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ON THE TRANSITIVE DOMINATION PRINCIPLE FOR CONTINUOUS FUNCTION-KERNELS

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1. Introduction

One of the interesting problems in the potential theory for a function-kernel is to investigate the relation between the domination principle and the balayage principle.

N. Ninomiya first proved the equivalence of the above two principles for a positive symmetric continuous kernel using the Gauss-Ninomiya variation (cf. [6]).

For a positive non-symmetric lower semi-continuous kernel, M. Kishi established a new existence theorem and proved in [4] that a kernel G satisfies the domination principle if and only if G satisfies the balayage principle under the additional condition that G and its adjoint $\check{G}$ satisfy the continuity principle.

For a continuous function-kernel (in the extended sense), it is well known that a kernel G itself satisfies the continuity principle when G satisfies the domination principle. Does the adjoint $\check{G}$ satisfy the continuity principle? M. Itô and the author proved in [2] that $\check{G}$ also satisfies the continuity principle when G satisfies the domination principle. We used there a result obtained in [3] where M. Itô developed the theory of generalized kernels by means of elementary kernels and resolvents.

Let G and N be continuous function-kernels. In this paper we shall prove that G satisfies the relative domination principle with respect to N if and only if $\check{G}$ satisfies the transitive domination principle with respect to $\check{N}$. The analogous result is first obtained by M. Kishi under the additional condition that G and $\check{G}$ satisfy the continuity principle (cf. [4]).

Our result shall give a simple proof of the theorem obtained in [2] that G satisfies the domination principle if and only if $\check{G}$ satisfies the

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domination principle. Therefore the continuity principle for $\check{G}$ follows immediately from the domination principle for G. It will be also shown that G satisfies the maximum principle if and only if $\check{G}$ satisfies the positive mass principle. All the proofs depend only on the existence theorem of Kishi.

2. Definitions

Let X be a locally compact Hausdorff space. A non-negative function $G(x, y)$ on $X \times X$ is called a continuous function-kernel on X if $G(x, y)$ is continuous in the extended sense on $X \times X$, finite except for the diagonal set of $X \times X$ and $0 < G(x, x) \leqq +\infty$ for any $x \in X$. The kernel $\check{G}$ defined by $\check{G}(x, y) = G(y, x)$ is called the adjoint kernel of G. The potential $G\mu(x)$ and the adjoint potential $\check{G}\mu(x)$ of a positive Radon measure μ in X are defined by

$$G\mu(x) = \int G(x, y)d\mu(y) \quad \text{and} \quad \check{G}\mu(x) = \int \check{G}(x, y)d\mu(y)$$

respectively.

The G-energy of μ is defined by $\int G\mu(x)d\mu(x)$. We denote by M_0 the family of all positive measures with compact support and by $E_0 = E_0(G)$ the family of all measures in M_0 with finite G-energy. Evidently $E_0(G) = E_0(\check{G})$.

We say that a property holds G-p.p.p. on a subset A of X if it holds on A except a set of the inner ν-measure 0 for every $\nu \in E_0$.

3. Existence theorem of Kishi and its modification

In the succeeding sections, the following existence theorem plays a fundamental role.

EXISTENCE THEOREM OF KISHI ([1], [5]). *Let K be a compact Hausdorff space, G be a finite continuous function-kernel on K and $u(x)$ be a non-negative finite continuous function on K. Then there exists a measure λ in M_0 such that*

$$G\lambda(x) \geqq u(x) \qquad on\ K\,,$$
$$G\lambda(x) = u(x) \qquad on\ S\lambda\,.$$

First we modify the above existence theorem into the following form.

LEMMA 1. *Let G be a continuous function-kernel on a locally compact Hausdorff space X, K be a compact subset of X and $u(x)$ be a non-negative finite continuous function on K. Take a sequence $\{G_n\}$ of finite continuous function-kernels such that $G_n \nearrow G$. Then there exists a vaguely bounded sequence $\{\lambda_n\}$ of positive measures supported by K satisfying*

$$(1) \qquad G_n\lambda_n(x) \geqq u(x) \qquad \text{on } K ,$$

$$(2) \qquad G_n\lambda_n(x) = u(x) \qquad \text{on } S\lambda_n .$$

And a vague cluster point λ of $\{\lambda_n\}$ satisfies

$$(3) \qquad G\lambda(x) \geqq u(x) \qquad \nu\text{-a.e. on } K \text{ for every } \nu \in C(\check{G}; K) ,$$

$$(4) \qquad G\lambda(x) \leqq u(x) \qquad \text{on } S\lambda ,$$

where $C(\check{G}; K)$ denotes the family of all measures in M_0 such that $\check{G}\nu(x)$ is finite continuous on K.

