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ON AREA INTEGRALS AND RADIAL VARIATIONS OF ANALYTIC FUNCTIONS IN THE UNIT DISK

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1. Introduction

We are concerned with the behaviour of analytic functions near the boundary. Let $\boldsymbol{T}$ and $\boldsymbol{D}$ be the unit circle $|z| = 1$ and the unit disk $|z| < 1$, respectively. The element of $\boldsymbol{T}$ is denoted by θ $(0 \leq \theta < 2\pi)$. Let $f(z) = \sum_{n=1}^{\infty} a_n z^n$ be analytic in $\boldsymbol{D}$. The area integral $A(f, \theta)$ of f at θ is defined by

$$A(f, \theta) = \iint_{\Gamma(\theta)} |f'(re^{i\varphi})|^2 \, r dr d\varphi ,$$

where $\Gamma(\theta) = \{z\,;\ |z| > \frac{1}{2}, |\arg(z - e^{i\theta})| < 1\}$. It represents the area of the image of $\Gamma(\theta)$. We know the following two relations:

(1) The finiteness of $A(f, \theta)$ reflects the existence of $\lim_{r\to 1} f(re^{i\theta})$.

(2) The infiniteness of $A(f, \theta)$ reflects the totality of $f(\Gamma(\theta))$, that is, $f(\Gamma(\theta)) = \{z\,;\ |z| < +\infty\}$.

So it is interesting to know whether $A(f, \theta)$ is finite or not. Our problems are to characterize the finiteness of $A(f, \theta)$ and to study these relations (1) and (2). But it is complicated to examine them for given f and $\theta \in \boldsymbol{T}$. So some authors studied them for a given f occasionally neglecting a small subset of $\boldsymbol{T}$. (cf. Theorem (1.1) in [4] p. 199) The author also took the same line at first. But, in this paper, we shall study them neglecting a class of functions. To define a negligible class of functions, we need a probability space.

Let $(\Omega, \mathfrak{B}, p)$ be a probability space, where Ω is a space, $\mathfrak{B}$ events and p a probability. Let $X = (X_n)_{n=1}^{\infty}$ be a sequence of independent random variables. Consider a class of analytic functions, so-called a random Taylor series by X, $f_X(z) = \sum_{n=1}^{\infty} X_n a_n z^n$. For a random Taylor series f_X, we shall neglect a class of functions in f_X with probability 0.

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From the point of view of random Taylor series, we shall consider the above problems. First, we remark the following fact. The property of the finiteness of $A(f_X, \theta)$ is an event and independent on the values of a finite number of $X_n a_n z^n$. By the zero-one law, we obtain that $A(f_X, \theta) < +\infty$ holds with probability 1 or 0.

We shall also treat by the same manner the generalized area integrals and the radial variations which are defined in the section 2.

2. Definitions

Let $\boldsymbol{C}$ be the complex plane. The element of $\boldsymbol{C}$ is denoted by $z = re^{i\varphi}, \zeta, \cdots$ etc. Let $\boldsymbol{T}$ and $\boldsymbol{D}$ be the unit circle and the unit open disk with center zero, respectively. The element of $\boldsymbol{T}$ is denoted by θ $(0 \leq \theta < 2\pi)$. Let $f(z) = \sum_{n=1}^{\infty} a_n z^n$ be analytic in $\boldsymbol{D}$.

The area integral $A(f, \theta, \beta)$ of f at θ is defined by

$$A(f, \theta, \beta) = \iint_{\Gamma_\beta(\theta)} |f'(re^{i\varphi})|^2 \, r dr d\varphi \, ,$$

where $\Gamma_\beta(\theta) = \{z;\ |z| > \frac{1}{2}, |\arg(z - e^{i\theta})| < \beta\}$ $(0 < \beta < \pi/2)$. We denote $A(f, \theta) = A(f, \theta, 1)$. We have two generalizations of $A(f, \theta)$.

The area integral $A_\alpha(f, \theta)$ of f of order α $(-1 < \alpha < 1)$ is defined by

$$A_\alpha(f, \theta) = \int_0^1 r(1-r)^{-\alpha} dr \int_{\theta-(1-r)}^{\theta+(1-r)} |f'(re^{i\varphi})|^2 \, d\varphi \, .$$

We know that $A_0(f, \theta)$ and $A(f, \theta)$ are equivalent in the following sense: There exist γ_1, γ_2 $(0 < \gamma_1,\ \gamma_2 < \pi/2)$ such that $c_1 A_0(f, \theta, \gamma) \leq A(f, \theta) \leq c_2 A_0(f, \theta, \gamma_2)$ for some positive constants c_1, c_2.

The area integral $\tilde{A}_\alpha(f, \theta)$ of f of tangency α $(0 \leq \alpha \leq \frac{1}{2})$ is defined by

$$\tilde{A}_\alpha(f, \theta) = \int_0^1 r dr \int_{\theta-(1-r)^{1-\alpha}}^{\theta+(1-r)^{1-\alpha}} |f'(re^{i\varphi})|^2 \, d\varphi \, .$$

The radial variation $V(f, \theta)$ of f is defined by

$$V(f, \theta) = \int_0^1 |f'(re^{i\theta})| \, dr \, .$$

For convenience sake, we write the following notation:

$$A_\alpha^t(f, \theta) = \int_0^t r(1-r)^{-\alpha} dr \int_{\theta-(1-r)}^{\theta+(1-r)} |f'(re^{i\varphi})|^2 \, d\varphi \qquad (0 < t < 1)$$

$$c_\alpha(n, m;\ t) = nm \int_0^t r^{n+m-1}(1-r)^{-\alpha} \int_{-1+r}^{1-r} \cos(n-m)\varphi d\varphi ,$$

where n, m are integers. We denote $c_\alpha(n, m) = c_\alpha(n, m;\ 1)$. Let $f(z) = \sum_{n=1}^\infty a_n z^n$ be analytic in $\boldsymbol{D}$. We have

$$\begin{aligned} A_\alpha^t(f, \theta) &= \int_0^t r(1-r)^{-\alpha} dr \int_{\theta-(1-r)}^{\theta+(1-r)} \left| \sum_{n=1}^\infty n a_n r^{n-1} e^{i(n-1)\varphi} \right|^2 d\varphi \\ &= \sum_{n=1}^\infty \sum_{m=1}^\infty c_\alpha(n, m;\ t) a_n e^{in\theta} \overline{a_m e^{im\theta}} . \end{aligned}$$

In this paper, we use the following notation: If the inequality $0 \le f(z) \le cg(z)$ holds for some positive constant c, we denote $f(z) \lesssim g(z)$. If the inequality $c_1 f(z) \le g(z) \le c_2 f(z)$ holds for some positive constants c_1, c_2, we denote $f(z) \approx g(z)$.

Next, we define the probability space $(\Omega, \mathfrak{B}, p)$ which is fixed throughout this paper. Let I be the interval $[0, 1)$ and let $(I, \mathfrak{B}_I, p_I)$ be the usual probability space. Set $\Omega = \prod_{n=1}^\infty I_n$, where $I_n = I$ for all n. Then the product space $(\Omega, \mathfrak{B}, p)$ is usually defined. The element of Ω is denoted by ω. The expectation is denoted by $\mathscr{E}[\ \cdot\ ]$. We consider a sequence $X = (X_n)_{n=1}^\infty$ of independent random variables which satisfies the following conditions:

(i) X_n is real-valued.
(ii) X_n is a random variable on I_n.
(iii) X_n is symmetric, that is, $p(X_n > c) = p(-X_n > c)$ for all $c \ge 0$.
(iv) $\sup_n \mathscr{E}[X_n^2] < +\infty$.
(v) $\sup_n \mathscr{E}[X_n^4]\mathscr{E}[X_n^2]^{-2} < +\infty$.

As a technique, we shall use a Rademacher series which is defined as follows. Let J be two points $\{-1, 1\}$. Set $\tilde{\Omega} = \prod_{n=1}^\infty J_n$, where $J_n = J$ for all n. Then the usual probability space $(\tilde{\Omega}, \tilde{\mathfrak{B}}, \tilde{p})$ is defined. The element of $\tilde{\Omega}$ is denoted by x. A Rademacher series $\varepsilon = (\varepsilon_n)_{n=1}^\infty$ is defined by

(a) ε_n is a random variable on J_n
(b) $\varepsilon_n(-1) = -1,\ \varepsilon_n(1) = 1$.

Then $\varepsilon = (\varepsilon_n)_{n=1}^\infty$ is a sequence of independent random variables with $\tilde{p}(\varepsilon_n = 1) = \tilde{p}(\varepsilon_n = -1) = \frac{1}{2}$ $(n = 1, 2, \cdots)$.

If some property $\boldsymbol{P}_1$ on Ω hold with probability 1, we say that $\boldsymbol{P}_1$

holds almost surely (a.s.). If some property $\boldsymbol{P}_2$ on $\boldsymbol{T}$ holds with Lebesgue measure 2π, we say that $\boldsymbol{P}_2$ holds almost everywhere (a.e.).

3. Immediate consequences and constructions of examples

We first show the following

PROPOSITION 1. *Let $|\alpha| < 1$ and let $f_X(z) = \sum_{n=1}^{\infty} X_n a_n z^n$ be a random Taylor series defined by $X = (X_n)_{n=1}^{\infty}$. Then $A_\alpha(f_X, 0) < +\infty$ a.s. if and only if $\sum_{n=1}^{\infty} \mathscr{E}[|X_n|^2] n^\alpha |a_n|^2 < +\infty \cdots (*)_\alpha$.*

For the proof, we prepare the following

LEMMA 1 ([1] p. 6). *Let Y be a positive random variable. Then for $0 < \lambda < 1$, we have*

$$p(Y \geq \lambda \mathscr{E}[Y]) \geq (1-\lambda)^2 \mathscr{E}[Y]^2 \mathscr{E}[Y^2]^{-1} .$$

Proof of Proposition 1. First we remark $\int_0^1 r^{2n-1}(1-r)^{1-\alpha} dr \approx n^{\alpha-2}$. Assume that $(*)_\alpha$ holds. From the hypothesis (v), we have, with some constant c, $\mathscr{E}[X_n^4] \leq c\mathscr{E}[X_n^2]^2$. Since

$$A_\alpha^t(f_X, 0) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} X_n X_m c_\alpha(n, m;\ t) a_n \bar{a}_m ,$$

it follows from (iii) that

$$\begin{aligned} \mathscr{E}[A_\alpha^t(f_X, 0)] &= \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \mathscr{E}[X_n X_m] c_\alpha(n, m;\ t) a_n \bar{a}_m \\ &= \sum_{n=1}^{\infty} \mathscr{E}[X_n^2] c_\alpha(n, n;\ t) |a_n|^2 . \end{aligned}$$

Letting t tend to 1, we have

$$\mathscr{E}[A_\alpha(f_X, 0)] = \sum_{n=1}^{\infty} \mathscr{E}[X_n^2] c_\alpha(n, n) |a_n|^2 \approx \sum_{n=1}^{\infty} \mathscr{E}[X_n^2] n^\alpha |a_n|^2 < +\infty .$$

Hence $A_\alpha(f_X, 0) < +\infty$ a.s..

