A REMARK ON THE CONTINUITY OF THE DUAL PROCESS

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§ 1. Introduction

Let S be a locally compact (not compact) Hausdorff space satisfying the second axiom of countability and let $\mathscr{B}$ be the σ-field of all Borel subsets of S and let $\mathscr{A}$ be the σ-field of all the subsets of S which, for each finite measure μ defined on $(S, \mathscr{A})$, are in the completed σ-field of $\mathscr{B}$ relative to μ. We denote by C_0 the Banach space of continuous functions vanishing at infinity with the uniform norm and B_k the space of bounded $\mathscr{A}$-measurable functions with compact support in S.

Let $X = (x_t, \zeta, M_t, P_x)$ be a standard process [1] on S. Let us set

$$G_\alpha(x, B) = \int_0^{+\infty} e^{-\alpha t} P_x(x_t \in B) dt, \quad \alpha \geqq 0,$$

where B is a $\mathscr{A}$-measurable set and set $G_\alpha f(x) = \int_S f(y) G_\alpha(x, dy)$ for each bounded $\mathscr{A}$-measurable function f. We say that the standard process X satisfies *the regularity condition with respect to a locally finite measure* $m(dx)$, if the following holds :

(**i**) $G_0(x, K)$ *is bounded on every compact set when* K *is compact.*

(**ii**) $G_\alpha(x, K)$ *is absolutely continuous with respect to* $m(dx)$ *for each* $\alpha \geqq 0$ *and for each* $x \in S$.

(**iii**) $G_\alpha f(x)$ *is finite and contiuous for each* $f \in B_k$.

We say that two standard processes on S $X = (x_t, \zeta, M_t, P_x)$ and $\hat{X} = (\hat{x}_t, \zeta, \hat{M}_t, \hat{P}_x)$ are *in the relation of duality with respect to a locally finite measure* $m(dx)$, if each of them satisfies the regularity condition with respect to $m(dx)$ and it holds that for each $\alpha \geqq 0$

$$\int_S g(x) G_\alpha f(x) m(dx) = \int_S f(x) \hat{G}_\alpha g(x) m(dx), \quad f, g \in B_k,$$

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1) For the definition see H. Kunita and T. Watanabe [3].

where

$$\hat{G}_\alpha(x,B)=\int_0^{+\infty}e^{-\alpha t}\hat{P}_x(\hat{x}_t\in B)dt,\quad \hat{G}_\alpha f(x)=\int_S f(y)\hat{G}_\alpha(x,dy).$$

Our aim is to show the following theorem.

THEOREM. *Let X and $\hat{X}$ be standard processes on S in the relation of duality with respect to a locally finite measure $m(dx)$. Further let us suppose that the semi-group $\{T_t\}_{t\geq 0}$ of X and $\{\hat{T}_t\}_{t\geq 0}$ of $\hat{X}$ are strongly continuous operators on C_0. Then the process $\hat{X}$ is a continuous process, if and only if X is a continuous process.*

In the author's previous paper [2], we studied the process X connected with the following strictly elliptic differential operator on the ball Ω of R^d $(d\geq 3)$

$$D=\sum_{i,j=1}^d a_{ij}(x)\frac{\partial^2}{\partial x_i\partial x_j}+\sum_{i=1}^d a_i(x)\frac{\partial}{\partial x_i}$$

and the process $\hat{X}$ connected with the formal adjoint operator

$$\hat{D}=\sum_{i\cdot j=1}^d\frac{\partial^2}{\partial x_i\partial x_j}(a_{ij}(x)\cdot)-\sum_{i=1}^d\frac{\partial}{\partial x_i}(a_i(x)\cdot),$$

where D is assumed to satisfy the condition (L), that is,

$$-\int_\Omega Dv(x)dx\geq 0$$

holds for every non-negative C^2-functions v with compact support in Ω and the coefficients $\{a_{ij},\ i\cdot j=1,2,\cdots,d\}$ and $\{a_i\ ;\ i=1,2,\cdots,d\}$ are bounded and uniformly Hölder continuous such that $a_{ij}=a_{ji}$.

