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Nagoya Math. J.
Vol. 54 (1974), 7–20

ON A CLASS OF NON-ELLIPTIC BOUNDARY PROBLEMS

Dedicated to Professor Minoru Kurita on his 60th birthday

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Introduction.

Let Ω be a bounded domain in $\boldsymbol{R}^l$ $(l \geq 2)$ with C^∞ boundary Γ of dimension $l-1$ and let there be given a second order elliptic differential equation

$$(1) \qquad Au = -\sum_{i,j=1}^{l} \partial_i(a_{ij}\partial_j u) + \sum_{i=1}^{l} a_i \partial_i u + au = f \quad \text{in } \Omega ,$$

where $\partial_j = \partial/\partial x_i$ and all coefficients are assumed, for the sake of simplicity, to be real-valued and C^∞ on $\bar{\Omega} = \Omega \cup \Gamma$. It is also assumed that $a_{ij} = a_{ji}$ on Ω and that there exists a positive constant c_0 such that

$$\sum_{i,j=1}^{l} a_{ij}(x)\xi_i\xi_j \geq c_0 |\xi|^2$$

holds for all $x \in \bar{\Omega}$ and $\xi \in \boldsymbol{R}^l$.

Then we consider a boundary condition

$$(2) \qquad Bu = \alpha\partial_\nu u + \gamma u + \beta u = \varphi \quad \text{on} \quad \Gamma ,$$

where α, β are real-valued C^∞ functions on Γ, γ is a C^∞ real vector field tangent to Γ, and $\partial_\nu u$ denotes the conormal derivative of u, i.e.,

$$\partial_\nu u = \sum_{i,j=1}^{l} a_{ij} n_i \partial_j u ,$$

$n = (n_1, \cdots, n_l)$ being the exterior normal of Γ. Moreover, throughout this paper, we assume $\alpha \geq 0$ on Γ.

In case $\gamma = 0$ on Γ, the boundary problem (1)–(2) was discussed in [2, 3] by using the Hilbert space technique and the elliptic regularization. This paper is a continuation of their studies and is especially nothing but a slight improvement of [2].

Received December 13, 1973.

Now we state the results obtained. The notations appearing will be made clear in §1.

THEOREM 1. *If we assume that*

$$\frac{1}{2}\gamma^*(1) + \beta > 0 \quad on \quad \Gamma_0 = \{x \in \Gamma ; \alpha(x) = 0\} , \tag{3}$$

it then follows that for every $f \in H^{k-2}(\Omega ; p)$ *and every* $\varphi \in H^{k-1}(\Gamma)$ (k *integer* ≥ 2), *the boundary problem*

$$\begin{cases} (A + \lambda)u = f & in\ \Omega \\ (B + t)u = \varphi & on\ \Gamma \end{cases} \tag{4}$$

has the unique solution u *in* $H^k(\Omega ; p)$, *provided* $\lambda \geq \lambda_0$, *a number which is a constant not depending on* k, *and* $t \geq t_k$, *a number which is a constant depending in general on* k.

Moreover it follows that there exists a constant $C_k > 0$ *independent of* $t \geq t_k$ *such that*

$$\|u ; p\|_k \leq C_k(\|f ; p\|_{k-2} + \|\varphi\|_{k-1,\Gamma}) . \tag{5}$$

COROLLARY. *Assume, in addition to* (3), *that*

$$\gamma = 0 \qquad in\ a\ neighbourhood\ of\ \Gamma_0. \tag{6}$$

Then we can take as $t_k = 0$ *for every* k.

The following example shows us that condition (6) is necessary for Theorem 1 to be valid for $t_k = 0$.

EXAMPLE. Let Ω be a bounded domain in the (x, y)-plane whose boundary Γ is a C^∞ curve and contains an open interval $\omega \ni (0, 0)$ in the x-axis. In (1) and (2) we take as $A = \Delta$, $\alpha = 0$ in ω, $\gamma = -x\partial/\partial x$ in ω, $\beta \geq 1$ integer and $\varphi = \alpha \partial v/\partial n + \gamma v + \beta v$, where v is a harmonic function whose boundary value is C^∞ except the origin and is equal to $|x|^\beta$ in ω. Clearly we have $\varphi \in C^\infty(\Gamma)$.

Then $u = v$ is a solution belonging to $C^{\beta-1}(\bar{\Omega})$ of the problem

$$\begin{cases} -\Delta u = 0 & \text{in } \Omega \\ \alpha\dfrac{\partial u}{\partial n} + \gamma u + \beta u = \varphi & \text{on } \Gamma , \end{cases}$$

but does not belong to $C^\beta(\bar{\Omega})$. Here it is easily seen that (3) is satisfied but not (6).

THEOREM 2. *If Γ_0 is a C^∞ manifold of dimension $l-2$ and γ is transversal to Γ_0, it then follows that for every $f \in H^{k-2}(\Omega; p)$ and every $\varphi \in H^{k-1}(\Gamma)$ (k integer ≥ 2) the problem* (4) *with $t=0$ has the unique solution u in $H^k(\Omega; p)$, provided $\lambda \geq \lambda_1$ which is a constant not depending on k. Moreover the u satisfies* (5).

In case $\beta = 0$, this is nothing but a class of the oblique derivative problems, which was already discussed in [1] by the slightly different manner (cf. § 7 of [1]).