Conversely, assume that $A_\alpha(f_X, 0) < +\infty$ hold a.s.. We shall apply the above lemma to the random variable $A_\alpha^t(f_X, 0)$. We have

$$\mathscr{E}[A_\alpha^t(f_x, 0)]^2 = \left(\sum_{n=1}^{\infty} \mathscr{E}[X_n^2] c_\alpha(n, n; t) |a_n|^2\right)^2$$

and

$$\mathscr{E}[A_\alpha^t(f_X, 0)^2] = \mathscr{E}\left[\left(\sum_{n,m} X_n X_m c_\alpha(n, m;\ t) a_n \bar{a}_m\right)^2\right]$$

$$= \mathscr{E}\left[\sum_{n_1 m_1 n_2 m_2} X_{n_1} X_{m_1} X_{n_2} X_{m_2} c_\alpha(n_1, m_1; t) c_\alpha(n_2, m_2; t) a_{n_1} \bar{a}_{m_1} a_{n_2} \bar{a}_{m_2}\right]$$

$$= \sum_{n_1 m_1 n_2 m_2} \mathscr{E}[X_{n_1} X_{m_1} X_{n_2} X_{m_2}] c_\alpha(n_1, m_1; t) c_\alpha(n_2, m_2; t) a_{n_1} \bar{a}_{m_1} a_{n_2} \bar{a}_{m_2}$$

$$\leq \sum_{n,m} \mathscr{E}[X_n^2 X_m^2] c_\alpha(n, n; t) c_\alpha(m, m; t) |a_n|^2 |a_m|^2$$

$$+ \sum_{n,m} \mathscr{E}[X_n^2 X_m^2] c_\alpha(n, m; t)^2 |a_n|^2 |a_m|^2 .$$

Since we have

$$\mathscr{E}[X_n^2 X_m^2] \leq \sqrt{\mathscr{E}[X_n^4]}\, \sqrt{\mathscr{E}[X_m^4]} \leq c \mathscr{E}[X_n^2] \mathscr{E}[X_m^2]$$

and

$$c_\alpha(n, m; t) \leq nm \int_0^1 r^{m+m-1}(1-r)^{-\alpha} dr \int_{-1+r}^{1-r} d\varphi \leq \sqrt{c_\alpha(n, n; t)}\, \sqrt{c_\alpha(m, m; t)} ,$$

we obtain

$$\mathscr{E}[A_\alpha^t(f_X, 0)^2] \leq 2c \left(\sum_{n=1}^\infty \mathscr{E}[X_n^2] c_\alpha(n, n; t) |a_n|^2\right)^2 .$$

Therefore

$$\mathscr{E}[A_\alpha^t(f_X, 0)]^2 \mathscr{E}[A_\alpha^t(f_X, 0)^2]^{-2} \geq \frac{1}{2c} .$$

By Lemma 1, we have

$$p(A_\alpha^t(f_X, 0) \geq \frac{1}{2} \mathscr{E}[A_\alpha^t(f_X, 0)]) \geq \left(1 - \left(\frac{1}{2}\right)\right) \frac{1}{2c} (=\eta) > 0 .$$

Choose a sequence $(t_n)_{n=1}^\infty$ such that $0 < t_n < 1$ and $t_n \uparrow 1$. Set

$$E_n = \{A_\alpha^{t_n}(f_X, 0) \geq \tfrac{1}{2} \mathscr{E}[A_\alpha^{t_n}(f_X, 0)]\} .$$

Since $p(E_n) \geq \eta$ for all n, we have $p(\limsup_{n\to\infty} E_n) \geq \eta$. By the assumption, there exists $\omega \in \limsup_{n\to\infty} E_n$ such that $A_\alpha(f_{X(\omega)}, 0) < +\infty$. Then we have

$$\sum_{n=1}^\infty \mathscr{E}[X_n^2] n^\alpha |a_n|^2 \approx \mathscr{E}[A_\alpha(f_X, 0)] = \lim_{n\to\infty} \mathscr{E}[A_\alpha^{t_n}(f_X, 0)]$$

$$\leq \lim_{n\to\infty} A_\alpha^{t_n}(f_{X(\omega)}, 0) = A_\alpha(f_{X(\omega)}, 0) < +\infty .$$

This completes the proof.

COROLLARY 1. *Let* $|\alpha| < 1$ *and* f_X *be the same as in Proposition* 1. *Then* $A_\alpha(f_X, \theta) < +\infty$ *a.e. holds a.s. if and only if* $(*)_\alpha$ *holds.*

Proof. Consider the product space $(\Omega \times \boldsymbol{T}, \mathfrak{B} \times \mathfrak{B}_T, p \times d\theta)$. We denote by $\tilde{\mathscr{E}}[\,\cdot\,]$ the expectation. Define a sequence $Y = (Y_n)_{n=1}^{\infty}$ of random variables on $\Omega \times \boldsymbol{T}$ by $Y_n(\omega, \theta) = X_n(\omega)e^{in\theta}$. Then we have

$$\sup_n \tilde{\mathscr{E}}[|Y_n|^4]\tilde{\mathscr{E}}[|Y_n|^2]^{-2} = \sup_n \mathscr{E}[X_n^4]\mathscr{E}[X_n^2]^{-2} < +\infty$$

and

$$\tilde{\mathscr{E}}[Y_{n_1}Y_{m_1}\overline{Y}_{n_2}\overline{Y}_{m_2}] = 2\pi\mathscr{E}[X_{n_1}X_{m_1}X_{n_2}X_{m_2}]\delta_{n_1+m_1, n_2+m_2} ,$$

where $\delta_{n,m}$ means Kronecker's. By the same method as in Proposition 1, we know that $A_\alpha(f_Y, 0) < +\infty$ a.s. $(p \times d\theta)$ if and only if $(*)_\alpha$ holds. Since $A_\alpha(f_{Y(\omega,\theta)}, 0) = A_\alpha(f_{X(\omega)}, \theta)$, we know that $A_\alpha(f_X, \theta) < +\infty$ a.e. holds a.s. if and only if $(*)_\alpha$ holds, this completes the proof.

PROPOSITION 1′. *Let* $f_X(z) = \sum_{n=1}^{\infty} X_n a_n z^n$ *be a random Taylor series. Set* $s_j = (\sum_{2^j \le n < 2^{j+1}} \mathscr{E}[X_n^2]\,|a_n|^2)^{1/2}$. *If* $\sum_{j=0}^{\infty} s_j < +\infty$, *then* $V(f_X, 0) < +\infty$ a.s..

Proof. We have

$$V(f_X, 0) = \int_0^1 |f_X'(r)|\,dr \le \sum_{j=0}^{\infty} \int_0^1 \Big|\sum_{2^j \le n < 2^{j+1}} nX_n a_n r^{n-1}\Big|\,dr .$$

Since we have

$$\mathscr{E}\Big[\Big|\sum_{2^j \le n < 2^{j+1}} X_n n a_n r^{n-1}\Big|\Big] \le \mathscr{E}\Big[\sum_{2^j \le n, m < 2^{j+1}} X_n X_m n m a_n \bar{a}_m r^{n+m-2}\Big]^{1/2}$$
$$\le \Big(\sum_{2^j \le n < 2^{j+1}} \mathscr{E}[X_n^2] n^2 |a_n|^2 r^{2n-2}\Big)^{1/2} \le 2^{j+1} r^{2^j - 1} s_j ,$$

we obtain

$$\mathscr{E}[V(f_X, 0)] \le \sum_{j=0}^{\infty} s_j 2^{j+1} \int_0^1 r^{2^j - 1} dr \approx \sum_{j=0}^{\infty} s_j < +\infty .$$

Therefore we have $V(f_X, 0) < +\infty$ a.s.. This completes the proof.

COROLLARY 1′. *If* $\sum_{j=0}^{\infty} s_j < +\infty$, *then* $V(f_X, \theta) < +\infty$ *a.e. holds a.s..*

This is easily proved by the same method as in Proposition 1′. Hence we omit the proof.

Remark 1. The similar assertion as in Proposition 1 for $\tilde{A}_\alpha$ $(0 < \alpha$

$\leq \frac{1}{2}$) holds. Now, choose a sequence $(a_n)_{n=1}^{\infty}$ such that $\sum_{n=1}^{\infty} n^{\alpha} |a_n|^2 < +\infty$ and $\sum_{n=1}^{\infty} n^{\beta} |a_n|^2 = +\infty$ $(0 \leq \alpha < \beta \leq \frac{1}{2})$. Consider a random Taylor series $f_{\varepsilon}(z) = \sum_{n=1}^{\infty} \varepsilon_n a_n z^n$. Then we have almost surely $A_{\alpha}(f_{\varepsilon}, \theta) < +\infty$, $A_{\beta}(f_{\varepsilon}, \theta) = +\infty$, $\tilde{A}_{\alpha}(f_{\varepsilon}, \theta) < +\infty$ and $\tilde{A}_{\alpha}(f_{\varepsilon}, \theta) = +\infty$ a.e..

PROPOSITION 2. *Let $X = (X_n)_{n=1}^{\infty}$ be a sequence of independent real-valued normal Gaussian variables (i.e. $p(X_n < t) = 1/\sqrt{2\pi}\int_{-\infty}^{t} e^{-s^2/2}ds$) and let $f_X(z) = \sum_{n=1}^{\infty} X_n a_n z^n$ be a random Taylor series. Then $V(f_X, 0) < +\infty$ a.s. if and only if $\int_0^1 \sqrt{\sum_{n=1}^{\infty} n^2 |a_n|^2 r^{2n-2}}\, dr < +\infty$.*

Proof. We can assume that a_n's are real. We have

$$\mathscr{E}[V(f_X, 0)] = \int_0^1 \mathscr{E}\left[\left|\sum_{n=1}^{\infty} X_n n a_n r^{n-1}\right|\right] dr = \sqrt{\frac{2}{\pi}} \int_0^1 \sqrt{\sum_{n=1}^{\infty} n^2 |a_n|^2 r^{2n-2}}\, dr\ .$$

Hence 'if' part holds. Set $V^t(f_X, 0) = \int_0^t |f_X'|\, dr$. We shall show that $\mathscr{E}[V^t(f_X, 0)^2]\mathscr{E}[V^t(f_X, 0)]^{-2} \leq 4$ for all $0 < t < 1$. We have

$$\mathscr{E}[V^t(f_X, 0)]^2 = \frac{2}{\pi}\left(\int_0^t \sqrt{\sum_{n=1}^{\infty} n^2 |a_n|^2 r^{2n-2}}\, dr\right)^2$$

and

$$\begin{aligned}
&\mathscr{E}[V^t(f_X, 0)^2] \\
&= \int_0^t\int_0^t \mathscr{E}\left[\left|\sum_{n=1}^{\infty} X_n n a_n r^{n-1}\right|\left|\sum_{n=1}^{\infty} X_n n a_n s^{n-1}\right|\right] drds \\
&= \int_0^t\int_0^t drds \frac{1}{\sqrt{AB - C^2}} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |x|\,|y| \exp\left(-\pi\frac{Bx^2 + Ay^2 - 2Cxy}{AB - C^2}\right) dxdy,
\end{aligned}$$

where

$$A = \mathscr{E}\left[\left|\sum_{n=1}^{\infty} X_n n a_n r^{n-1}\right|^2\right] = \sum_{n=1}^{\infty} n^2 |a_n|^2 r^{2n-2}\ , \qquad B = \sum_{n=1}^{\infty} n^2 |a_n|^2 s^{2n-2}$$

and

$$C = \mathscr{E}\left[\sum_{n=1}^{\infty} X_n n a_n r^{n-1} \sum_{n=1}^{\infty} X_n n a_n s^{n-1}\right] = \sum_{n=1}^{\infty} n^2 |a_n|^2 r^{n-1} s^{n-1}\ .$$

Since

$$\begin{aligned}
&\frac{1}{\sqrt{AB - C^2}} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |x|\,|y| \exp\left(-\pi\frac{Bx^2 + Ay^2 - 2Cxy}{AB - C^2}\right) dxdy \\
&\qquad \leq 4\sqrt{AB - C^2} \leq 4\sqrt{AB}\ ,
\end{aligned}$$

we have

$$\mathscr{E}[V^t(f_X,0)^2] \leq 4\left(\int_0^t \sqrt{\sum_{n=1}^{\infty} n^2\,|a_n|^2\, r^{2n-2}}\; dr\right)^2 .$$

Therefore $\mathscr{E}[V^t(f_X,0)^2]\mathscr{E}[V^t(f_X,0)]^{-2} \leq 4$. Hence the rest of the proof follows in the same manner as in Proposition 1. This completes the proof.