By using the above theorem and the proposition in § 4, we can show the following

COROLLARY. *The process $\hat{X}$ connected with the operator $\hat{D}$ which is mentioned above is a continuous process.*

§ 2. Resolvent kernels.

Throughout this section we use the notations in H. Kunita and T. Watanabe [3].

A function $R_\alpha(x,A)$, defined for $\alpha>O$, x of S and A of $\mathscr{A}$, is said to be *a resolvent kernel* if it satisfies the following conditions (**a**)-(**d**). (**a**).

For each $\alpha > O$ and x of $S, R_\alpha(x, \cdot)$ is a locally finite measure. (**b**). Let f be a bounded $\mathscr{A}$-measurable function of compact support, then $R_\alpha f$ is $\mathscr{A}$-measurable and bounded on every compact set, where we write $R_\alpha f$ for $\int f(y) R_\alpha(\cdot, dy)$. (**c**). The resolvent equation $R_\alpha f - R_\beta f + (\alpha - \beta) R_\alpha R_\beta f = O$ is satisfied and (**d**). $\lim\limits_{\alpha \to +\infty} R_\alpha f(x) = O$ for each x and for each bounded $\mathscr{A}$-measurable function f of compact support.

Since $R_\alpha(x, A) \geqq R_\beta(x, A)$ for each A of $\mathscr{A}$

$$R_0(x, A) = \lim{}_{\alpha \to 0} R_\alpha(x, A)$$

exists for each A of $\mathscr{A}$ and defines a measure on $\mathscr{A}$.

Let $\{R_\alpha(x, A)\}$ be a resolvent kernel and m, a measure defined over $(S, \mathscr{A})$.

$\{R_\alpha(x, A)\}$ is said to be *dominated by* m if, for each $\alpha > O$, $R_\alpha(x, A)$ satisfies the condition (ii) of § **1.**

$\{R_\alpha(x, A)\}$ is said to be *integrable* if $R_0(\cdot, A)$ satisfies the condition (i) of § **1.**

$\{R_\alpha(x, A)\}$ is said to be *regular* if, for each continuous function f of compact support, $\alpha R_\alpha f$ converges boundedly on every compact set to f as $\alpha \to +\infty$.

A resolvent kernel $\{\hat{R}_\alpha(x, A)\}$ is called the *co-resolvent kernel of* $\{R_\alpha(x, A)\}$ *with respect to* $m(dx)$ if, for each f, g of B_k and for each $\alpha > O$

$$\int_S f(x) R_\alpha g(x) dx = \int_S g(x) \hat{R}_\alpha f(x) dx.$$

A non-negative $\mathscr{A}$-measurable function u is said to be (R, α)-*excessive* if $\beta R_{\alpha+\beta} u \leqq u$ for all $\beta > 0$ and if $\lim\limits_{\beta \to +\infty} \beta R_{\alpha+\beta} u = u$.

Given a number $\alpha \geqq O$, a jointly $(= \mathscr{A} \times \mathscr{A})$ measurable function $R_\alpha(x, y)$ is said to be *the potential kernel of exponent* α if the following conditions are satisfied : (**a**) $R_\alpha(x, dy) = R_\alpha(x, y) m(dy)$; (**b**) $\hat{R}_\alpha(y, dx) = R_\alpha(x, y) m(dx)$; (**c**) $R_\alpha(\cdot, y)$ is (R, α)-excessive for each fixed y and (**d**) $R_\alpha(x, \cdot)$ is $(\hat{R}, \alpha)$-excessive for each fixed x.

The following lemma is Theorem 1 in H. Kunita and T. Watanabe [3].