Proof. Let G' be the restriction of G to $K \times K$. Then G' is evidently a continuous function-kernel on K and $G'\mu(x) = G\mu(x)$ on K for any positive Radon measure μ in K. Therefore we have only to prove it in the case that $X = K$.

By the existence theorem of Kishi, there exists for any n, a measure $\lambda_n \in M_0$ satisfying

$$G_n\lambda_n(x) \geqq u(x) \qquad \text{on } K ,$$
$$G_n\lambda_n(x) = u(x) \qquad \text{on } S\lambda_n .$$

The strictly positiveness of G on the diagonal set asserts that $\{\lambda_n\}$ is vaguely bounded. Therefore a subnet $\{\lambda_\omega ; \omega \in D, D \text{ is a directed set}\}$ of the sequence $\{\lambda_n\}$ converges vaguely to a positive measure λ supported by K. λ fulfils (3) and (4). In fact, for any $\nu \in C(\check{G}; K)$, we have

$$\int G\lambda d\nu = \int \check{G}\nu d\lambda = \lim_\omega \int \check{G}\nu d\lambda_\omega = \lim_\omega \int G\lambda_\omega d\nu$$
$$\geqq \overline{\lim_\omega} \int G_\omega\lambda_\omega d\nu \geqq \int u d\nu ,$$

hence (3). Let $\omega_0 \in D$ and $x_0 \in S\lambda$ be fixed. There exists a net $\{x_\omega\}$ converging to x_0 with $x_\omega \in S\lambda_\omega$. Then we obtain

$$G_{\omega_0}\lambda(x_0) = \lim_\omega G_{\omega_0}\lambda_\omega(x_\omega) \leqq \underline{\lim}_\omega G_\omega\lambda_\omega(x_\omega)$$
$$\leqq \lim_\omega u(x_\omega) = u(x_0) ,$$

hence (4).

Remark 1. The analogous result holds for a lower semi-continuous function-kernel G and a bounded upper semi-continuous function $u(x)$.

4. Relative domination principle and transitive domination principle

Let us start with the definitions of principles.

(I) Continuity principle: A kernel G is said to satisfy the continuity principle when, for any $\mu \in M_0$, the finite continuity of the restriction of $G\mu(x)$ to $S\mu$ implies the finite continuity of $G\mu(x)$ on the whole space X.

(II) Relative domination principle: G is said to satisfy the relative domination principle with respect to N (written simply $G \prec N$) when, for $\mu \in E_0(G)$ and $\nu \in M_0$, an inequality $G\mu(x) \leqq N\nu(x)$ on $S\mu$ implies the same inequality on X. In the case that it holds only when $\nu = \varepsilon_{x_0}$ for any $x_0 \in CS\mu$, we call it the elementary relative domination principle.

(III) Transitive domination principle: We say that G satisfies the transitive domination principle with respect to N (written simply $G \sqsubset N$) when, for μ and ν in M_0 with $S\mu \cap S\nu = \phi$, an inequality $G\mu(x) \leqq G\nu(x)$ on $S\mu$ implies the inequality $N\mu(x) \leqq N\nu(x)$ on X.

Remark 2. It is well known, for strictly positive continuous function-kernels G and N, that G satisfies the continuity principle if G satisfies the elementary relative domination principle with respect to N. This is true for our continuous function-kernels. In fact, let x_0 be any non-isolated point in X. Then there exist a point y_0 ($y_0 \neq x_0$) and an open neighbourhood $V(x_0)$ of x_0 such that $N(x, y_0) > 0$ for any $x \in V(x_0)$, because $N(x_0, x_0) > 0$. Consequently, the continuity principle for G follows by the same way as in the case of strictly positive kernels.

