T. Kubota
Nagoya Math. J.
Vol. 61 (1976), 113-116

A COMPLEX AIRY INTEGRAL

Dedicated to Professor Tikao Tatuzawa on his 60th birthday

TOMIO KUBOTA

The Airy integral is a formula concerning the Fourier transform of a function like $\sin x^3$ or $\cos x^3$, and is written, for instance in [2], as

$$\int_0^\infty \cos(t^3 - xt)dt = \frac{1}{3}\pi\sqrt{\frac{1}{3}x}\left[J_{-1/3}\left(\frac{2x\sqrt{x}}{3\sqrt{3}}\right) + J_{1/3}\left(\frac{2x\sqrt{x}}{3\sqrt{3}}\right)\right]$$

for $x \geqq 0$.

In this paper, we shall prove a similar formula

$$(1) \quad \int_C e(z^3 - 3zw)dV(z) = \frac{1}{3}\pi^2\left(\sin\frac{\pi}{3}\right)^{-1}|w|(|J_{-1/3}(2\pi w^{3/2})|^2 - |J_{1/3}(2\pi w^{3/2})|^2)$$

containing same Bessel functions and the exponential function $e(z) = \exp(\pi\sqrt{-1}(z + \bar{z}))$, where $dV(z)$ is the usual euclidean measure, and the integral $\int_C$ should be interpreted as $\lim_{Y\to\infty}\int_{|z|<Y}$. This is a byproduct of the results in [1].

The proof of our main result (1) is reduced to an equality between Mellin transforms of certain functions. Let us first treat the purely computational part of the proof. If $z = r\exp(\sqrt{-1}\theta)$ and $w = r'\exp(\sqrt{-1}\theta')$, $(r, r' \geqq 0,\ \theta, \theta' \in \boldsymbol{R})$, are polar expressions of complex numbers z and w, then a general formula on the Bessel function J_m says

$$e(z) = \sum_{m=-\infty}^{\infty} \sqrt{-1}^m J_m(2\pi r)\exp(\sqrt{-1}m\theta) .$$

This implies

$$e(z^3) = \sum_{m=-\infty}^{\infty} \sqrt{-1}^m J_m(2\pi r^3)\exp(\sqrt{-1}3m\theta) ,$$

$$e(-3zw) = \sum_{m=-\infty}^{\infty} (-\sqrt{-1})^m J_m(6\pi rr')\exp(\sqrt{-1}m(\theta + \theta')) ,$$

Received June 2, 1975.
This research is supported by NSF Grant GP-43950 (SK-CUCB).

and consequently

$$(2)\qquad \int_C e(z^3)e(-3zw)dV(z)$$

$$= \sum_{m=-\infty}^{\infty}\left[2\pi\int_0^\infty J_{-m}(2\pi r^3)J_{3m}(6\pi rr')rdr\right]\exp(\sqrt{-1}3m\theta')$$

$$= \sum_{m=-\infty}^{\infty}(-1)^m\left[2\pi\int_0^\infty J_m(2\pi r^3)J_{3m}(6\pi rr')rdr\right]\exp(\sqrt{-1}3m\theta')\ ,$$

where $\int_0^\infty$ is in the sense of $\lim\limits_{Y\to\infty}\int_0^Y$.

Denote in general by

$$M(f,s) = \int_0^\infty f(y)y^s\frac{dy}{y}$$

the Mellin transform of a function f. Then, there are well-known formulas

$$M(J_m(\alpha r),s) = \alpha^{-s}\frac{2^{s-1}\Gamma(s/2+m/2)}{\Gamma(1-s/2+m/2)}\ ,$$

($\alpha > 0$), and

$$M(J_m(2\pi r^3),s) = \frac{1}{3}(2\pi)^{-s/3}\frac{2^{s/3-1}\Gamma(s/6+m/2)}{\Gamma(1-s/6+m/2)}\ .$$

On the other hand, Γ-function satisfies the multiplication formula $\Gamma(s) = (2\pi)^{-1}3^{s-1/3}\Gamma(s/3)\Gamma(s/3+1/3)\Gamma(s/3+2/3)$. Using these facts, we can compute the Mellin transform of the function

$$b_m(r') = 2\pi\int_0^\infty J_m(2\pi r^3)J_{3m}(6\pi rr')rdr$$

of r' appearing in (2) as follows:

$$M(b_m,s) = 2\pi\int_0^\infty\int_0^\infty J_m(2\pi r^3)J_{3m}(6\pi rr')rdr\,r'^s\frac{dr'}{r'}$$

$$= 2\pi\int_0^\infty\int_0^\infty J_m(2\pi r^3)J_{3m}(6\pi r')rdr\frac{r'^s}{r^s}\frac{dr'}{r'}$$