LEMMA 1. (H. Kunita and T. Watanabe). *Let* $\{R_\alpha(x, A)\}$ *be a resolvent kernel and* $\{\hat{R}_\alpha(x, A)\}$ *be the co-resolvent kernel of* $\{R_\alpha(x, A)\}$. *Assume that*

$\{R_\alpha(x, A)\}$ and $\{\hat{R}_\alpha(x, A)\}$ are dominated by the locally finite measure $m(dx)$. Then there is a unique potential kernel of exponent α for $\alpha \geqq O$.

§ 3. Fundamental lemmas.

Throughout this section we treat two standard processes on S $X = (x_t, \zeta, M_t, P_t)$ and $\hat{X} = (\hat{x}_t, \hat{\zeta}, \hat{M}_t, \hat{P}_x)$ which are in the relation of duality with respect to a locally finite measure $m(dx)$ without special mentioning.

Evidently $\{G_\alpha(x, A)\}$ is a resolvent kernel and $\{\hat{G}_\alpha(x, A)\}$ is the co-resolvent kernel which are dominated by $m(dx)$ by the condition (ii) in § 1. Hence the following lemma is a direct consequence of lemma 1.

LEMMA 2. *There is a unique potential kernel $G_\alpha(x, y)$ of exponent α for all $\alpha \geqq 0$.*

Let E be an analytic set in S and let us set $\sigma_E = inf(t > 0,\ x_t \in E)$, $= +\infty$ if the set $(t > O,\ x_t \in E)$ is empty.

The next lemma plays an essential role in this paper, which is first shown by G.A. Hunt [1] under his assumptions (F) and (G) and P.A. Meyer [4] has next shown it under a little different assumption. Our case follows directly from P.A. Meyer's result.

LEMMA 3. *Suppose that the semi-groups $\{T_t\}_{t \geqq 0}$ and $\{\hat{T}_t\}_{t \geqq 0}$ of the processes X and $\hat{X}$ respectively are the strongly continuous operators on C_0. Then, for each analytic set E in S, it holds that*

$$\int_S G_0(x, z)\hat{P}_y(\hat{x}_{\sigma_E} \in dz) = \int_S G_0(z, y)P_x(x_{\sigma_E} \in dz)$$

for eaxch x and y in S.

Proof. Let us note that the notion "*(G, α)-excessive*" is equivalent to the notion "*α-excessive with respect to $\{T_t$*" 2). Then the semi-group $\{T_t\}$ of X and $\{\hat{T}_t\}$ of $\hat{X}$ are *in the relation of duality in Meyer' s sense* by Lemma 2. Therefore, for each $\alpha > 0$ it holds that

$$\int_S G_\alpha(z, y)P_E^\alpha(x, dz) = \int_S G_\alpha(x, z)\hat{P}_E^\alpha(y, dz),$$

where $P_E^\alpha(x, dz) = E_x(e^{-\alpha\sigma_E}\ ;\ x_{\sigma_E} \in dz)$, $\hat{P}_E^\alpha(y, dz) = \hat{E}_y\,(e^{-\alpha\sigma_E}\ ;\ \hat{x}_{\sigma_E} \in dz)$. Noting that $\lim_{\alpha \to 0} G_\alpha(x, y) \uparrow G_0(x, y)$, we have

2) We say that a non-negative $\mathscr{A}$-measurable function $u(x)$ is *α-excessive with respect to $\{T_t\}$*, if $E_x(e^{-\alpha t}u(x_t),\ t < \zeta) \leqq u(x)$ for each $t > 0$ and $\lim_{t \to 0} E_x(t^{-\alpha t}u(x_t),\ t < \zeta) = u(x)$.

$$\text{(1)} \qquad \int_S G_\alpha(z, y) P_E^\alpha(x, dz) \leqq \int_S G_0(z, y) P_x(x_{\sigma_E} \in dz),$$
$$\int_S G_\alpha(x, z) \hat{P}_E^\alpha(y, dz) \leqq \int_S G_0(x, z) P_y(\hat{x}_{\sigma_E} \in dz).$$