Remark 3. If $\check{G}$ satisfies the continuity principle, Lusin's theorem asserts that the inequality (3) in Lemma 1 is equivalent to the following inequality (3′).

$$(3') \qquad G\lambda(x) \geqq u(x) \qquad G\text{-p.p.p. on } K\,.$$

LEMMA 2. *Let G and N be continuous function-kernels on X. Suppose that $\check{G}$ satisfies the transitive domination principle with respect to $\check{N}$. Then G satisfies the continuity principle.*

Proof. It suffices to prove that G satisfies the elementary relative domination principle with respect to N (cf. Remark 2).

Suppose that for $\mu \in E_0(G)$ and $x_0 \in CS\mu$, an inequality $G\mu(x) \leqq N\varepsilon_{x_0}(x)$ holds on $S\mu$. We shall show that $G\mu(y) \leqq N\varepsilon_{x_0}(y)$ holds for any $y \in CS\mu$. By Lemma 1, there exists a vaguely bounded sequence $\{\varepsilon'_{yn}\}$ of positive measures supported by $S\mu$ satisfying

$$\check{G}_n\varepsilon'_{yn}(x) \geqq \check{G}\varepsilon_y(x) \qquad \text{on } S\mu$$

and

$$\check{G}_n\varepsilon'_{yn}(x) = \check{G}\varepsilon_y(x) \qquad \text{on } S\varepsilon'_{yn} ,$$

because $\check{G}\varepsilon_y(x)$ is finite continuous on $S\mu$. A vague cluster point ε'_y of $\{\varepsilon'_{yn}\}$ fulfils

$$\check{G}\varepsilon'_y(x) \leqq \check{G}\varepsilon_y(x) \qquad \text{on } S\varepsilon'_y \subset S\mu .$$

$\check{G} \sqsubset \check{N}$ asserts

$$\check{N}\varepsilon'_y(x) \leqq \check{N}\varepsilon_y(x) \qquad \text{on } X .$$

Then we obtain

$$\begin{aligned} G\mu(y) &= \int G\mu(x)d\varepsilon_y(x) = \int \check{G}\varepsilon_y d\mu \leqq \int \check{G}_n\varepsilon'_{yn}d\mu \\ &\leqq \int G\mu d\varepsilon'_{yn} \leqq \int N\varepsilon_{x_0}d\varepsilon'_{yn} \qquad \text{for any } n . \end{aligned}$$

ε'_y being a vague cluster point of $\{\varepsilon'_{yn}\}$, we have

$$\begin{aligned} G\mu(y) &\leqq \int N\varepsilon_{x_0}d\varepsilon'_y = \int \check{N}\varepsilon'_y d\varepsilon_{x_0} \\ &\leqq \int \check{N}\varepsilon_y d\varepsilon_{x_0} = N\varepsilon_{x_0}(y) . \end{aligned}$$

Consequently G satisfies the elementary relative domination principle with respect to N.

THEOREM 1. *Let G and N be continuous function-kernels on a locally compact Hausdorff space X. Then the following statements are equivalent.*

(a) *G satisfies the relative domination principle with respect to N.*

(b) *G satisfies the elementary relative domination principle with respect to N.*

(c) *$\check{G}$ satisfies the transitive domination principle with respect to $\check{N}$.*

Proof. Evidently (a) $\to$ (b).

(b) $\to$ (c). Suppose that for μ and ν in M_0 with $S\mu \cap S\nu = \phi$, an inequality $\check{G}\mu(x) \leqq \check{G}\nu(x)$ holds on $S\mu$. We have only to obtain the inequality $\check{N}_p\mu(y) \leqq \check{N}\nu(y)$ for any positive number p and for any $y \in X$, where $\check{N}_p = \check{N}_p(x, y) = \inf\{\check{N}(x, y), p\}$. By Lemma 1, there exists a sequence $\{\varepsilon'_{ypn}\}$ of measures in $E_0(G)$ supported by $S\mu$ satisfying

$$G_n\varepsilon'_{ypn}(x) \geqq N_p\varepsilon_y(x) \qquad \text{on } S\mu$$

and

$$G_n\varepsilon'_{ypn}(x) = N_p\varepsilon_y(x) \qquad \text{on } S\varepsilon_{ypn} \, .$$