To discuss the sure properties, we consider lacunary series. Let $(\ell_\alpha(k))_{k=0}^{\infty}$ $(0 < \alpha < 1)$ be a sequence of positive integers such that $(1-\alpha)\ell_\alpha(k+1) \geq 2\ell_\alpha(k)$. We denote by $N_\alpha(k) = 2^{\ell_\alpha(k)}$ and $N(k) = 2^{2^k}$ throughout this paper.

PROPOSITION 5. *Let $0 < \alpha < 1$ and let $(a_n)_{n=1}^{\infty}$ be a bounded sequence such that $a_n = 0$ for $n \neq N_\alpha(k)$ $(k = 0, 1, \cdots)$. Set $f(z) = \sum_{n=1}^{\infty} a_n z^n$. Then $A_\alpha(f,\theta) < +\infty$ for all θ or $A_\alpha(f,\theta) = +\infty$ for all θ according to $\sum_{n=1}^{\infty} n^\alpha |a_n|^2 < +\infty$ or $= +\infty$.*

Proof. We can assume $|a_n| \leq 1$ for all n. We have

$$\begin{aligned} A_\alpha^t(f,\theta) &= \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} c_\alpha(n,m\,;t)a_n e^{in\theta}\overline{a_m e^{im\theta}} \\ &= \sum_{k=0}^{\infty} c_\alpha(N_\alpha(k),N_\alpha(k)\,;t)\,|a_{N_\alpha(k)}|^2 \\ &\quad + 2\,\mathrm{Re}\left(\sum_{k=1}^{\infty}\sum_{k'=0}^{k-1} c_\alpha(N_\alpha(k),N_\alpha(k')\,;t)a_{N_\alpha(k)}\bar{a}_{N_\alpha(k')}e^{i(N_\alpha(k)-N_\alpha(k'))\theta}\right). \end{aligned}$$

We have the following estimation:

$$\begin{aligned} |(\text{The second term})| &\lesssim \sum_{k=1}^{\infty}\sum_{k'=0}^{k-1} N_\alpha(k)N_\alpha(k')(N_\alpha(k)+N_\alpha(k'))^{\alpha-2} \\ &\leq \sum_{k=1}^{\infty} N_\alpha(k)^{\alpha-1}\cdot k\cdot N_\alpha(k-1) < +\infty . \end{aligned}$$

Letting t tend to 1, we have $A_\alpha(f,\theta) \approx \sum_{n=1}^{\infty} n^\alpha |a_n|^2 + 0(1)$. This completes the proof.

PROPOSITION 5′. *Let $0 < \alpha < 1$ and let $(a_n)_{n=1}^{\infty}$ be an absolutely convergent sequence such that $a_n = 0$ for $n \neq N(k)$ $(k = 0, 1, \cdots)$. Set $f(z) = \sum_{n=1}^{\infty} a_n z^n$. Then $A_\alpha(f,\theta) < +\infty$ for all θ or $A_\alpha(f,\theta) = +\infty$ for all θ according to $\sum_{n=1}^{\infty} n^\alpha |a_n|^2 < +\infty$ or $= +\infty$.*

By using the following estimation, we have $A_\alpha(f,\theta) \approx \sum_{n=1}^{\infty} n^\alpha |a_n|^2 + 0(1)$.

$$\begin{aligned}
&\left|\sum_{k=1}^{\infty}\sum_{k'=0}^{\infty} c_\alpha(N(k), N(k'); t) a_{N(k)} \bar{a}_{N(k')} e^{i(N(k)-N(k'))\theta}\right| \\
&\quad \lesssim \sum_{k=1}^{\infty} |a_{N(k)}| \sum_{k'=0}^{k-1} |a_{N(k')}|\, N(k)N(k')(N(k) + N(k'))^{\alpha-2} \\
&\quad \leq \left(\sum_{k=1}^{\infty} |a_{N(k)}|\right)^2 < +\infty\,.
\end{aligned}$$

COROLLARY 2. *There exists an absolutely convergent Taylor series* $f(z) = \sum_{n=1}^{\infty} a_n z^n$ *such that* $A_\alpha(f, \theta) = +\infty$ *for all* θ *and all* $0 < \alpha < 1$.

Proof. Let $(a_n)_{n=1}^{\infty}$ be a sequence such that $a_{N(k)} = (k+1)^{-2}$ $(k = 0, 1, \cdots)$ $a_n = 0$ $n \neq N(k)$. Then $\sum_{n=0}^{\infty} n^\alpha |a_n|^2 = +\infty$ for all $0 < \alpha < 1$. By Proposition 2′, $A_\alpha(f, \theta) = +\infty$ for all θ and $0 < \alpha < 1$. This completes the proof.

Remark 2. By [2], $\theta \in \boldsymbol{T}$ is called a Lusin point of f if $\tilde{A}_{1/2}(f, \theta, t) = \iint_{|z-te^{i\theta}|<1-t} |f'(z)|^2\, r dr d\varphi$ diverges for all $0 < t < 1$. We know that there exists a bounded function such that every point $\theta \in \boldsymbol{T}$ is a Lusin point of it ([2]). Let f be the function in Corollary 2. Then every point $\theta \in \boldsymbol{T}$ is a Lusin point of f. We shall show it. We have $\tilde{A}_{1/2}(f, \theta) = +\infty$ for each θ. We can assume $t > \frac{1}{2}$. If we choose suitable constants $\beta_t, \gamma_{t,f}$, we have, for each θ,

$$\begin{aligned}
\tilde{A}_{1/2}(f, \theta, t) &= \iint_{\substack{|z-te^{i\theta}|<1-t \\ r<t}} |f'(z)|^2\, r dr d\varphi \\
&\quad + \int_t^1 r dr \int_{|\varphi-\theta|<\arccos(2t-1+r^2)(2rt)^{-1}} |f'|^2\, d\varphi \\
&\geq \int_t^1 r dr \int_{|\varphi-\theta|<\beta_t\sqrt{1-r}} |f'|^2\, d\varphi \approx A_{1/2}(f, \theta) + \gamma_{t,f} = +\infty\,.
\end{aligned}$$

Therefore $\tilde{A}_{1/2}(f, \theta, t) = +\infty$ for all $\theta \in \boldsymbol{T}$ and all $0 < t < 1$. But there exists $g(z) = \sum_{n=1}^{\infty} b_n z^n$ such that each $\theta \in \boldsymbol{T}$ is not a Lusin point of g and $A_\alpha(g, \theta) = +\infty$ for all θ and all $\alpha > \frac{1}{2}$. For example, put $b_{N(k)} = k^{-1/2} N(k)^{-1/4}$ $(k = 1, 2, \cdots)$ and $b_n = 0$ for $n \neq N(k)$.

EXAMPLE. There exists an analytic function f such that $V(f, \theta) = +\infty$ and $A_0(f, \theta) < +\infty$ for all θ.

Put $b_{N(k)} = k^{-1/2} N(k)^{1/2}$ $(k = 1, 2, \cdots)$ and $b_n = 0$ for $n \neq N(k)$ $(k = 1, 2, \cdots)$. Consider $f(z) = \int_0^z \left(\sum_{n=0}^{\infty} b_n \zeta^n\right)^2 d\zeta$. We show that f satisfies the required conditions. We have

$$\begin{aligned}
V(f,\theta) &= \int_0^1 \left|\sum_{n=1}^{\infty} b_n r^n e^{in\theta}\right|^2 dr \\
&= \sum_{k=1}^{\infty}\sum_{k'=1}^{\infty} b_{N(k)} b_{N(k')} (N(k)+N(k')+1)^{-1} e^{i(N(k)-N(k'))\theta} \\
&= \sum_{k=1}^{\infty} b_{N(k)}^2 (2N(k)+1)^{-1} \\
&\quad + \sum_{k=1}^{\infty}\sum_{k'\neq k}^{\infty} b_{N(k)} b_{N(k')} (N(k)+N(k')+1)^{-1} e^{i(N(k)-N(k'))}\theta .
\end{aligned}$$

We have the following estimation:

$$\begin{aligned}
&(\text{The first term}) \approx \sum_{k=1}^{\infty} k^{-1} = +\infty \\
&|(\text{The second term})| \lesssim \sum_{k=1}^{\infty} b_{N(k)} N(k)^{-1} \sum_{k'=1}^{k-1} b_{N(k')} \leq \sum_{k=2}^{\infty} N(k-2)^{-1} < +\infty .
\end{aligned}$$

Therefore we have $V(f,\theta) = +\infty$ for all θ. On the other hand, we have

$$\begin{aligned}
A_0(f,\theta) &= \int_0^1 dr \int_{\theta-(1-r)}^{\theta+(1-r)} \left|\sum_{n=1}^{\infty} b_n r^n e^{in\varphi}\right|^4 d\varphi \\
&= \int_0^1 dr \int_{\theta-(1-r)}^{\theta+(1-r)} \sum_{k_1,k_2,k_3,k_4=1}^{\infty} b_{N(k_1)} b_{N(k_2)} b_{N(k_3)} b_{N(k_4)} r^{N(k_1)+N(k_2)+N(k_3)+N(k_4)} \\
&\qquad\qquad \times e^{i(N(k_1)+N(k_2)-N(k_3)-N(k_4))\varphi} d\varphi \\
&\approx \sum_{k_1,k_2,k_3,k_4=1}^{\infty} b_{N(k_1)} b_{N(k_2)} b_{N(k_3)} b_{N(k_4)} (N(k_1)+N(k_2)+N(k_3)+N(k_4))^{-2} \\
&\lesssim \sum_{k=1}^{\infty} b_{N(k)}^4 N(k)^{-2} + \sum_{k=1}^{\infty} b_{N(k)}^3 N(k)^{-2} \sum_{k'=1}^{k-1} b_{N(k')} + \sum_{k=1}^{\infty} b_{N(k)}^2 N(k)^{-2} \left(\sum_{k'=1}^{k-1} b_{N(k')}\right)^2 \\
&\quad + \sum_{k=1}^{\infty} b_{N(k)} N(k)^{-2} \left(\sum_{k'=1}^{k-1} b_{N(k')}\right)^3 \\
&\lesssim \sum_{k=1}^{\infty} k^{-2} + \sum_{k=2}^{\infty} k^{-1} N(k-2)^{-1} + \sum_{k=1}^{\infty} N(k-1)^{-1} + \sum_{k=2}^{\infty} k^2 N(k-2)^{-2} \\
&\quad + O(1) < +\infty .
\end{aligned}$$

Therefore we have $A_0(f,\theta) < +\infty$ for all θ.