On the other hand, we have for each fixed $\beta > 0$

$$\int_S G_0(z, y) P_E^\beta(x, dz) = \lim_{\alpha \to 0} \int_S G_\alpha(z, y) P_E^\beta(x, dz) \leqq \lim_{\alpha \to 0} \int_S G_\alpha(z, y) P_E^\alpha(x, dz),$$
$$\int_S G_0(x, z) \hat{P}_E^\beta(y, dz) = \lim_{\alpha \to 0} \int_S G_\alpha(x, z) \hat{P}_E^\beta(y, dz) \leqq \lim_{\alpha \to 0} \int_S G_\alpha(x, z) \hat{P}_E^\alpha(y, dz).$$

Hence by tending β to zero we can show that

$$\text{(2)} \qquad \int_S G_0(z, y) P_x(x_{\sigma_E} \in dz) \leqq \lim_{\alpha \to 0} \int_S G_\alpha(z, y) P_E^\alpha(x, dz)$$
$$\int_S G_0(x, z) \hat{P}_y(x_{\sigma_E} \in dz) \leqq \lim_{\alpha \to 0} \int_S G_\alpha(x, z) \hat{P}_E^\alpha(y, dz).$$

From the inegqualities (1) and (2) we can prove the lemma.

Now, let us note that $\{G_\alpha(x, A)\}$ and $\{\hat{G}_\alpha(x, A)\}$ satisfy *the hypothesis* (B) in H. Kunita and T. Watanabe **[3]**, that is, $G_\alpha(x, A)$ is integrable and dominated by a locally finite measure $m(dx)$, $\{\hat{G}_\alpha(x, A)\}$ is regular and $\hat{G}_\alpha f$, $\alpha \geqq O$ is continuous and finite everywhere for each f of B_K. Then, by Theorem 7 in H. Kunita and T. Watanabe **[3]**, Proposition 7. 11 in **[3]** is valid for the processes X and $\hat{X}$. Hence we have the following

LEMMA 4. *If the measures* μ_1 *and* μ_2 *define the same potential, i.e.* $\int_S G_0(x, y) \mu_1(dy) = \int_S G_0(x, y) \mu_2(dy)$, *which is integrable over each compact set, then we have* $\mu_1 = \mu_2$.

§ 4. Proof of the Theorem.

In this section we always treat the processes X and $\hat{X}$ mentioned in the Theorem.

Let us assume that X is a continuous process. Since $m(dx)$ can be considered as a reference measure by the regularity condition, according to Corollary to Theorem 4. 2 in S. Watanabe [5], for the proof of the continuity of the process X, we have only to show that

3) $\{\infty\}$ is adjoined to S and $S \cup \{\infty\}$ denotes the one-point compactification of S. For each function f we set $f(\{\infty\} = 0$.

(3) $$\hat{P}_x(\hat{x}_{\hat{\sigma}_{Q^c}} \in \partial Q \cup \{\infty\}\ ;\ \hat{\sigma}_{Q^c} < +\infty) = \hat{P}_x(\hat{\sigma}_{Q^c} < +\infty),\quad x \in Q$$

for a bounded and non-empty open set Q, where $Q^c = S - Q$. We shall first prove the following equality

(4) $$\int_S G_0(z, x_0) P_x(x_{\sigma_{Q^c}} \in dz) = \int_S G_0(z, x_0) P_x(x_{\sigma_{\partial Q}} \in dz)$$

for each x and $x_0 \in Q$. When x is in the interior of Q, the equality (4) holds by the continuity of the path of X. In case $x \in \bar{Q}^c \cup \partial Q^{reg}$, where $\bar{Q}$ denotes the closure of Q and ∂Q^{reg} denotes the set of all regular points of ∂Q for X, the lef-hand side of (4) equals to $G_0(x, x_0)$. For $x \in \partial Q^{reg}$ it is clear that the right-hand side of (4) equals $G_0(x, x_0)$ too. When $x \in \bar{Q}^c$, by the continuity of the path we have

