POSITIVELY INFINITE SINGULARITIES OF A SUPERHARMONIC FUNCTION

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1. Let E be a compact set of logarithmic capacity zero in the complex plane. Then the following is well-known as Evans-Selberg's theorem [1] [8]: there is a measure with support contained in E such that its logarithmic potential is positively infinite at each point of E. But such a potential does not exist for E of logarithmic positive capacity. Now suppose that E is contained in the circumference of the unit disc $|z|<1$ and is of linear measure zero. Then there is a function $u(z)$, an analogue of Evans-Selberg's potential, such that $u(z)$ is positive and harmonic in $|z|<1$ and the boundary value of $u(z)$ at every point of E is positively infinite, even if the logarithmic capacity of E is positive (F. and M. Riesz [7]). This shows that the existence of an analogous function $u(z)$ of Evans-Selberg's potential depends not only on E but also on the domain D where $u(z)$ is defined. To seek the conditions on E and D, under which $u(z)$ exists, is an interesting problem in itself. Moreover such $u(z)$ is a useful tool to investigate the covering properties of meromorphic functions, as we see, for instance, in the study of functions of the class (U) in Seidel's sense (cf. Noshiro [4]). In the below, we shall give a sufficient condition to this problem and some applications to the cluster sets of meromorphic functions.

2. Let D be the unit disc and let ρ_j $(j = 1, 2, \dots, n)$ and ρ be radial segments $a_j \leqq r \leqq b_j$, $\theta = \theta_j$ $(j = 1, 2, \dots, n)$ and the union of radial segments $a_j \leqq r \leqq b_j$, $\theta = 0$ $(j=1, 2, \dots, n)$ respectively, where $z = re^{i\theta}$, $0 < a_1 < b_1 \leqq a_2 < b_2 \leqq \dots \leqq a_n < b_n \leqq 1$ and $0 \leqq \theta_j < 2\pi$. We denote by $\tilde{\omega}_\rho(z)$ and $\omega_\rho(z)$ the harmonic measure of the unit circumference with respect to the domain $D - \bigcup_{j=1}^{n} \rho_j$ and that with respect to the domain $D - \rho$, respectively. The following lemmas are given in [2].

LEMMA 1. $$\tilde{\omega}_\rho(0) \leqq \omega_\rho(0),$$

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where the equality holds if and only if $\bigcup_j \rho_j$ *coincides with* ρ *or some rotation of* ρ *around the origin.*

LEMMA 2. *Let* $G(r, \ell)$ $(0 < r < \ell)$ *denote the domain obtained by deleting the closed disc* $|z| \leqq r$ *and the segment on the real axis* $r \leqq x \leqq \ell$, $y = 0$ $(z = x + iy)$ *from the extended z-plane. Then the harmonic measure* $\omega(z; r, \ell)$ *of the circumference* $|z| = r$ *with respect to the domain* $G(r, \ell)$ *satisfies that*

$$\omega(\infty; r, \ell) = O(\sqrt{r/\ell})$$

for every sufficiently small r/ℓ.

3. Let E be a compact set of the 1/2-dimensional Hausdorff measure zero in the z-plane. We shall give a sufficient condition for the domain D in order that there exists a positive superharmonic function $u(z)$ in D being positively infinite at each point of E. For a point $z_0 = x_0 + iy_0$ of E we consider the set of all the points each of which is the rotation around the origin of a point of $\mathscr{C}D$[1] on the half line $x \geqq x_0$, $y = y_0$. If there is some $\ell > 0$ such that the segment $x_0 \leqq x \leqq x_0 + \ell$, $y = y_0$ is filled with this set, we say that the point z_0 has the rotation radius ℓ relative to the domain D. We shall prove.

THEOREM 1. *If every point of* E, *a compact set of the* 1/2*-dimensional Hausdorff measure zero on the boundary of a domain* D, *has a positive rotation radius relative to the domain* D, *then there exists a positive superharmonic function* $u(z)$ *in* D *being positively infinite at each point of* E.

Proof. We may assume without any loss of generality that D contains the point at infinity. Let E_n be the subset of E each point of which has a rotation radius greater than $1/n$. Then obviously E_n is a compact set of the 1/2-dimensional Hausdorff measure zero and E is the union of these E_n. We shall show the existence of $u(z)$ for each E_n.

