - u^*_{\mu'}(x+y))d\sigma(y)$$

is a locally summable function in ω, and hence $\mu'^{(1)}$ is absolutely continuous for ξ. This completes the proof.

Similarly as in Theorem 4, we obtain that $\mu'^{(1)}$ is a function of class C^∞ in ω if and only if σ is a function of class C^∞ in $R^n - \{0\}$, where D is a special Dirichlet space on the n-dimensional Euclidean space $R^n (n \geq 1)$. (Cf. [7])

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