$$\int_S G_0(z, x_0) P_x(x_{\sigma_{\partial Q}} \in dz) = \int_S G_0(z, x_0) P_x(x_{\sigma_{\bar{Q}}} \in dz),$$

and by lemma 3

$$\int_S G_0(z, x_0) P_x(x_{\sigma_{\bar{Q}}} \in dz) = \int_S G_0(x, z) \hat{P}_{x_0}(\hat{x}_{\sigma_{\bar{Q}}} \in dz).$$

Noting that $P_{x_0}(x_{\sigma_{\bar{Q}}} \in dz) = \delta_{x_0}(dz)$, where $\delta_{x_0}(dz)$ is the Dirac measure at x_0, we have

$$\int_S G_0(z, x_0) P_x(x_{\sigma_{\partial Q}} \in dz) = G_0(x, x_0).$$

Hence the equality (4) holds on $S - (\partial Q - \partial Q^{reg})$. On the other hand we have $m(\partial Q - \partial Q^{reg}) = 0$. Indeed, $G_0(x, \partial Q - \partial Q^{reg}) = 0$ for each $x \in S$, because $\partial Q - \partial Q^{reg}$ is a semi-polar set, therefore $G_\alpha(x, \partial Q - \partial Q^{reg}) = 0$ for each $x \in S$ and $\alpha \geqq O$, because $G_0 \geqq G_\alpha$. Noting that $\lim_{\alpha \to +\infty} \alpha \hat{G}_\alpha f(x) = f(x)$ uniformly for each $f \in C_0$, we can choose *a* function $f \in C_0$ and $\alpha > O$ such that $\hat{G}_\alpha f(x) > \delta > O$ on $\partial Q - \partial Q^{reg}$. Hence it holds that

$$\begin{aligned} O &= \int_S G_\alpha(x, \partial Q - \partial Q^{reg}) f(x) m(dx) = \int_{\partial Q - \partial Q^{reg}} \hat{G}_\alpha f(y) m(dy) \\ &\geqq \delta m(\partial Q - \partial Q^{reg}), \end{aligned}$$

which implies $m(\partial Q - \partial Q^{reg}) = 0$. Therefore the equality (4) holds (m)-almost everywhere. Since the both sides of (4) are (G, O)-excessive, the equality holds everywhere. Applying Lemma 3 to the equality (4), we have

$$\int_S G_0(x,z)\hat{P}_{x_0}(\hat{x}_{\sigma_{Q^c}}\in dz)=\int_S G_0(x,z)\hat{P}_{x_0}(\hat{x}_{\sigma_{\partial Q}}\in dz),\quad x_0\in Q,$$

for each x. Hence by Lemma 4 we have

$$\hat{P}_{x_0}(\hat{x}_{\sigma_{Q^c}}\in dz)=\hat{P}_{x_0}(\hat{x}_{\sigma_{\partial Q}}\in dz),\quad x_0\in Q,$$

which implies (3). We complete the proof.

§ 5. Green function and standard processes in the relation of duality

Let $G(x,y)$ be a Green function on the domain $\Omega\subseteq R^d(d\geqq 3)$ in the sense of [2], p. 46, with the condition (S), i.e.,

$$\frac{C_1}{|x-y|^{d-\alpha}}\geqq G(x,y)\geqq\frac{C_2}{|x-y|^{d-\alpha}},\quad x,y\in K,\ d>\alpha>0,$$

where K is a compact set in Ω and C_1, C_2 are strictly positive constants depending only on K. We say that $G(x,y)$ is *quasi-symmetric*, if $G(x,y)$ is continuous in $\Omega\times\Omega$ except on the diagonal set and both $Gf(x)$ and $\hat{G}f(x)$ maps B_K into C_0, where $Gf(x)=\int_\Omega G(x,y)f(y)dy$ and $\hat{G}f(x)=\int_\Omega \hat{G}(x,y)f(y)dy$, $\hat{G}(x,y)=G(y,x)$ and further G and $\hat{G}$ satisfy the weak principle of the positive maximum.[4)]

Lemma 5. *For a Green function $G(x,y)$, with the condition (S) in Ω in the sense of* [2], *there corresponds a standard process such that*

$$\int_0^{\pm\infty}T_tf(x)dt=Gf(x),\quad f\in B_K.$$

This lemma is shown in [2].