Let r be a positive small number, for which the assertion of Lemma 2 holds good with $\ell = 1/n$, and let ε be a positive number, arbitrarily small. Then by the definition of the 1/2-dimensional Hausdorff measure, there exist finitely many open discs δ_i in the z-plane such that

(1) the radius r_i of δ_i is smaller than r,

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[1] $\mathscr{C}D$ denotes the complement of D with respect to the extended z-plane.

(2) their union $\bigcup_i \delta_i$ covers E_n,

(3) $\sum_i \sqrt{r_i} < \varepsilon\sqrt{1/n}$.

We denote by $D(i)$ the connected component of the open set $D-\bar{\delta}_i$ which contains the point at infinity and by $D(\infty)$ that of the open set $D-\bigcup_i \bar{\delta}_i$. Further we denote by $\tilde{\omega}_i(z)$ and $\tilde{\omega}_\infty(z)$ the harmonic measure of the part of the boundary of $D(i)$ contained in the circumference c_i of δ_i with respect to $D(i)$ and that of the part of the boundary of $D(\infty)$ contained in $\bigcup_i c_i$ with respect to $D(\infty)$, respectiviely. Then

$$\tilde{\omega}_\infty(z) \leqq \sum_i \tilde{\omega}_i(z) \quad \text{in} \quad D(\infty).$$

Now we estimate each $\tilde{\omega}_i(\infty)$. Denoting by z_i the centre of δ_i, we consider for any r'_i, $r_i < r'_i < 1/n$, the part γ_i of $\mathscr{C}D$ lying outside of the disc $|z-z_i| \leqq r'_i$. Let $\varDelta_i$ be the connected component of $\mathscr{C}\gamma_i$ containing the point at infinty, and let $\{\varDelta_{ik}\}_{k=0,1,2,\dots}$ be a normal exhaustion of $\varDelta_i$ such that $\varDelta_{i0} \supset \delta_i$. Then the harmonic measure $\tilde{w}_{ik}(z)$ of c_i with respect to $\varDelta_{ik}-\delta_i$ converges as $k\to\infty$ uniformly on each relatively compact subset of $\varDelta_i-\delta_i$ to that $\tilde{w}_i(z)$ with respect to $\varDelta_i-\delta_i$, and

$$\tilde{\omega}_i(z) \leqq \tilde{w}_i(z) \qquad \text{in } D(i).$$

Therefore for any $\varepsilon'>0$, arbitrarily small, there is a k such that

$$\tilde{w}_i(\infty) - \varepsilon' < \tilde{w}_{ik}(\infty)$$

The complement $\mathscr{C}\bar{\varDelta}_{ik}$ of $\bar{\varDelta}_{ik}$ is an open set containing γ_i and hence we can find in $\mathscr{C}\bar{\varDelta}_{ik}$ a finite set of segments ρ_j: $a_j \leqq |z-z_i| \leqq a_{j+1}$, $\arg(z-z_i)=\theta_i$ $(j=1,2,\dots,m)$, where $a_1=r'_i<a_2<\cdots<a_m=1/n$, because we may assume that z_i is a point of E_n so that it has a rotation radius greater than $1/n$. Map the outside of c_i on the unit disc $|\zeta|<1$ by $\zeta = r_i/(z-z_i)$ and use Lemma 1. Then we see that

$$\tilde{w}_{ik}(\infty) \leqq \tilde{w}_\rho(\infty) < w_\rho(\infty),$$

where $\tilde{w}_\rho(z)$ and $w_\rho(z)$ are the harmonic measures of c_i with respect to the domain $\{|z-z_i|>r_i\}-\bigcup_j \rho_j$ and the domain $\{|z-z_i|>r_i\}-\rho$ (ρ: $r'_i \leqq |z-z_i| \leqq 1/n$, $\arg(z-z_i)=0$), respectively. Because of the arbitrariness of $\varepsilon'>0$, we have thus