4. Almost sure property for all θ

THEOREM 1. *Let* $|\alpha| < 1$ *and* $f_X(z) = \sum_{n=1}^{\infty} X_n a_n z^n$ *be a random Taylor series. Set* $s_j = \sqrt{\sum_{2^j \leq n < 2^{j+1}} \mathscr{E}[X_n^2] n^{\alpha} |a_n|^2}$ $(j = 0, 1, \cdots)$. *If* $s_j \downarrow 0$ *and* $\sum_{j=0}^{\infty} s_j < +\infty$, *then* $A_\alpha(f_X, \theta)$ *is bounded ((as a function of* θ) *a.s..*

We denote by $\|P\|_\infty = \sup_{\theta\in T} |P(\theta)|$ for a continuous function P on T. We use the following

LEMMA 2. ([1] p. 55) *Let* $(P_n)_{n=1}^{\ell}$ *be a sequence of trigonometric polynomials of degree* $\leq N$. *Set* $P_\varepsilon = \sum_{n=1}^{\ell} \varepsilon_n P_n$. *Then we have, with positive constants* c_1, c_2,

$$\tilde{p}\left((\|P_\varepsilon\|_\infty \geq c_1(\log N)^{1/2}\left(\sum_{n=1}^{\ell} \|P_n\|_\infty^2\right)^{1/2}\right) \leq c_2 N^{-2} .$$

Proof of Theorem 1. First we consider the case of a Rademacher series. We denote $R_{\varepsilon k}(z) = \sum_{N(k) \leq n < N(k+1)} \varepsilon_n a_n z^n$ $(k = 0, 1, \cdots)$. We have

$$\sqrt{A_\alpha(f_\varepsilon, \theta)} \leq \sqrt{A_\alpha(a_1 z, \theta)} + \sum_{k=0}^{\infty} \sqrt{A_\alpha(R_{\varepsilon k}, \theta)} .$$

We show

$$\tilde{p}\left(\sqrt{\|A_\alpha(R_{\varepsilon k}, \cdot)\|_\infty} \geq c_1(\log N(k+1))^{1/2}\left(\sum_{N(k) \leq n < N(k+1)} c_\alpha(n, n)\, |a_n|^2\right)^{1/2}\right)$$
$$\leq c_2 N(k+1)^{-1} .$$

Set $\ell(k) = N(k+1) - N(k)$, $\tilde{\varepsilon}_\mu = \varepsilon_{N(k)-1+\mu}$, $b_\mu = a_{N(k)-1+\mu}$ and $b_\mu(\theta) = a_{N(k)-1+\mu} e^{i(N(k)-1+\mu)\theta}$ $(\mu = 1, \cdots, \ell(k))$. We denote by $b_\varepsilon(\theta) = (\tilde{\varepsilon}_1 b_1(\theta), \cdots, \tilde{\varepsilon}_{\ell(k)} b_{\ell(k)}(\theta))$ and

$$C = (c_{\mu\nu})_{\mu,\nu=1,\cdots,\ell(k)}$$
$$= \begin{pmatrix} c_\alpha(N(k), N(k)), \cdots c_\alpha(N(k), N(k+1)-1) \\ \vdots \\ c_\alpha(N(k+1)-1, N(k)), \cdots c_\alpha(N(k+1)-1, N(k+1)-1) \end{pmatrix} .$$

Since C is positive definite, there exists a unitary matrix $U = (u_{\mu\nu})_{\mu\nu=1, \cdots, \ell(k)}$ such that $U^* C U = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_{\ell(k)} \end{pmatrix}$, where $\{\lambda_\mu\}_{\mu=1}^{\ell(k)}$ are eigen values of C. Set $d_{\tilde{\varepsilon}\nu}(\theta) = \sum_{\mu=1}^{\ell(k)} \tilde{\varepsilon}_\mu b_\mu(\theta) u_{\mu\nu}$ $(\nu = 1, \cdots, \ell(k))$. Then we have

$$A_\alpha(R_{\varepsilon k}, \theta) = \boldsymbol{b}_\varepsilon(\theta) C \boldsymbol{b}_\varepsilon^*(\theta) = \sum_{\nu=1}^{\ell(k)} \lambda_\nu\, |d_\nu(\theta)|^2 .$$

Since $\deg b_\nu(\theta) \leq N(k+1)$, we have

$$\tilde{p}\left(\|d_{\tilde{\varepsilon}\nu}\|_\infty \geq c_1(\log N(k+1))^{1/2}\left(\sum_{\mu=1}^{\ell(k)} |b_\mu|^2\, |u_{\mu\nu}|^2\right)^{1/2}\right) \leq c_2(N(k+1))^{-2} .$$

Therefore we have

$$\tilde{p}\left(\|d_{\varepsilon\nu}\|_\infty \geq c_1(\log N(k+1))^{1/2}\left(\sum_{\mu=1}^{\ell(k)} |b_\mu|^2\, |u_{\mu\nu}|^2\right)^{1/2} \text{ for some } \nu\ (1 \leq \nu \leq \ell(k))\right)$$
$$\leq c_2 N(k+1)^{-1} .$$

Since

$$\|A_\alpha(R_{\varepsilon k},\cdot)\|_\infty \le \sum_{\nu=1}^{\ell(k)} \lambda_\mu \|d_{\tilde{\varepsilon}\nu}\|_\infty$$

and

$$\sum_{\nu=1}^{\ell(k)} \lambda_\mu \sum_{\mu=1}^{\ell(k)} |b_\mu|^2 |u_{\mu\nu}|^2 = \sum_{\mu=1}^{\ell(k)} |b_\mu|^2 \sum_{\mu=1}^{\ell(k)} \lambda_\mu |u_{\mu\nu}|^2 = \sum_{\mu=1}^{\ell(k)} |b_\mu|^2 c_{\mu\mu}$$
$$= \sum_{N(k)\le n<N(k+1)} c_\alpha(n,n)\,|a_n|^2\,,$$

we have

$$\tilde{p}\left(\sqrt{\|A_\alpha(R_{\varepsilon k},\cdot)\|_\infty} \ge c_1(\log N(k+1))^{1/2}\left(\sum_{N(k)\le n<N(k+1)} c_\alpha(n,n)\,|a_n|^2\right)^{1/2}\right)$$
$$\le c_2 N(k+1)^{-1}\,.$$

By the Borel-Cantelli lemma, we have

$$\sqrt{\|A_\alpha(R_{\varepsilon k},\cdot)\|_\infty} = O\left((\log N(k+1))^{1/2}\left(\sum_{N(k)\le n<N(k+1)} c_\alpha(n,n)\,|a_n|^2\right)^{1/2}\right) \qquad \text{a.s. } (\tilde{p}).$$

Since

$$\sum_{k=0}^{\infty} (\log N(k+1))^{1/2}\left(\sum_{N(k)\le n<N(k+1)} c_\alpha(n,n)\,|a_n|^2\right)^{1/2} \approx \sum_{k=0}^{\infty} 2^{k/2}\left(\sum_{2^k\le j<2^{k+1}} s_j^2\right)^{1/2}$$
$$\le \sum_{k=0}^{\infty} 2^k s_{2^k} \le \sum_{j=0}^{\infty} s_j + s_0 < +\infty\,,$$

we have $\|A_\alpha(f_\varepsilon,\cdot)\|_\infty < +\infty$ a.s. $(\tilde{p})$. We show this in the general case. Consider a random Taylor series $f_{\varepsilon X}(z) = \sum_{n=1}^{\infty} \varepsilon_n X_n a_n z^n$. Set

$$T_k(\omega) = 2^{k/2}\left(\sum_{N(k)\le n<N(k+1)} X_n(\omega)^2 c_\alpha(n,n)\,|a_n|^2\right)^{1/2}.$$

Then we have

$$\mathscr{E}\left[\sum_{k=0}^{\infty} T_k(\omega)\right] \le \mathscr{E}\left[\sqrt{\sum_{k=0}^{\infty} T_k(\omega)^2(\mathscr{E}[T_k(\omega)^2])^{-1/2}}\ \sqrt{\sum_{k=0}^{\infty} (\mathscr{E}[T_k(\omega)^2])^{1/2}}\right.$$
$$\le \sum_{k=0}^{\infty} (\mathscr{E}[T_k(\omega)^2])^{1/2} \approx \sum_{k=0}^{\infty} 2^{k/2}\left(\sum_{2^k\le j<2^{k+1}} s_j^2\right)^{1/2}$$
$$\le \sum_{j=0}^{\infty} s_j + s_0 < +\infty\,.$$

Consequently $\sum_{k=0}^{\infty} T_k(\omega) < +\infty$ a.s. (p). Therefore we have $\|A_\alpha(f_{\varepsilon X},\cdot)\|_\infty < +\infty$ a.s. $(\tilde{p})$ for each ω such that $\sum_{k=0}^{\infty} T_k(\omega) < +\infty$. Hence $\|A_\alpha(f_X,\cdot)\|_\infty$

$< +\infty$ a.s. $(\tilde{p} \times p)$. There exists a sequence $\overset{*}{\varepsilon} = (\overset{*}{\varepsilon}_n)_{n=1}^{\infty}$ of numbers 1 or -1 such that $\|A_\alpha(f_{\overset{*}{\varepsilon}X}, \cdot)\|_\infty < +\infty$ a.s. (p). For positive integers N, ℓ and k,

$$F_{\ell,k}^N = \left\{(x_1, \cdots, x_N)\,;\ \sup_\theta \left|\sum_{n=1}^N x_n x_m a_n \bar{a}_m e^{i(n-m)\theta} c_\alpha\left(n, m\,;\ 1 - \frac{1}{k}\right)\right| < \ell\right\}$$

$$E_{\ell,k}^N = \{\omega \in \Omega\,;\ (X_1(\omega), \cdots, X_N(\omega)) \in F_{\ell,k}^N\}$$

and

$$\overset{*}{E}{}_{\ell,k}^N = \{\omega \in \Omega\,;\ (\overset{*}{\varepsilon}_1 X_1(\omega), \cdots, \overset{*}{\varepsilon}_N X_N(\omega)) \in F_{\ell,k}^N\}\ .$$

If $F_{\ell,k}^N$ is a cylinder set, $p(E_{\ell,k}^N) = p(\overset{*}{E}{}_{\ell,k}^N)$ (since $X_n's$ are symmetric). In the general case, using a limit process, we have $p(E_{\ell,k}^N) = p(\overset{*}{E}{}_{\ell,k}^N)$. Since $\lim_{\ell\to\infty} \lim_{k\to\infty} \lim_{N\to\infty} p(E_{\ell,k}^N) = \lim_{\ell\to\infty} \lim_{k\to\infty} \lim_{N\to\infty} p(\overset{*}{E}{}_{\ell,k}^N) = 1$, we have $\|A_\alpha(f_X, \cdot)\|_\infty < +\infty$ a.s.. This completes the proof.

COROLLARY 3. *Let $f_X(z) = \sum_{n=1}^\infty X_n a_n z^n$ be a random Taylor series. Set $s_j = (\sum_{2^j \le n < 2^{j+1}} \mathscr{E}(X_n^2)\,|a_m|^2)^{1/2}$ $(j = 0, 1, \cdots)$. If $(s_j)_{j=0}^\infty$ is a decreasing sequence and f_X is bounded a.s., then $A_0(f_X, \cdot)$ is also bounded a.s..*

Proof. It is known that if f_X is bounded a.s., then $\sum_{j=0}^\infty s_j < +\infty$ ([1] p. 72). By Theorem 1, we have $\|A_0(f_X, \cdot)\|_\infty < +\infty$ a.s.. This completes the proof.

THEOREM 1′. *Let f_X and $(s_j)_{j=0}^\infty$ be the same as in Corollary 3. If $\sum_{j=0}^\infty j^{1/2} s_j < +\infty$, then $V(f_X, \cdot)$ is bounded a.s..*

Proof. First, we consider the case of Rademacher series. We denote by $Q_{\varepsilon k}(z) = \sum_{2^k \le n < 2^{k+1}} \varepsilon_n n a_n z^{n-2^k}$ and $\tilde{Q}_{\varepsilon k}(\theta) = Q_{\varepsilon k}(e^{i\theta})$ $(k = 0, 1, \cdots)$. Since

$$V(f_\varepsilon, \theta) \le \sum_{k=0}^\infty \int_0^1 r^{2^k - 1} |Q_{\varepsilon k}(z)|\, dr \le \sum_{k=0}^\infty 2^{-k} \|\tilde{Q}_{\varepsilon k}\|_\infty\ ,$$

it is sufficient to show that $\sum_{k=0}^\infty 2^{-k} \|\tilde{Q}_{\varepsilon k}\|_\infty < +\infty$ a.s. $(\tilde{p})$. By Lemma 2, we have

$$\tilde{p}\left(\|\tilde{Q}_{\varepsilon k}\|_\infty \ge c_1 k^{1/2} \left(\sum_{2^k \le n < 2^{k+1}} n^2 |a_n|^2\right)^{1/2}\right) \le c_2 2^{-2k}\ .$$

By the Borel-Cantelli lemma, we have

$$\|\tilde{Q}_{sk}\|_\infty = O\Big(k^{1/2}\Big(\sum_{2^k \le n < 2^{k+1}} n^2 |a_n|^2\Big)^{1/2}\Big) \qquad \text{a.s..}$$

Since

$$\sum_{k=0}^{\infty} 2^{-k} k^{1/2}\Big(\sum_{2^k \le n < 2^{k+1}} n^2 |a_n|^2\Big)^{1/2} \lesssim \sum_{k=0}^{\infty} k^{1/2} s_k < +\infty ,$$

we have $\sum_{k=0}^{\infty} 2^{-k} \|\tilde{Q}_{sk}\|_\infty < +\infty$ a.s.. In the general case, using the same method as in Theorem 1, we obtain the proof. Hence we omit the rest of the proof.