Proposition. *For a quasi-symmetric Green function $G(x,y)$ with the condition (S), there correspond standard processes $X=(x_t,\zeta,M_t,P_x)$ and $\hat{X}=(\hat{x}_t,\hat{\zeta},\hat{M}_t,\hat{P}_x)$ in the relation of duality with respect to Lebesque measure dx such that*

$$\int_0^{+\infty}T_tf(x)dt=Gf(x),\ \int_0^{+\infty}\hat{T}_tf(x)dt=\hat{G}f(x),\quad f\in B_K.$$

Further, let $G_0(x,y)$ be a kernel for $\alpha=0$ which is constructed in Lemma 2 by setting $m(dx)=dx$, then we have

4) We say that a kernel $G(x,y)$ satisfies *the weak principle of the positive maximum* if, for a continuous function f of compact support such that $Gf\geqq 0$, Gf attains its (strictly positive) maximum at a point of S where f is strictly positive.

$$G(x, y) = G_0(x, y).$$

Proof. The existence of such processes follows from Lemma 5. The relation of duality between X and $\hat{X}$ is evident by the definition of G and $\hat{G}$. Hence it is sufficient to show $G_0(x, y) = G(x, y)$. Since $G_0(x, y) = G(x, y)$ holds (m)-almost everywhere for fixed x $(m(dx) = dx)$, we have only to prove $G(x, y)$ is $(\hat{G}, O)$-excessive function of y for each fixed x.[5] For each $f \in B_K$, $\hat{G}f$ is $(\hat{G}, O)$-excessive, therefore $\alpha\hat{G}_\alpha\hat{G}f(x) \leqq \hat{G}f(x)$ for each $\alpha > O$ and hence

$$\int_\Omega \hat{G}(z, y)\alpha\hat{G}_\alpha(x, dz) \leqq \hat{G}(x, y) \tag{5}$$

holds for (m)-almost all y for each fixed x. As $\hat{G}(x, y)$ is continuous in $y(\neq x)$ and $\int_\Omega \hat{G}(z, y)\alpha\hat{G}_\alpha(x, dz)$ is lower semicontinuous in y, the inequality (5) holds everywhere. On the other hand, if we set $\hat{G}_{n,y}(x) = \min\{\hat{G}(x, y) \wedge n\}$, we have

$$\lim_{\alpha\to+\infty}\int_\Omega \hat{G}(z, y)\alpha\hat{G}_\alpha(x, dz) \geqq \lim_{\alpha\to+\infty}\int_\Omega \hat{G}_{n,y}(z)\alpha\hat{G}_\alpha(x, dz) = \hat{G}_{n,y}(x).$$

By tending n to infinity, we have

$$\lim_{\alpha\to+\infty}\int \alpha\hat{G}(z, y)\hat{G}_\alpha(x, dz) \geqq \hat{G}(x, y). \tag{6}$$

The inequalities (5) and (6) introduce $\lim_{\alpha\to+\infty}\int_\Omega \hat{G}(z, y)\alpha\hat{G}_\alpha(x, dz) = \hat{G}(x, y)$, which means $G(x, y)$ is a $(\hat{G}, O)$-excessive function of y for each fixed x. Thus we have proved the Proposition.

Remark. Also we can prove that $G(x, y)$ is a (G, O)-excessive function of x for each fixed y in the same way.

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5) The following proof is due to Prof. T. Watanabe.

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