$$\tilde{\omega}_i(\infty) \leqq w_\rho(\infty).$$

Let r_i' tend to r_i. Then ρ becomes the segment $r_i \leqq |z-z_i| \leqq 1/n$, $\arg(z-z_i)=0$ and hence the domain $\{|z-z_i|>r_i\}-\rho$ is conformally equivalent to the domain $G(r_i, 1/n)$ in the manner that the points at infinity correspond each other. Hence

$$\tilde{\omega}_i(\infty) \leqq \omega(\infty; r_i, 1/n),$$

and it follows from this and Lemma 2 that

$$\tilde{\omega}_i(\infty) = O(\sqrt{r_i/(1/n)}).$$

Hence by (3)

$$\tilde{\omega}_\infty(\infty) \leqq \sum_i \tilde{\omega}_i(\infty) = O(\sum_i \sqrt{r_i/(1/n)}) = O(\varepsilon).$$

Defining by $\tilde{\omega}_\infty(z)=1$ on $D-D(\infty)$, we obtain a positive superharmonic function in D, which we denote by the same notation $\tilde{\omega}_\infty(z)$.

Now take $\varepsilon_k \searrow 0$ so small that $O(\varepsilon_k) \leqq 1/2^{n+k}$ and consider the superharmonic functions $\tilde{\omega}_\infty^{(k)}(z)$ in D corresponding to ε_i. Then

$$u_n(z) = \sum_{k=1}^{\infty} \tilde{\omega}_\infty^{(k)}(z)$$

is positive superharmonic in D because $\sum_{k=1}^{\infty} \tilde{\omega}_\infty^{(k)}(\infty) \leqq \sum_{k=1}^{\infty} 1/2^{n+k} = 1/2^n$ and is positively infinite at each point of E_n. Since $\sum_{n=1}^{\infty} u_n(\infty) \leqq \sum_{n=1}^{\infty} 1/2^n = 1$, we can define a positive superharmonic function $u(z)$ in D by

$$u(z) = \sum_{n=1}^{\infty} u_n(z)$$

Obviously this $u(z)$ satisfies the conditions of the theorem.

4. Here we shall prove a theorem that generalizes McMillan's theorem [3]. Let $w=f(z)$ be a nonconstant meromorphic function in the unit disc $|z|<1$ and E_z be a point set on the unit circumference $|z|=1$ of positive linear measure. Suppose that $f(z)$ has an angular limit a_ζ at each point $\zeta \in E_z$ and set

$$E_w = \{a_\zeta; \zeta \in E_z\}.$$

It is well-known that E_w contains a closed set with positive logarithmic capacity (see Privalov [6, p. 210], Tsuji [9, p. 339]). McMillan's theorem asserts the following.

Let R denote the Riemannian image of $|z|<1$ under $w=f(z)$. For each $\zeta \in E_z$ and $h>0$, $R(\zeta, h)$ is the component of R over $\{|w-a_\zeta|<h\}$ such that $f(r\zeta) \in R(\zeta, h)$ for every $r<1$, sufficiently near 1, and $\varphi R(\zeta, h)$ is the projection of $R(\zeta, h)$ onto the extended w-plane. If for each $\zeta \in E_z$, there exists a Jordan arc γ_ζ in the w-plane such that its one endpoint is a_ζ and $\varphi R(\zeta, h_\zeta) \cap \gamma_\zeta = 0$ for some $h_\zeta>0$, then E_w contains a closed set of positive 1/2-dimensional Hausdorff's measure.

As a slight improvement of McMillan's theorem we prove

THEOREM 2. *If for each $\zeta \in E_z$, E_z being of positive linear measure, a_ζ has a positive rotation radius relative to $\varphi R(\zeta, h_\zeta)$ for some $h_\zeta>0$, then E_w contains a closed set of positive 1/2-dimensional Hausdorff's measure.*

Proof. To each $\zeta \in E_z$ we correspond an open disc U_ζ with rational radius and rational center such that $a_\zeta \in U_\zeta \subset \{|w-a_\zeta|<h_\zeta\}$ and consider the component R_ζ of R over U_ζ such that $f(r\zeta) \in R_\zeta$ for every $r<1$, sufficiently near 1. Since there are only countably many distinct R_ζ, there exists a $\zeta_0 \in E_z$ such that $\{\zeta \in E_z; R_\zeta = R_{\zeta_0}\}$ contains a set $E_z^{(1)}$ of positive linear measure.