Next, we prove the following:

THEOREM 2. *Let $|\alpha| < 1$. Let $X = (X_n)_{n=1}^{\infty}$ be a sequence of real valued normal Gaussian variables and $f_X(z) = \sum_{n=1}^{\infty} X_n a_n z^n$ a random Taylor series by X. If $\sum_{n=1}^{\infty} n^{\alpha}(\log n)|a_n|^2 < +\infty$, then $A_0(f_X, \cdot)$ is bounded a.s..*

LEMMA 3. *Let Y be a real valued Gaussian variable such that $\mathscr{E}[Y] = 0$ and $\mathscr{E}[Y^2] = \sigma$. Then for any $E \in \mathfrak{B}$, we have*

$$\int_E |Y|^2 \, dp(\omega) \le \sigma p(E)\Big(4 \log \frac{1}{p(E)} + \frac{e^{-1/2}}{\sqrt{\pi}}\Big) .$$

Proof. We have $se^{-s^2/4} \le \sqrt{2}\, e^{-1/2}$. We have

$$\int_E |Y|^2 \, dp(\omega) = \int_{E;\ |Y|^2 \le \sigma 4 \log (1/p(E))} + \int_{E;\ |Y|^2 > \sigma 4 \log (1/p(E))} = I_1 + I_2 ,$$

$$I_1 \le \sigma p(E) 4 \log \frac{1}{p(E)}$$

and

$$\begin{aligned} I_2 &\le \frac{2}{\sqrt{2\pi\sigma}} \int_{2\sqrt{\sigma}\sqrt{\log (1/p(E))}}^{\infty} s^2 e^{-s^2/2\sigma} ds = \frac{\sqrt{2}}{\pi} \sigma \int_{2\sqrt{\log (1/p(E))}}^{\infty} s^2 e^{-s^2/2} ds \\ &\le \frac{2}{\sqrt{\pi}} e^{-1/2} \sigma \int_{2\sqrt{\log (1/p(E))}}^{\infty} s e^{-s^2/4} ds = \frac{e^{-1/2}}{\sqrt{\pi}} \sigma p(E) . \end{aligned}$$

Therefore we have

$$\int_E |Y|^2 \, dp(\omega) \le \sigma p(E)\Big(4 \log \frac{1}{p(E)} + \frac{e^{-1/2}}{\sqrt{\pi}}\Big) .$$

LEMMA 4. *Set $r_j = 1 - 2^{-j}$ and*

$$A_{\alpha j}(f_X, \theta) = \int_{r_j}^{r_{j+1}} (1-r)^{-\alpha} r dr \int_{\theta-(1-r)}^{\theta+(1-r)} |f_X'(re^{i\psi})|^2 d\psi$$

$j = 0, 1, \cdots$. *Then we have, for* $\theta, \varphi \in T$ *such that* $|\theta - \varphi| < 1$.

$$A_{\alpha j}(f_X, \theta) \leq A_{\alpha j}(f_X, \varphi) + 2^{1+\alpha}\Big(|\theta-\varphi| 2^{j\alpha} + \frac{1}{1-\alpha}|\theta-\varphi|^{1-\alpha}\Big) \sum_{n=1}^{\infty} |X_n|^2 n^2 |a_n|^2 r_{j+1}^{n-1} .$$

Proof. We can assume $0 < \varphi < \theta < 1$. We have

$$\begin{aligned}
&A_{\alpha j}(f_X, \theta) - A_{\alpha j}(f_X, \varphi) \\
&= \int_{r_j}^{r_{j+1}} (1-r)^{-\alpha} r dr \left\{\int_{\theta-(1-r)}^{\theta+(1-r)} - \int_{\varphi-(1-r)}^{\varphi+(1-r)}\right\} |f_X'(re^{i\psi})|^2 d\psi \\
&= \int_{\substack{r_j \\ 0<r<1-(\theta-\varphi)/2}}^{r_{j+1}} (1-r)^{-\alpha} r dr \left\{\int_{\varphi-(1-r)}^{\theta+(1-r)} - \int_{\varphi-(1-r)}^{\theta-(1-r)}\right\} |f_X'(re^{i\psi})|^2 d\psi \\
&\quad + \int_{\substack{r_j \\ 1-(\theta-\varphi)/2<r<1}}^{r_{j+1}} (1-r)^{-\alpha} r dr \left\{\int_{\theta-(1-r)}^{\theta+(1-r)} - \int_{\varphi-(1-r)}^{\varphi+(1-r)}\right\} |f_X'(re^{i\psi})|^2 d\psi = J_1 + J_2 ,
\end{aligned}$$

$$\begin{aligned}
J_1 &\leq \int_{r_j}^{r_{j+1}} (1-r)^{-\alpha} r dr \left\{\int_{\varphi+(1-r)}^{\theta+(1-r)} + \int_{\varphi-(1-r)}^{\theta-(1-r)}\right\} \Big(\sum_{n=1}^{\infty} |X_n|^2 n^2 |a_n|^2 r^{n-1} \sum_{n=1}^{\infty} r^{n-1}\Big) d\psi \\
&\leq 2(\theta-\varphi) \sum_{n=1}^{\infty} |X_n|^2 n^2 |a_n|^2 r_{j+1}^{n-1} \int_{r_j}^{r_{j+1}} (1-r)^{-1-\alpha} dr \\
&\leq 2^{1+\alpha}(\theta-\varphi) 2^{j\alpha} \sum_{n=1}^{\infty} |X_n|^2 n^2 |a_n|^2 r_{j+1}^{n-1}
\end{aligned}$$

and

$$\begin{aligned}
J_2 &\leq 4 \int_{\substack{r_j \\ 1-(\theta-\varphi)/2<r<1}}^{r_{j+1}} (1-r)^{1-\alpha} r \cdot \sum_{n=1}^{\infty} |X_n|^2 n^2 |a_n|^2 r^{n-1} \cdot \sum_{n=1}^{\infty} r^{n-1} dr \\
&\leq 4 \sum_{n=1}^{\infty} |X_n|^2 n^2 |a_n|^2 r_{j+1}^{n-1} \int_{1-(\theta-\varphi)/2}^{1} (1-r)^{-\alpha} dr \\
&= \frac{2^{1+\alpha}}{1-\alpha} (\theta-\varphi)^{1-\alpha} \sum_{n=1}^{\infty} |X_n|^2 n^2 |a_n|^2 r_{j+1}^{n-1} .
\end{aligned}$$

This completes the proof.

Proof of Theorem 2. We may assume that a_n's are real. Since $n^\alpha |a_n|^2 = O(1)$, we can assume that $|a_n| \leq n$. If $\sum_{n=1}^{\infty} n^2 |a_n|^2 < +\infty$, we have

$$\mathscr{E}[\|A_\alpha(f_X, \cdot)\|_\infty] \leq \mathscr{E}\Big[2 \int_0^1 (1-r)^{1-\alpha} \cdot r \cdot \sum_{n=1}^{\infty} |X_n|^2 n^2 |a_n|^2 r^{n-1} \cdot \sum_{n=1}^{\infty} r^{n-1} dr\Big]$$

$$\leq \sum_{n=1}^{\infty} n^2 |a_n|^2 \cdot 2 \int_0^1 (1-r)^{-\alpha} dr = \frac{2}{1-\alpha} \sum_{n=1}^{\infty} n^2 |a_n|^2 < +\infty .$$

Therefore $\|A_\alpha(f_X, \cdot)\|_\infty < +\infty$ a.s.. Suppose $\sum_{n=1}^{\infty} n^2 |a_n|^2 = +\infty$. We have, for each j_0,

$$\|A_\alpha(f_X, \cdot)\|_\infty \leq \frac{2}{1-\alpha} \sum_{n=1}^{\infty} |X_n|^2 n^2 |a_n|^2 r_{j_0}^{n-1} + \sum_{j=j_0}^{\infty} \|A_{\alpha j}(f_X, \cdot)\|_\infty .$$

Since $\sum_{n=1}^{\infty} |X_n|^2 n^2 |a_n|^2 r_{j_0}^{n-1} < +\infty$ a.s. for each j_0, it is sufficient to show that $\sum_{j=j_0}^{\infty} \|A_{\alpha j}(f_X, \cdot)\|_\infty < +\infty$ a.s. for some j_0. There exists j_0 such that $\sum_{n=1}^{\infty} n^2 |a_n|^2 r_{j_0}^{2n-1} > 1$. For a positive integer ℓ, let $E_j(\ell)$ be the event:

$$\|A_{\alpha j}(f_X, \cdot)\|_\infty \geq \ell \log \frac{1}{1-r_j} \sum_{n=1}^{\infty} n^2 |a_n|^2 \int_{r_j}^{r_{j+1}} (1-r)^{1-\alpha} r^{2n-1} dr .$$

We shall show that $p(\limsup_{j\to\infty} E_j(\ell)) = 0$ for some $\ell > 0$. Choose a random variable $\theta_j(\omega)$ such that $A_{\alpha j}(_{X(\omega)}, \theta_j(\omega)) = \|A_{\alpha j}(f_{X(\omega)}, \cdot)\|_\infty$. Let N be an integer such that $2^N \geq 2^{16+4|\alpha|} \max(1, 1/(1-\alpha))$. Then $2^{-(j+1)|\alpha|} \geq 2^{11+\alpha} \max(1, 1/(1-\alpha)) 2^{(5+|\alpha|+N)j}$ for any $j \geq 1$. Set $K = 2^{jN}$ and $\psi_k = 2\pi(k/K)$ $(k = 0, 1, \cdots, K-1)$. Let $E_j(\ell, k)$ be the event: E_j and $\theta_j(\omega) \in (\psi_k - \pi/K, \psi_k + \pi/K)$. We prove $p(E_j(\ell, k)) \leq \exp(e^{-1/2}/(4\sqrt{\pi})) 2^{-(\ell/12)j}$ for $j \geq j_0$. Suppose $\omega \in E_j(\ell, k)$. By Lemma 4, we have

$$\begin{aligned} A_{\alpha j}(f_{X(\omega)}, \theta_j(\omega)) &\leq A_{\alpha j}(f_{X(\omega)}, \psi_k) \\ &\quad + 2^{1+\alpha}\left(2^{(-N+\alpha)j} + \frac{1}{1-\alpha} 2^{-N(1-\alpha)j}\right) \sum_{n=1}^{\infty} |X_n(\omega)|^2 n^2 |a_n|^2 r_{j+1}^{n-1} . \end{aligned}$$