A vague cluster point ε'_{yp} of $\{\varepsilon'_{ypn}\}_{n=1}^{\infty}$ fulfils

$$G\varepsilon'_{yp}(x) \leqq N_p\varepsilon_y(x) \leqq N\varepsilon_y(x) \qquad \text{on } S\varepsilon'_{yp} \subset S\mu \, .$$

(b) implies

$$G\varepsilon'_{yp}(x) \leqq N\varepsilon_y(x) \qquad \text{on } X \, .$$

Then

$$\check{N}_p\mu(y) = \int \check{N}_p\mu\, d\varepsilon_y = \int N_p\varepsilon_y d\mu \leqq \int G_n\varepsilon'_{ypn} d\mu$$
$$\leqq \int \check{G}\mu d\varepsilon'_{ypn} \leqq \int \check{G}\nu d\varepsilon'_{ypn} \qquad \text{for any } n \, .$$

And ε'_{yp} being a vague cluster point of $\{\varepsilon'_{ypn}\}_{n=1}^{\infty}$, we obtain

$$N_p\mu(y) \leqq \int \check{G}\nu d\varepsilon'_{yp} = \int G\varepsilon'_{yp} d\nu$$
$$\leqq \int N\varepsilon_y d\nu(y) = \check{N}\nu(y) \, .$$

(c) $\to$ (a). Assume that $G\mu(x) \leqq N\nu(x)$ holds on $S\mu$ for $\mu \in E_0$ and $\nu \in M_0$. By virtue of Lemma 2, (c) asserts that G satisfies the continuity principle and so we can find, for any n and any $y \in CS\mu$, a measure $\tilde{\varepsilon}_{yn} \in E_0$ supported by $S\mu$ satisfying

$$\check{G}\tilde{\varepsilon}_{yn}(x) \geqq \check{G}_n\varepsilon_y(x) \qquad G\text{-p.p.p. on } S\mu$$

and

$$\check{G}\tilde{\varepsilon}_{yn}(x) \leqq \check{G}_n\varepsilon_y(x) \leqq \check{G}\varepsilon_y(x) \qquad \text{on } S\tilde{\varepsilon}_{yn}$$

(cf. (3′) in Remark 3). (c) implies

$$\check{N}\tilde{\varepsilon}_{yn}(x) \leqq \check{N}\varepsilon_y(x) \qquad \text{on } X .$$

Consequently we obtain

$$\begin{aligned} G_n\mu(y) &= \int \check{G}_n\varepsilon_y d\mu \leqq \int \check{G}\tilde{\varepsilon}_{yn} d\mu \leqq \int G\mu d\tilde{\varepsilon}_{yn} \\ &\leqq \int N\nu d\tilde{\varepsilon}_{yn} \leqq \int \check{N}\tilde{\varepsilon}_{yn} d\nu \leqq \int \check{N}\varepsilon_y d\nu = N\nu(y) . \end{aligned}$$

Letting n tend to $+\infty$, we have $G\mu(y) \leqq N\nu(y)$. This completes the proof.

Remark 4. M. Kishi proved this theorem under the additional condition that G and $\check{G}$ satisfy the continuity principle. But by the above argument this condition can be omitted.

5. Domination principle and continuity principle

As an application of Theorem 1, we consider the domination, the continuity and the balayage principles for a continuous function-kernel G and its adjoint $\check{G}$.

(IV) Domination principle: For $\mu \in E_0$ and $\nu \in M_0$, an inequality $G\mu(x) \leqq G\nu(x)$ on $S\mu$ implies the same inequality on X. Especially we call it the elementary domination principle when $\nu = \varepsilon_{x_0}$ for any $x_0 \in CS\mu$.

(V) Balayage principle: For a given compact set K and a given measure μ in M_0, there exists a measure μ' in M_0 supported by K such that

$$\begin{aligned} G\mu'(x) &= G\mu(x) && G\text{-p.p.p. on } K , \\ G\mu'(x) &\leqq G\mu(x) && \text{on } X . \end{aligned}$$

THEOREM 2. *Let G be a continuous function-kernel on X. Then both G and $\check{G}$ satisfy the continuity principle if either G or $\check{G}$ satisfies the domination principle.*

Proof. We have only to prove that $\check{G}$ satisfies the continuity principle when G satisfies the domination principle. The domination principle for G implies that $G \prec G$ and also implies, by virtue of Theorem 1, that $\check{G} \sqsubset \check{G}$. Therefore $\check{G}$ satisfies the elementary domination principle and, by Remark 2, $\check{G}$ satisfies the continuity principle.