For $\zeta \in E_z^{(1)}$, we denote by $\Delta(\zeta)$ the Stolz domain whose vertex is at ζ and is bounded by two lines through ζ making the angle $\pi/4$ with the radius of $|z|=1$ at ζ. By Egoroff's theorem, we may assume that $E_z^{(1)}$ is a closed set of positive linear measure and $f(z)$ tends uniformly to a_ζ when z tends to any $\zeta \in E_z^{(1)}$ from the inside of $\Delta(\zeta)$. Hence, denoting by $\Delta_\rho(\zeta)$ the part of $\Delta(\zeta)$ lying in $0<\rho<|z|<1$ and setting $\Delta_\rho = \bigcup_{\zeta \in E_z^{(1)}} \Delta_\rho(\zeta)$ we see that $f(z)$ is continuous on the closure $\bar{\Delta}_\rho$ of Δ_ρ, if we define by $f(\zeta)=a_\zeta$ on $E_z^{(1)}$, and that the set $\{a_\zeta; \zeta \in E_z^{(1)}\}$ is a closed set contained in U_{ζ_0} as the continuous image of a closed set $E_z^{(1)}$. Therefore we can choose ρ so near 1 that the image of Δ_ρ is contained in U_{ζ_0}.

We take a component Δ of the open set Δ_ρ such that its boundary contains a closed subset $E_z^{(2)}$ of $E_z^{(1)}$ of positive linear measure (the existence of such a Δ follows from the fact that the number of components of Δ_ρ is at most countably infinite). The domain Δ is bounded by a rectifiable Jourdan curve, so that if we map Δ conformally on $|z'|<1$, then the image E' of $E_z^{\zeta(2)}$ is of positive linear measure. We denote by $z = z(z')$ this mapping function and set $g(z') = f(z(z'))$. The function $g(z')$ has the boundary value $a'_{\zeta'} = a_{z(\zeta')}$ at each $\zeta' \in E'$. The Riemannian image R' of $|z'| < 1$ under $w = g(z')$ is a subdomain of R_{ζ_0} and hence $a'_{\zeta'}$ has a positive rotation radius relative to $\varphi R'$ for every $\zeta' \in E'$. Now suppose that the closed subset $\{a'_{\zeta'} : \zeta' \in E'\}$ of E_w is of 1/2-dimensional Hausdorff's measure zero. Then, by Theorem 1, there exists a positive superharmonic function $u(w)$ in $\varphi R'$ being positively infinite at each $a'_{\zeta'}$. It is easy to see that the harmonic measure $\omega(z')$ of E' with respect to $|z'| < 1$ is dominated by $u(g(z'))/n$ for any positive integer n, so that it must be identically zero. This contradicts that E' is of positive linear measure and the theorem is proved.

5. As another application of Theorem 1, we shall prove some theorems on cluster sets. First we shall prove

THEOREM 3. *Let D be an arbitrary domain, Γ its boundary, E a compact set of 1/2-dimensional Hausdorff's measure zero on Γ and z_0 a point of E. We assume that E satisfies the following condition: If for a point $\zeta \in E$, every neighborhood of ζ contains a subset of E of positive logarithmic capacity, then ζ has a positive rotation radius relative to D. Suppose that $w = f(z)$ is nonconstant, single-valued and meromorphic in D and $C_D(f, z_0) - C_{\Gamma - E}(f, z_0)$ is not empty. Then for $\alpha \in C_D(f, z) - C_{\Gamma - E}(f, z_0)$ and for any neighborhood U of z_0, there is a $\rho_0 > 0$ such that the counter-image of (c_ρ): $|w - \alpha| < \rho$, $0 < \rho < \rho_0$, has at least one connected component in U and $f(z)$ takes on each value of (c_ρ) in any connected component lying in U with possible exception of logarithmic capacity zero.*