We integrate each term by $dp|_{E_j(\ell,k)}$ and use Lemma 3. Then we have

$$\begin{aligned} &\int_{E_j(\ell,k)} A_{\alpha j}(f_{X(\omega)}, \theta_j(\omega)) dp(\omega) \\ &\quad \leq \int_{E_j(\ell,k)} A_{\alpha j}(f_{X(\omega)}, \psi_k) dp(\omega) + 2^{1+\alpha}\left(2^{(-N+\alpha)j} + \frac{1}{1-\alpha} 2^{-N(1-\alpha)j}\right) \\ &\quad\quad \times \sum_{n=1}^{\infty} n^2 |a_n|^2 r_{j+1}^{n-1} \int_{E_j(\ell,k)} |X_{n(\omega)}|^2 dp(\omega) = I_1 + I_2 , \\ &I_1 = \int_{r_j}^{r_{j+1}} (1-r)^{-\alpha} r dr \int_{\psi_k-(1-r)}^{\psi_k+(1-r)} d\psi \left\{ \int_{E_j(\ell,k)} \left| \sum_{n=1}^{\infty} X_n n a_n r^{n-1} \cos(n-1)\psi \right|^2 dp(\omega) \right. \\ &\quad\quad \left. + \int_{E_j(\ell,k)} \left| \sum_{n=1}^{\infty} X_n n a_n r^{n-1} \sin(n-1)\psi \right|^2 dp(\omega) \right\} \\ &\quad \leq 2 \int_{r_j}^{r_{j+1}} (1-r)^{1-\alpha} r \cdot \sum_{n=1}^{\infty} n^2 |a_n|^2 r^{2n-2} dr \, p(E_j(\ell, k)) \end{aligned}$$

$$\times\left(4\log\frac{1}{p(E_j(\ell,k))}+\frac{e^{-1/2}}{\sqrt{\pi}}\right),$$

and

$$I_2\le 2^{1+\alpha}\max\left(1,\frac{1}{1-\alpha}\right)2^{(|\alpha|-N)j}\left(\sum_{n=1}^{\infty}n^4r_{j+1}^{n-1}\right)p(E_j(\ell,k))$$
$$\times\left(4\log\frac{1}{p(E_j(\ell,k))}+\frac{e^{-1/2}}{\sqrt{\pi}}\right)$$
$$\le 2^{11+\alpha}\max\left(1,\frac{1}{1-\alpha}\right)2^{(5+|\alpha|-N)j}p(E_j(\ell,k))$$
$$\times\left(4\log\frac{1}{p(E_j(\ell,k))}+\frac{e^{-1/2}}{\sqrt{\pi}}\right)$$

$$\left(\text{since }\sum_{n=1}^{\infty}n^4r_{j+1}^{n-1}\le\sum_{n=1}^{\infty}n(n+1)(n+2)(n+3)r_{j+1}^{n-1}\le\frac{2^5}{(1-r_{j+1})^5}=2^{10}\cdot 2^{5j}\right).$$

For $j\ge j_0$, we have

$$\int_{r_j}^{r_{j+1}}(1-r)^{1-\alpha}\sum_{n=1}^{\infty}n^2|a_n|^2r^{2n-1}dr\ge\int_{r_j}^{r_{j+1}}(1-r)^{1-\alpha}dr\ge 2^{-(j+1)|\alpha|}$$
$$\ge 2^{11+\alpha}\max\left(1,\frac{1}{1-\alpha}\right)2^{(5+|\alpha|-N)j}.$$

Therefore we have, for $j\ge j_0$,

$$\int_{E_j(\ell,k)}A_{\alpha j}(f_{X(\omega)},\theta_j(\omega))dp(\omega)$$
$$\le 3\sum_{n=1}^{\infty}n^2|a_n|^2\int_{r_j}^{r_{j+1}}(1-r)^{1-\alpha}r^{2n-1}dr\,p(E_j(\ell,k))$$
$$\times\left(4\log\frac{1}{E_j(\ell,k)}+\frac{e^{-1/2}}{\sqrt{\pi}}\right).$$

On the other hand, we have

$$\int_{E_j(\ell,k)}A_{\alpha j}(f_{X(\omega)},\theta_j(\omega))dp(\omega)$$
$$\ge \ell p(E_j(\ell,k))\log\frac{1}{1-r_j}\sum_{n=1}^{\infty}n^2|a_n|^2\int_{r_j}^{r_{j+1}}(1-r)^{1-\alpha}r^{2n-1}dr.$$

Therefore $p(E_j(\ell,k))\le\exp(e^{-1/2}/(4\sqrt{\pi}))2^{-(\ell/12)j}$ for $j\ge j_0$. Consequently, we have $p(E_j(\ell))\le\exp((e^{-1/2}/(4\sqrt{\pi}))2^{(N-(\ell/12))j}$ for $j\ge j_0$. Choose $\ell_0=12N+12$. Then $p(E_j(\ell_0))\le\exp((e^{-1/2}/(4\sqrt{\pi}))2^{-j}$ for $j\ge j_0$. By the Borel-Cantelli lemma, we have $(\limsup_{j\to\infty\ j>j_0}E_j(\ell_0))=0$. So we have

$$\|A_\alpha(f_X,\cdot)\|_\infty = O\Big(\log\frac{1}{1-r_j}\sum_{n=1}^{\infty} n^2|a_n|^2\int_{r_j}^{r_{j+1}}(1-r)^{1-\alpha}r^{2n-1}dr\Big)$$
$$= O\Big(\sum_{n=1}^{\infty} n^2|a_n|^2\int_{r_j}^{r_{j+1}}(1-r)^{1-\alpha}r^{2n-1}\log\frac{1}{1-r}dr\Big)$$
$$j = j_0, j_0+1, \cdots \qquad \text{a.s..}$$

Since

$$\sum_{j=0}^{\infty}\int_{r_j}^{r_{j+1}}(1-r)^{1-\alpha}r^{2n-1}\log\frac{1}{1-r}dr$$
$$\leq \int_0^1 (1-r)^{1-\alpha}r^{2n-1}\log\frac{1}{1-r}dr$$
$$= \sum_{m=1}^{\infty}\frac{1}{m}\int_0^1(1-r)^{1-\alpha}r^{2n+m-1}dr \approx \sum_{m=1}^{\infty}\frac{1}{m(n+m)^{2-\alpha}}$$
$$\leq \frac{1}{n^{2-\alpha}}\sum_{m=1}^{n}\frac{1}{m} + \frac{1}{n^{1-\alpha}}\sum_{m=n}^{\infty}\frac{1}{m^2} \approx n^{\alpha-2}\log n\ ,$$

we have

$$\sum_{j=0}^{\infty}\sum_{n=1}^{\infty} n^2|a_n|^2\int_{r_j}^{r_{j+1}}(1-r)^{1-\alpha}r^{2n-1}\log\frac{1}{1-r}dr \lesssim \sum_{n=1}^{\infty} n^\alpha(\log n)|a_n|^2 < +\infty\ .$$

Therefore $\sum_{j=j_0}^{\infty}\|A_{\alpha j}(f_X,\cdot)\|_\infty < +\infty$ a.s.. This completes the proof.

By Theorem 2, we can answer the converse problem to Corollary 3. That is, we can show that there exists a random Taylor series f_X such that $\|f_X\|_\infty = +\infty$ and $\|A_0(f_X,\cdot)\|_\infty < +\infty$ a.s.. For example, set $a_{2^j} = 1/(j\log j)$ $(j=2,\cdots)$ and $a_n = 0$ for $n \neq 2^j$ $(j=2,\cdots)$. Let $X=(X_n)_{n=1}^{\infty}$ be the same as in Theorem 2. Then $\sum_{j=0}^{\infty}(\sum_{2^j\leq n<2^{j+1}}|a_n|^2)^{1/2} = \sum_{j=0}^{\infty} a_{2^j} = +\infty$. Therefore $f_X(z) = \sum_{n=1}^{\infty} X_n a_n z^n$ is unbounded a.s.. On the other hand, since $\sum_{n=1}^{\infty}(\log n)|a_n|^2 < +\infty$, we have $\|A_0(f_X,\cdot)\|_\infty < +\infty$ a.s..

The method of the proof is usual. But it has many applications. Since the case of $V(f_X,\cdot)$ is typical, we show some applications for $V(f_X,\cdot)$.

PROPOSITION 6. *Let $X=(X_n)_{n=1}^{\infty}$ and f_X be the same as in Theorem 2. For any $m \geq 1$, we have with constant c_1,*

$$p\Big(V(f_X,0) \geq c_1 m\int_0^1\sqrt{\sum_{n=1}^{\infty} n^2|a_n|^2 r^{2n-2}}dr\Big) \leq e^{-m^2}\ .$$

LEMMA 5. *Let Y be the same as in Lemma 3. Then for any $E \in \mathfrak{B}$, we have*

$$\int_E |Y|\, dp(\omega) \leq \sqrt{\sigma}\, p(E)\left(\sqrt{2}\sqrt{\log \frac{1}{p(E)}} + \sqrt{\frac{2}{\pi}}\right).$$

Proof. We have

$$\begin{aligned}\int_E |Y|\, dp(\omega) &\leq \int_{E;|Y|\leq\sqrt{\sigma}\sqrt{2\log 1/(p(E))}} + \int_{E;|Y|>\sqrt{\sigma}\sqrt{2\log 1/(p(E))}} \\ &\leq \sqrt{\sigma} p(E)\sqrt{2\log \frac{1}{p(E)}} + \frac{2}{\sqrt{2\pi\sigma}}\int_{\sqrt{\sigma}\sqrt{2\log 1/(p/(E))}}^{\infty} se^{-s^2/2\sigma}ds \\ &= \sqrt{\sigma} p(E)\left(\sqrt{2}\sqrt{\log \frac{1}{p(E)}} + \sqrt{\frac{2}{\pi}}\right).\end{aligned}$$

Proof of Proposition 6. Let E be the event:

$$V(f_X, 0) \geq 4\sqrt{2}\, m \int_0^1 \sqrt{\sum_{n=1}^{\infty} n^2 |a_n|^2 r^{2n-2}}\, dr .$$

Then we have

$$\begin{aligned}&p(E)4\sqrt{2}\, m \int_0^1 \sqrt{\sum_{n=1}^{\infty} n^2 |a_n|^2 r^{2n-2}}\, dr \\ &\leq \int_E V(f_X, 0) dp(\omega) \\ &\leq 2\int_0^1 \sqrt{\sum_{n=1}^{\infty} n^2 |a_n|^2 r^{2n-2}}\, dr\; p(E)\left(\sqrt{2}\sqrt{\log \frac{1}{p(E)}} + \sqrt{\frac{2}{\pi}}\right).\end{aligned}$$

Therefore $p(E) \leq e^{-(2m-1/\sqrt{\pi})^2} \leq e^{-m^2}$.

PROPOSITION 7. *Under the same hypothesis of Proposition* 6, *for any* $m < 1$, *we have, with constant* c_2,

$$p\left(V(f_X, 0) \leq c_2 m \int_0^1 \sqrt{\sum_{n=1}^{\infty} n^2 |a_n|^2 r^{2n-2}}\, dr\right) \geq 1 - m .$$

LEMMA 6. *Let Y be the same as in Lamma* 3. *Then for any* $E \in \mathfrak{B}$, *we have*

$$\int_E |Y|\, dp(\omega) \geq \sqrt{\frac{\pi}{8}}\sqrt{\sigma}\, p(E)^2 .$$

Proof. Choose a such that $p(|Y| \leq a) = \frac{1}{2}p(E)$. Then we have

$$a \geq \int_0^a e^{-s^2/2}ds = \sqrt{\frac{\pi}{2}}\sqrt{\sigma}\, p(|Y| \leq a) = \sqrt{\frac{2\pi}{4}}\sqrt{\sigma}\, p(E) .$$

Then we have

$$\int_E |Y|\,dp(\omega) \geq \int_{E:|Y|\geq a} |Y|\,dp(\omega)$$
$$\geq ap(E\,;|Y|\geq a)\; a\tfrac{1}{2}p(E) \geq \frac{\sqrt{2\pi}}{8}\sqrt{\sigma}\,p(E)^2 .$$

Proof of Proposition 7. Let E be the event:

$$V(f_X, 0) \leq \frac{\sqrt{2\pi}}{16} m \int_0^1 \sqrt{\sum_{n=1}^{\infty} n^2 |a_n|^2 r^{2n-2}}\, dr .$$

We may assume

$$\int_0^1 \sqrt{\sum_{n=1}^{\infty} n^2 (\mathrm{Re}\, a_n)^2 r^{2n-2}}\, dr \geq \frac{1}{2}\int_0^1 \sqrt{\sum_{n=1}^{\infty} n^2 |a_n|^2 r^{2n-2}}\, dr .$$

Then we have

$$p(E)\frac{\sqrt{2\pi}}{16} m \int_0^1 \sqrt{\sum_{n=1}^{\infty} n^2 |a_n|^2 r^{2n-2}}\, dr$$
$$\geq \int_E V(f_X, 0)\,dp(\omega) \geq \int_0^1 dr \int_E \left|\sum_{n=1}^{\infty} X_n n(\mathrm{Re}\, a_n) r^{n-1}\right| dp(\omega)$$
$$\geq \frac{\sqrt{2\pi}}{8}\int_0^1 \sqrt{\sum_{n=1}^{\infty} n^2(\mathrm{Re}\, a_n)^2 r^{2n-2}}\cdot p(E)^2$$
$$\geq \frac{\sqrt{2\pi}}{16}\int_0^1 \sqrt{\sum_{n=1}^{\infty} n^2 |a_n|^2 r^{2n-2}}\, dr\; p(E)^2 .$$

Therefore we have $p(E) \leq m$. Consequently, we have

$$p\left(V(f_X, 0) \geq \frac{\sqrt{2\pi}}{16} m \int_0^1 \sqrt{\sum_{n=1}^{\infty} n^2 |a_n|^2 r^{2n-2}}\, dr\right) \geq 1 - m .$$

THEOREM 2′. *Let $X = (X_n)_{n=1}^{\infty}$ and f_X be the same as in Theorem 2. If*

$$\int_0^1 \sqrt{\sum_{n=1}^{\infty} n^2 |a_n|^2 r^{2n-2} \log\frac{1}{1-r}}\, dr < +\infty ,$$

then $\|V(f_X, \cdot)\|_\infty < +\infty$ a.s..