THEOREM 3. *Let G be a continuous function-kernel on X. Then G satisfies the domination principle if and only if $\check{G}$ does.*

Proof. By Theorem 1, $G \prec G$ implies that $\check{G} \sqcup \check{G}$ and therefore $\check{G}$ satisfies the elementary domination principle. The equivalence of (a) and (b) in Theorem 1 asserts that $\check{G}$ satisfies the domination principle. The converse is also true.

Similarly the following theorem of Kishi is an immediate consequence of Theorem 2 and Remark 3.

THEOREM 4. *Let G be a continuous function-kernel on X. Then G satisfies the balayage principle if and only if G satisfies the domination principle.*

Remark 5. Theorem 3 and Theorem 4 were first proved by M. Kishi in [4] under the condition that G and $\check{G}$ satisfy the continuity principle and were proved by M. Itô and the author in [2] without that condition using a result of generalized kernels established by M. Itô in [3]. The above argument gives a simple proof depending only on Theorem 1.

6. Maximum principle and positive mass principle

Finally we investigate the relation between the maximum and the positive mass principle.

(VI) Maximum principle: For $\mu \in M_0$, an inequality $G\mu(x) \leqq 1$ on $S\mu$ implies the same inequality on X.

(VII) Complete maximum principle: For $\mu \in E_0, \nu \in M_0$ and for a non-negative number a, an inequality $G\mu(x) \leqq G\nu(x) + a$ on $S\mu$ implies the same inequality on X.

(VIII) Positive mass principle: For μ and ν in M_0 with $S\mu \cap S\nu = \phi$, an inequality $G\mu(x) \leqq G\nu(x)$ on $S\mu$ implies the inequality $\int d\mu \leqq \int d\nu$.

THEOREM 5. *Let G be a continuous function-kernel on X. Then G satisfies the maximum principle if and only if $\check{G}$ satisfies the positive mass principle.*

Proof. The maximum principle for G implies that $G \prec 1$, where 1 is the constant kernel. By theorem 1, $G \prec 1$ if and only if $\check{G} \sqcup \check{1} = 1$. $\check{G} \sqcup 1$ is just the positive mass principle for $\check{G}$.

By this theorem, we have the following corollaries without assuming that G and $\check{G}$ satisfy the continuity principle.

COROLLARY 1. *Let G be a continuous function-kernel on X. Then G satisfies the complete maximum principle if and only if G satisfies the domination principle and the maximum principle.*

COROLLARY 2. *Let G and N be continuous function-kernels on X such that G satisfies the relative domination principle with respect to N. Then G satisfies the maximum principle if N does.*

Proof. It is sufficient to prove that $\check{G}$ satisfies the positive mass principle. Suppose that $\check{G}\mu(x) \leqq \check{G}\nu(x)$ on $S\mu$ for μ and ν in M_0 with $S\mu \cap S\nu = \phi$. $G \prec N$ implies, by Theorem 1, that $\check{G} \sqsubset \check{N}$. Consequently we have $\check{N}\mu(x) \leqq \check{N}\nu(x)$ on X. On the other hand $N \prec 1$ asserts, by Theorem 5, that $\check{N} \sqsubset 1$ and hence that $\int d\mu \leqq \int d\nu$. Therefore G satisfies the positive mass principle.

REFERENCES

[1] R. Durier: Thèse, Faculté des Sciences d'Orsay, Université de Paris, 1969.

[2] I. Higuchi and M. Itô: On the theorem of Kishi for a continuous function-kernel, Nagoya Math. J., **53** (1974), p. 127–135.

[3] M. Itô: Sur les principes divers du maximum et le type positif, Nagoya Math. J., **44** (1971), p. 133–164.

[4] M. Kishi: Maximum principles in the potential theory, Nagoya Math. J., **23** (1963), p. 165–187.

[5] M. Kishi: An existence theorem in potential theory, Nagoya Math. J., **27** (1966), p. 133–137.

[6] N. Ninomiya: Etude sur la théorie du potentiel pris par rapport au noyau symétrique, J. Inst. Polytechnics, Osaka City Univ., **8** (1957), p. 147–178.

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