Proof. It is sufficient to consider the case that z_0 is an accumulation point of $\Gamma - E$, for otherwise, there is a neighborhood of z_0 such that the part of E contained in this neighborhood is of logarithmic capacity zero. We take a small $r > 0$ such that K: $|z - z_0| = r$ is contained in U, $K \cap E = 0$ and $f(z) \neq \alpha$ on $K \cap D$ and further the closure M_r of the union $\cup_\zeta C_D(f, \zeta)$ for ζ belonging to $(\Gamma - E) \cap \overline{(K)}$, $\overline{(K)}$: $|z - z_0| \leqq r$, does not contain α. Then there is $\rho_1 > 0$ such that $|f(z) - \alpha| \geqq \rho_1$. Let ρ_2 be the distance of α from M_r and ρ a positive number less than $\rho_0 = \min\{\rho_1, \rho_2\}$. Since α is a cluster value of $w = f(z)$ at z_0, there exists a sequence of points z_n $(n = 1, 2. \ldots)$ inside $(K) \cap D$ converging to z_0 such that $w_n = f(z_n) \to \alpha$.

Now we consider the counter-image D_0 of (c_ρ) inside $(K) \cap D$. Choose a point $w_n \in (c_\rho)$ and denote by $\varDelta_0$ the connected component of D_0 containing z_n. Then the boundary of the domain $\varDelta_0$ consists of a closed subset E_0 of E (may be empty) and at most a countable number of analytic curves γ_0 (boundary relative to the open set $(K) \cap D$). By our assumption, if for $\zeta \in E_0$, every neighborhood of ζ contains a subset of E_0 of positive logarithmic capacity, then ζ has a positive rotation radius relative to $\varDelta_0$. Hence by Evans-Selberg's theorem and Theorem 1, there is a positive superharmonic function $u(z)$ in $\varDelta_0$ being positively infinite at each point $\zeta \in E_0$. Now contrary suppose that the set e of values in (c_ρ), which are not taken by $f(z)$ in $\varDelta_0$, is of positive logarithmic capacity. We take a closed subset e_0 of e of positive logarithmic capacity. Then there is a positive bounded harmonic function $v(w)$ in $(c_\rho) - e_0$ vanishing continuously on c_ρ: $|w - \alpha| = \rho$. Since for $z \in \gamma_0$, $f(z)$ falls in c_ρ, we see that $v(f(z)) \leq u(z)/n$ in $\varDelta_0$ for every positive integer n. Hence $v(f(z)) \equiv 0$ in $\varDelta_0$. This contradicts that $v(w) = v(f(z_n)) > 0$, and the theorem is proved.

6. Using Theorem 3 and the usual argument, we can prove

THEOREM 4. *Let D, Γ, E, z_0 and $w = f(z)$ be the same as in Theorem 3. Suppose that $C_D(f, z_0) - C_{\Gamma - E}(f, z_0)$ is not empty and $\alpha \in C_D(f, z_0) - C_{\Gamma - E}(f, z_0)$ is taken by $f(z)$ in the intersection of a neighborhood of z_0 and D only at most finitely often. Then either α is an asymptotic value of $f(z)$ at z_0 or there exists a sequence $\zeta_n \in E$ $(n = 1, 2, \ldots)$ tending to z_0 such that α is an asymptotic value of $f(z)$ at each ζ_n.*

THEOREM 5. *Let D, Γ, E, z_0 and $w = f(z)$ be the same as in Theorem 3. Suppose that $U(z_0) \cap (\Gamma - E) \neq 0$ for every neighborhood $U(z_0)$ of z_0. Then the set*

$$\Omega = C_D(f, z_0) - C_{\Gamma - E}(f, z_0)$$

is empty or open.

THEOREM 6. *Let D, Γ, E, z_0 and $w = f(z)$ be the same as in Theorem 3. Suppose that $U(z_0) \cap (\Gamma - E) \neq 0$ for every neighborhood $U(z_0)$ of z_0 and the open set $\Omega = C_D(f, z_0) - C_{\Gamma - E}(f, z_0)$ is not empty. Then every value of Ω is taken by $w = f(z)$ infinitely often in the intersection of any neighborhood of z_0 and D except for a possible set of values of logarithmic capacity zero.*

Remark. Recently Noshiro [5] has given extensions of some theorems on cluster sets, which are closely related to ours.

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