Proof. The proof is analogous as in Theorem 2. For the sake of completeness, we give the proof. We can assume that a_n's are real and $|a_n| \leq 1$. There is nothing to prove in the case of $\sum_{n=1}^{\infty} n^2 |a_n|^2 < +\infty$. Suppose that $\sum_{n=1}^{\infty} n^2 |a_n|^2 = +\infty$. Let E_j be the event:

$$\max_\theta \int_{r_j}^{r_{j+1}} |f_X'(re^{i\theta})|\, dr \geq 15\sqrt{2}\sqrt{\log\frac{1}{1-r_j}} \int_{r_j}^{r_{j+1}} \sqrt{\sum_{n=1}^\infty n^2 |a_n|^2 r^{2n-2}} dr .$$

We shall show that $p(E_j) \leq \exp(1/(3\sqrt{\pi}))2^{-j}$ for large j. Set $K = 2^{4j}$ and $\psi_k = 2\pi(k/K)$ $(k = 0, 1, \cdots, K-1)$. Choose a random variable $\theta_j(\omega)$ such that

$$\int_{r_j}^{r_{j+1}} |f_{X(\omega)}'(re^{i\theta_j(\omega)})|\, dr = \max_\theta \int_{r_j}^{r_{j+1}} |f_{X(\omega)}'(re^{i\theta})|\, dr .$$

Let $E_j(k)$ $(k = 0, \cdots, K-1)$ be the event: E_j and $\theta_j(\omega) \in [\psi_k - \pi/K, \psi_k + \pi/K)$. We prove $p(E_j(k)) \leq \exp(1/(3\sqrt{\pi}))2^{-5j}$ for large j. Suppose $\omega \in E_j(k)$. Then

$$|f_{X(\omega)}'(re^{i\theta_j(\omega)})| \leq |f_{X(\omega)}'(re^{i\psi_k})| + \frac{\pi}{K}\sum_{n=1}^\infty |X_n(\omega)|\, n^2 |a_n|\, r^{n-2} .$$

Therefore

$$\begin{aligned}&\int_{r_j}^{r_{j+1}} |f_{X(\omega)}'(re^{i\theta_j(\omega)})|\, dr \\ &\quad \leq \int_{r_j}^{r_{j+1}} |f_{X(\omega)}'(re^{i\psi_k})|\, dr + \frac{\pi}{K} 2^{-j-1}\sum_{n=2}^\infty |X_n(\omega)|\, n^2 |a_n|\, r_{j+1}^{n-2} .\end{aligned}$$

Integrate each term by $dp|_{E_j(k)}$ and use Proposition 6. Then we have

$$\begin{aligned}&\int_{E_j(k)} dp(\omega) \int_{r_j}^{r_{j+1}} |f_{X(\omega)}'(re^{i\theta_j(\omega)})|\, dr \\ &\quad \leq \left(2\int_{r_j}^{r_{j+1}} \sqrt{\sum_{n=1}^\infty n^2 |a_n|^2 r^{2n-2}} dr + \pi 2^{-5j-1} \sum_{n=2}^\infty n^2 |a_n|\, r_{j+1}^{n-2}\right) \\ &\qquad \times p(E_j(k))\left(\sqrt{2}\sqrt{\log\frac{1}{p(E_j(k))}} + \sqrt{\frac{2}{\pi}}\right) .\end{aligned}$$

Since $\sum_{n=1}^\infty n^2 |a_n|^2 = +\infty$, there exists j_0 such that

$$\int_{r_j}^{r_{j+1}} \sqrt{\sum_{n=1}^\infty n^2 |a_n|^2 r^{2n-2}}\, dr \geq \pi 2^{-5j-1} \sum_{n=2}^\infty n^2 |a_n|\, r_{j+1}^{n-2}$$

for all $j \geq j_0$. Then we have, for $j \geq j_0$

$$\begin{aligned}&p(E_j(k))15\sqrt{2} \int_{r_j}^{r_{j+1}} \sqrt{\sum_{n=1}^\infty n^2 |a_n|^2 r^{2n-2}}\, dr \sqrt{\log\frac{1}{1-r_j}} \\ &\quad \leq 3\int_{r_j}^{r_{j+1}} \sqrt{\sum_{n=1}^\infty n^2 |a_n|^2 r^{2n-2}}\, dr p(E_j(k)) \\ &\qquad \times \left(\sqrt{2}\sqrt{\log\frac{1}{p(E_j(k))}} + \sqrt{\frac{2}{\pi}}\right) .\end{aligned}$$

Therefore $p(E_j(k)) \leq \exp(1/(3\sqrt{\pi}))2^{-5j}$ for $j \geq j_0$. Consequently, $p(E_j) \leq \exp(1/(3\sqrt{\pi}))2^{-j}$ for $j \geq j_0$. So we have

$$\max_{\theta} \int_{r_j}^{r_{j+1}} |f'_X(re^{i\theta})|\, dr = O\left(\sqrt{\log \frac{1}{1-r_j}} \int_{r_j}^{r_{j+1}} \sqrt{\sum_{n=1}^{\infty} n^2 |a_n|^2 r^{2n-2}}\, dr\right)$$
$$= O\left(\int_{r_j}^{r_{j+1}} \sqrt{\sum_{n=1}^{\infty} n^2 |a_n|^2 r^{2n-2} \log \frac{1}{1-r}}\, dr\right)$$
$$j = j_0, j_0+1, \cdots \text{ a.s..}$$

Since $\int_0^1 \sqrt{\sum_{n=1}^{\infty} n^2 |a_n|^2 r^{2n-2} \log \frac{1}{1-r}}\, dr < +\infty$, we have

$$\|V(f_X, \cdot)\|_\infty \leq \sum_{n=0}^{\infty} |X_n|\, n\, |a_n|\, r_{j_0}^{n-1} + \sum_{j=j_0}^{\infty} \max_{\theta} \int_{r_j}^{r_{j+1}} |f'_X(re^{i\theta})|\, dr < +\infty \text{ a.s..}$$

This completes the proof.

Next, we consider one of converse problems for Theorem 2.

THEOREM 3. *Let $|\alpha| < 1$ and let $f_X(z) = \sum_{n=1}^{\infty} X_n a_n z^n$ be a random Taylor series by $X = (X_n)_{n=1}^{\infty}$. If $\limsup_{N\to\infty} (\log N)^{-1} \sum_{n=1}^{N} \mathscr{E}[X_n^2] n^\alpha |a_n|^2 = +\infty$ and $n^\alpha |a_n|^2 = O(1)$, then $\limsup_{N\to\infty} A_\alpha(f_X^N, \theta) = +\infty$ for all θ a.s..*

For the proof, we use the probability space $(\tilde{\Omega} \times \Omega, \tilde{\mathfrak{B}} \times \mathfrak{B}, \tilde{p} \times p)$. We denote by $\tilde{\mathscr{E}}[\,\cdot\,]$ the expectation. Define a sequence $Y = (Y_n)_{n=1}^{\infty}$ of random variables on $\tilde{\Omega} \times \Omega$ by $Y_n(x, \omega) = \varepsilon_n(x) X_n(\omega)$.

LEMMA 7. *Let $(\nu_j)_{j=0}^{\infty}$ $(\nu_0 = 1)$ be an increasing sequence of positive integers. Set $P_{Yj}(\theta) = A_\alpha(f_Y^{\nu_j}, \theta) - A_\alpha(f_Y^{\nu_{j-1}}, \theta)$ and*

$$q_j = \left(\sum_{\nu_{j-1} < n \leq \nu_j} \tilde{\mathscr{E}}(Y_n^2) c_\alpha(n, n) |a_n|^2\right)^{1/2} \qquad (j = 1, 2, \cdots).$$

Let E_μ be the event:

There exists θ such that $P_{Yj}(\theta) \leq \frac{1}{4} q_j^2$ for $j = 1, \cdots, \mu$.

Then we have, with positive constants B, β $(0 < \beta < 1)$,

$$\tilde{p} \times p(E_\mu) \leq B \mu \nu_\mu^2 \left(\sum_{j=1}^{\mu} q_j^2\right)^{1/2} \sup\{q_j^{-1}; j = 1, \cdots, \mu\} \beta^\mu.$$

Proof. We denote by $(\Omega', \mathfrak{B}', p') = (\tilde{\Omega} \times \Omega, \tilde{\mathfrak{B}} \times \mathfrak{B}, \tilde{p} \times p)$. Set $\Omega'_j = \prod_{\nu_{j-1} < n \leq \nu_j} J_n \times I_n$. The element is denoted by (x_j, ω_j). Let $(\Omega'_j, \mathfrak{B}'_j, p'_j)$ be the usual probability space. We consider $(\Omega', \mathfrak{B}', p')$ as the product space $(\prod_{j=1}^{\infty} \Omega'_j, \prod_{j=1}^{\infty} \mathfrak{B}'_j, \prod_{j=1}^{\infty} p'_j)$. Set

$$Q_{Yj}(\theta) = Q_{Yj}(x_j, \omega_j)(\theta) = A_\alpha(f_Y^{\nu_j} - f_Y^{\nu_{j-1}}, \theta)$$

and

$$\begin{aligned} R_{Yj}(\theta) &= R_{Yj}[(x_1, \omega_1), \cdots, (x_j, \omega_j)](\theta) \\ &= 2 \operatorname{Re}\left(\sum_{\nu_{j-1} < n \le \nu_j} Y_n a_n e^{in\theta} \overline{\sum_{m \le \nu_{j-1}} Y_m a_m c_\alpha(n, m) e^{im\theta}} \right). \end{aligned}$$

Then we have $P_{Yj}(\theta) = Q_{Yj}(\theta) + R_{Yj}(\theta)$. Let $E(\theta, j)$ be the event: $Q_{Yj}(\theta) < \frac{1}{2}q_j^2$ or $R_{Yj}(\theta) < 0$. We show $p'(\bigcap_{j=1}^{\mu} E(\theta, j)) \le \gamma^\mu$ for some γ $(0 < \gamma < 1)$. For any $\{(x_k^*, \omega_k^*)\}_{k=1}^{j-1}$, let $E[(x_k^*, \omega_k^*);\ k = 1, \cdots, j-1](\theta)$ be the event:

$$Q_{Yj}(x_j, \omega_j)(\theta) < \tfrac{1}{2}q_j^2 \quad \text{or} \quad R_{Yj}[(x_1^*, \omega_1^*), \cdots, (x_{j-1}^*, \omega_{j-1}^*), (x_j, \omega_j)](\theta) < 0 .$$