$$= 2\pi\int_0^\infty J_m(2\pi r^3)r^{2-s}\frac{dr}{r}\int_0^\infty J_{3m}(6\pi r')r'^s\frac{dr'}{r'}$$

$$= 2\pi M(J_m(2\pi r^3), 2-s)M(J_{3m}(6\pi r), s)$$

$$= 2\pi\cdot\frac{1}{3}(2\pi)^{-(2-s)/3}\frac{2^{(2-s)/3-1}\Gamma((2-s)/6+m/2)}{\Gamma(1-(2-s)/6+m/2)}(6\pi)^{-s}\frac{2^{s-1}\Gamma(s/2+3m/2)}{\Gamma(1-s/2+3m/2)}$$

$$= 2\pi\cdot\frac{1}{3}(2\pi)^{-(2-s)/3}(6\pi)^{-s}2^{(2-s)/3-1}2^{s-1}\frac{\Gamma(1/3-s/6+m/2)}{\Gamma(2/3+s/6+m/2)}\cdot$$

$$\cdot\frac{3^{s/2+3m/2-1/2}\Gamma(s/6+m/2)\Gamma(s/6+m/2+1/3)\Gamma(s/6+m/2+2/3)}{3^{1-s/2+3m/2-1/2}\Gamma(1/3-s/6+m/2)\Gamma(2/3-s/6+m/2)\Gamma(1-s/6+m/2)}$$

$$= \frac{1}{18\pi}\pi^{-(2s-4)/3}\frac{\Gamma(s/6+m/2)\Gamma(s/6+m/2+1/3)}{\Gamma(2/3-s/6+m/2)\Gamma(1-s/6+m/2)}\,.$$

Comparing this result with Proposition 1 of [1], one sees by Theorem 1 of [1] that the coefficients of $\exp(\sqrt{-1}m\theta')$ in the Fourier series expansion with respect to θ', $(w = r'\exp(\sqrt{-1}\theta'))$, of the both hand sides of (1) have a common Mellin transform for $m \geqq 0$. Since, however, $e(z) = e(\bar{z})$ implies that the left hand side of (1) is invariant by $w \to \overline{w}$, the same situation holds for $m < 0$, too.

To complete the proof, we now need only a few supplements to the above computation. Introducing a parameter ρ, let us consider the integral

$$(3) \qquad 2\pi\int_0^\infty\int_0^\infty J_m(2\pi r^3)J_{3m}(6\pi rr')r^\rho dr\, r'^s\frac{dr'}{r'}\,.$$

Then, under the condition, for instance, $0 < \operatorname{Re} s < \varepsilon$ and $-\varepsilon < \operatorname{Re}\rho < 0$ with a small positive ε, $\operatorname{Re}(1+\rho-s)$ is slightly smaller than 1. (As a matter of fact, it will be enough that $\operatorname{Re}(1+\rho-s)$ is close to 1.) Therefore, the same computation as above shows that (3) is equal to the absolutely convergent integral

$$2\pi\int_0^\infty J_m(2\pi r^3)r^{1+\rho-s}\frac{dr}{r}\int_0^\infty J_{3m}(6\pi r')r'^s\frac{dr'}{r'}\,,$$

which can be expressed as

$$2\pi M(J_m(2\pi r^3), 1+\rho-s)M(J_{3m}(6\pi r), s)$$
$$= 2\pi\cdot\frac{1}{3}(2\pi)^{-(1+\rho-s)/3}\frac{2^{(1+\rho-s)/3}\Gamma((1+\rho-s)/6+m/2)}{\Gamma(1-(1+\rho-s)/6+m/2)}(6\pi)^{-s}\frac{2^{s-1}\Gamma(s/2+3m/2)}{\Gamma(1-s/2+3m/2)}$$

in terms of Γ-functions, and has the inverse Mellin transform

$$2\pi\int_0^\infty J_m(2\pi r^3)J_{3m}(6\pi rr')r^\rho dr$$

in the region determined by $0 < \operatorname{Re} s < 1$, say. Considering the analytic continuation to $\rho = 1$, we see now that b_m is actually the inverse Mellin transform in the region $0 < \operatorname{Re} s < \varepsilon$ of $M(b_m, s)$, which has been computed formally.

Remark. A simpler integral similar to (1) is

$$\int_C e(z^2)e(zw)dV(z) = \tfrac{1}{2}e(-\tfrac{1}{4}w^2).$$

REFERENCES

[1] T. Kubota, On a generalized Fourier transformation, to appear in J. Fac. Sci. Univ. Tokyo.

[2] G. N. Watson, A treatise on the theory of Bessel functions, Cambridge University Press, 1966.

Department of Mathematics, Nagoya University
Department of Mathematics, University of Maryland