By the Lemma 1, we have, with constant η $(0 < \eta < 1)$,

$$p_j'(Q_{Yj}(\theta) \ge \tfrac{1}{2}q_j^2) \ge \eta .$$

Suppose $Q_{Yj}(\tilde{x}_j, \tilde{\omega}_j)(\theta) \ge \frac{1}{2}q_j^2$ and $R_{Yj}[(x_1^*, \omega_1^*), \cdots, (x_{j-1}^*, \omega_{j-1}^*), (\tilde{x}_j, \tilde{\omega}_j)](\theta) < 0$ for some $(\tilde{x}_j, \tilde{\omega}_j)$. Then we have $Q_{Yj}(-\tilde{x}_j, \tilde{\omega}_j)(\theta) \ge \frac{1}{2}q_j^2$ and

$$R_{Yj}[(x_1^*, \omega_1^*), \cdots, (x_{j-1}^*, \omega_{j-1}^*), (-\tilde{x}_j, \tilde{\omega}_j)](\theta) > 0 .$$

Therefore we have

$$\begin{aligned} &p_j'(Q_{Yj}(\theta) \ge \tfrac{1}{2}q_j^2 \quad \text{and} \\ &R_{Yj}[(x_1^*, \omega_1^*), \cdots, (x_{j-1}^*, \omega_{j-1}^*), (x_j, \omega_j)](\theta) \ge 0) \ge \tfrac{1}{2}\eta . \end{aligned}$$

That is, $p_j'(E[(x_k^*, \omega_k^*);\ k = 1, \cdots, j-1](\theta)) \le 1 - \frac{1}{2}\eta$ $(=\gamma)$. We have

$$\begin{aligned} p'\left(\bigcap_{j=1}^{\mu} E(\theta, j)\right) &= p_1' \times \cdots \times p_j'\left(\bigcap_{j=1}^{\mu} E(\theta, j)\right) \\ &= \int_{\bigcap_{j=1}^{\mu-1} E(\theta, j)} p_\mu'(E[(x_k, \omega_k);\ k = 1, \cdots, \mu-1]) d(p_1' \times \cdots \times p_{\mu-1}') \\ &\le \gamma p_1' \times \cdots \times p_{\mu-1}'\left(\bigcap_{j=1}^{\mu-1} E(\theta, j)\right) \le \cdots \le \gamma^\mu . \end{aligned}$$

Let $F(\theta, j)$ be the event: $P_{Yj}(\theta) < \frac{1}{2}q_j^2$. Then $F(\theta, j) \subset E(\theta, j)$. Therefore $\bigcap_{j=1}^{\mu} F(\theta, j) \subset \bigcap_{j=1}^{\mu} E(\theta, j)$. We write $\psi_k = 2\pi(k/K)$ $(k = 0, \cdots, K-1)$, where K is an integer which will be determined later. Then we have $p'(\bigcup_{k=0}^{K-1} \bigcap_{j=1}^{\mu} F(\psi_k, j)) \le K\gamma^\mu$. Next, we estimate $\|P_{Yj}'\|_\infty$. We have

$$\begin{aligned} P_{Yj}(\theta) = &\sum_{\nu_{j-1} < n \le \nu_j} Y_n a_n e^{in\theta} \overline{\sum_{m \le \nu_j} Y_m c_\alpha(n, m) a_m e^{im\theta}} \\ &+ \sum_{n \le \nu_{j-1}} Y_n a_n e^{in\theta} \overline{\sum_{\nu_{j-1} < m \le \nu_j} Y_m c_\alpha(n, m) a_m e^{im\theta}} . \end{aligned}$$

Therefore we have

$$\begin{aligned}\|P'_{Yj}\|_\infty &\le 4\nu_j \sum_{n\le\nu_j} |Y_n||a_n| \sum_{\nu_{j-1}<m\le\nu_j} |Y_m||a_m|\, c_\alpha(n,m) \\ &\le 4\nu_j \sum_{n\le\nu_j} |Y_n||a_n| \sqrt{c_\alpha(n,n)} \sum_{\nu_{j-1}<m\le\nu_j} |Y_m||a_m| \sqrt{c_\alpha(m,m)} \\ &\le 4\nu_j^2 \sqrt{\sum_{n\le\nu_j} Y_n^2 |a_n|^2 c_\alpha(n,n)} \sqrt{\sum_{\nu_{j-1}<m\le\nu_j} Y_m^2 |a_m|^2 c_\alpha(m,m)}\,.\end{aligned}$$

We have $\mathscr{E}[\|P'_{Yj}\|_\infty] \le 4\nu_j^2(\sum_{k=1}^j q_k^2)^{1/2} q_j$. Consequently, we have

$$p'(\|P'_{Yj}\|_\infty \ge (4\pi)^{-1}Kq_j^2) \le 16\pi K^{-1}\nu_j^2\Big(\sum_{k=1}^j q_k^2\Big)^{1/2} q_j^{-1}\,.$$

Let F_μ be the event: $\|P'_{Yj}\|_\infty \le (4\pi)^{-1}Kq_j^2$ for $j=1,\cdots,\mu$. Then

$$p'(F_\mu^c) \le 16\pi K^{-1}\mu\nu_\mu^2\Big(\sum_{k=1}^\mu q_k^2\Big)^{1/2} \sup\{q_j^{-1};\, j=1,\cdots,\mu\}\,.$$

For any θ, there exists k such that $|P_{Yj}(\theta) - P_{Yj}(\psi_k)| \le \pi K^{-1}\|P'_{Yj}\|_\infty$. Therefore $P_{Yj}(\psi_k) \le \pi K^{-1}\|P'_{Yj}\|_\infty + P_{Yj}(\theta)$. If $(x,\omega)\in E_\mu \cap F_\mu$, then we have $\pi K^{-1}\|P'_{Y(x,\omega)j}\|_\infty \le \frac{1}{4}q_j^2$ and $P_{Yj}(\theta) \le \frac{1}{4}q_j^2$ for some θ and $j=1,\cdots,\mu$. Therefore we have for some k, $P_{Y(x,\omega)j}(\psi_k) \le \frac{1}{2}q_j^2$ $(j=1,\cdots,\mu)$. Hence we have $E_\mu \cap F_\mu \subset \bigcup_{k=0}^{k-1}\bigcap_{j=1}^\mu F(\psi_k, j)$. That is, $E_\mu \subset F_\mu^c \cup \bigcup_{k=1}^{k-1}\bigcap_{j=1}^\mu F(\psi_k, j)$. Consequently, we have

$$p'(E_\mu) \le K\gamma^\mu + 16\pi K^{-1}\mu\nu_\mu^2\Big(\sum_{j=1}^\mu q_j^2\Big)^{1/2} \sup\{q_j^{-1};\, j=1,\cdots,\mu\}\,.$$

Let K be the integer part of $\gamma^{-\mu/2}$. Then we have, with positive constant B,

$$p'(E_\mu) \le B\mu\nu_\mu^2\Big(\sum_{j=1}^\mu q_j^2\Big)^{1/2} \sup\{q_j^{-1};\, j=1,\cdots,\mu\}\gamma^{\mu/2}\,.$$

This completes the proof.

Proof of Theorem 3. We can assume $\tilde{\mathscr{E}}[Y_n^2] = \mathscr{E}[X_n^2] \le 1$ and $c_\alpha(n,n)|a_n|^2 \le 1$ for all n. Let ℓ $(\ell \ge 2)$ be an integer. We define a sequence $(\nu_j)_{j=1}^\infty$ of integers, inductively. Set $\nu_0 = 1$. Assume that $\{\nu_j\}_{j=1}^{\mu-1}$ are already chosen. Then let ν_μ be the smallest integer such that $\nu_\mu > \nu_{\mu-1}$ and $\sum_{\nu_{\mu-1}<n\le\nu_\mu} \tilde{\mathscr{E}}[Y_n^2]c_\alpha(n,n)|a_n|^2$ $(=q_\mu^2) \ge \ell$. Set $c_\mu = (\log\nu_\mu)^{-1}\sum_{k=1}^\mu q_k^2$. By the assumption $\limsup_{N\to\infty}(\log N)^{-1}\sum_{n=1}^N \tilde{\mathscr{E}}[Y_n^2]n^\alpha|a_n|^2 = +\infty$ and $q_j^2 \le \ell+1$ $(j=1,2,\cdots)$, we have $\limsup_{\mu\to\infty} c_\mu = +\infty$. We have

$$\mu\nu_\mu^2\Big(\sum_{j=1}^\mu q_j^2\Big)^{1/2} \sup\{q_j^{-1};\ j=1,\cdots,\mu\}\beta^\mu \le (\ell-1)^{-1}\mu\nu_\mu^3\beta^\mu$$

$$= (\ell - 1)^{-1}\mu \exp\left(3 \sum_{j=1}^{\mu} q_j^2 \frac{1}{c_\mu} - \mu \log \frac{1}{\beta}\right)$$
$$\leq (\ell - 1)^{-1}\mu \exp\left(3(\ell + 1)\frac{1}{c_\mu} - \log \frac{1}{\beta}\right)\mu .$$

Since $\liminf_{\mu\to\infty} c_\mu^{-1} = 0$, we have

$$\liminf_{\mu\to\infty} \mu\nu_\mu^2\left(\sum_{j=1}^{\mu} q_q^2\right)^{1/2} \sup\{q_j^{-1}; j = 1, \cdots, \mu\}\beta^\mu = 0 .$$

By Lemma 7, we have $\liminf_{\mu\to\infty} p'(E_\mu) = 0$. Let $G(\ell, m)$ be the event: there exists θ such that $P_{Yj}(\theta) \leq \frac{1}{4}\ell$ for $j = m, m + 1, \cdots$. Since $G(\ell, 1) \subset E_\mu$ for all μ, we have $p'(G(\ell, 1)) = 0$. By the same method, we have $p'(G(\ell, m)) = 0$ for all m, ℓ ($m, \ell = 2, 3, \cdots$). Therefore $p'(\bigcup_{\ell=2}^{\infty} \bigcup_{m=1}^{\infty} G(\ell, m)) = 0$. This show that $\limsup_{j\to\infty} P_{Yj}(\theta) = +\infty$ holds for all θ a.s. $(\tilde{p} \times p)$. Since $A_\alpha(f_Y^{\nu_j}, \theta) = P_{Yj}(\theta) + A_\alpha(f_Y^{\nu_{j-1}}, \theta) \geq P_{Yj}(\theta)$, we have

$$\limsup_{N\to\infty} A_\alpha(f_Y^N, \theta) = +\infty \qquad \text{for all } \theta \text{ a.s. } (\tilde{p} \times p) .$$

There exists $\varepsilon^* = (\varepsilon_n^*)_{n=1}^{\infty}$ ($\varepsilon_n^* = 1$ or -1) such that $\limsup_{n\to\infty} A_\alpha(f_{\varepsilon^* X}^N, \theta) = +\infty$ for all θ a.s.. Since $\{X_n\}_{n=1}^{\infty}$ are symmetric, (by the similar method as in Theorem 1,) we have $\limsup_{N\to\infty} A_\alpha(f_X^N, \theta) = +\infty$ for all θ a.s.. This completes the proof.

REFERENCES

[1] Kahane, J. P.; Some random series of functions, Raytheon education company, 1968.

[2] Lohwater, A. J. and Piranian, G.; On a conjecture of Lusin, Michigan Math. J., (1955), p. 63–68.

[3] Offord, A. C.; Distribution of values of random function in the unit disk, Studia Math., (1972), p. 71–106.

[4] Zygmund, A.; Trigonometric series II, Cambridge, 1959.

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