The plan of the paper is as follows. § 1 is devoted to preliminaries of the proof of Theorem 1, which will be given in § 2. Corollary and Theorem 2 will be briefly proved in §§ 3 and 4, respectively, by the similar argument as in Theorem 1.

§ 1. Preliminaries.

Let γ be a C^∞ real vector field tangent to Γ. The adjoint γ^* of γ is defined by the identity

$$\int_\Gamma \gamma u \cdot v d\sigma = \int_\Gamma u \cdot \gamma^* v d\sigma \ , \qquad u, v \in C^\infty(\Gamma) \ ,$$

where $d\sigma$ is the Lebesque measure on Γ.

Let $\{U_j\}$, $j = 1, \cdots, N$, be a family of open subsets of $\boldsymbol{R}^l$, covering Γ, and assume that there exists a C^∞ coordinate transformation $y = \kappa_j(x)$ on U_j such that $\Omega \cap U_j$ is mapped in a one-to-one way onto an open portion Σ_j of a half space $y_l < 0$ and $\Gamma_j = \Gamma \cap U_j$ is transformed onto an open portion τ_j of $y_l = 0$. Moreover assume that $dy = J_j dx$ and $d\sigma = K_j dy'$ ($y' = y_1, \cdots, y_{l-1}$)).

Let $\{\zeta_j(x)\}$ be a partition of unity of Γ belonging to $\{U_j\}$, i.e., $\zeta_j \in C_0^\infty(U_j)$, $\zeta_j \geq 0$ and $\sum_{j=1}^N \zeta_j(x) = 1$ on Γ. Using the partition of unity $\{U_j, \zeta_j\}$, we can easily prove

LEMMA 1. *There exists a C^∞ function $b(x)$ on Γ such that $\gamma^* = -\gamma + b(x)$.*

Proof. We assume that by the transformation κ_j the vector field γ is altered to

$$\delta_j = \sum_{k=1}^{l-1} c_{jk}\partial_k \qquad (\partial_k = \partial/\partial y_k) .$$

Then we have

$$\begin{aligned}
\int_\Gamma \gamma u\cdot v d\sigma &= \sum_{j=1}^{N}\int_{\Gamma_j} \gamma(\zeta_j u)\cdot v d\sigma \\
&= \sum_j \int_{y_l=0} \sum_k c_{jk}\partial_k(\zeta_j u)\cdot v K_j dy' = -\sum_j \int_{y_l=0} \sum_k \zeta_j u\cdot \partial_k(c_{jk}K_j v)dy' \\
&= -\sum_j \int_{y_l=0} \sum_k \zeta_j u\{c_{jk}K_j\partial_k v + \partial_k(c_{jk}K_j)v\}dy' \\
&= -\sum_j \int_{y_l=0} \zeta_j u\cdot \sum_k (c_{jk}\partial_k v)K_j dy' \\
&\quad - \sum_j \int_{y_l=0} \zeta_j u\cdot \sum_k \partial_k(c_{jk}K_j)K_j^{-1}vK_j dy' \\
&= -\sum_j \int_\Gamma \zeta_j u\cdot \gamma v d\sigma - \sum_j \int_\Gamma u\{\zeta_j K_j^{-1}\sum \partial_k(c_{jk}K_j)\}v d\sigma ,
\end{aligned}$$

which completes the proof.

The following lemma can be easily proved. So we omit the proof.

LEMMA 2. *Under condition* (3) *we can find a function* $q(x) \in C^\infty(\bar{\Omega})$ *satisfying*

(i) $q > 0$ *in* Ω *and* $q = \alpha$ *on* Γ.

(ii) *There exist two positive constants* C *and* d *such that* $C \operatorname{dis}(x, \Gamma) \le q(x)$ *in* $\Omega_d = \{x \in \bar{\Omega}; \operatorname{dis}(x, \Gamma) < d\}$.

(iii) *There exists a positive constant* c_1 *such that*

$$\frac{1}{2}\partial_\nu q + \frac{1}{2}\gamma^*(1) + \beta \ge c_1 \quad on \quad \Gamma .$$

LEMMA 3. *For any* $\delta > 0$ *there exists a constant* $C_\delta > 0$ *such that*

$$\|u\|_{0,\Omega}^2 \le \delta \|p\partial u\|_{0,\Omega}^2 + C_\delta \|pu\|_{0,\Omega}^2 , \qquad u \in C^\infty(\bar{\Omega}) ,$$

where $p = \sqrt{q}$, $\|u\|_{0,\Omega}^2 = \int_\Omega |u|^2 dx$ *and*

$$\|p\partial u\|_{0,\Omega}^2 = \sum_{j=1}^{l} \int q\,|\partial_j u|^2 dx .$$

Proof. This lemma is due to [2]. Let $\zeta_0(x) \in C_0^\infty(\Omega)$ such that $\zeta_0 = 1 - \sum_{j=1}^N \zeta_j$ in Ω and $=0$ outside of $\bar{\Omega}$. Then $u = \sum_{j=1}^N \zeta_j u + \zeta_0 u$ in Ω. Hence we have

$$\|u\|_{0,\Omega}^2 \leq \left(\sum_{j=1}^N \|\zeta_j u\|_{0,\Omega} + \|\zeta_0 u\|_{0,\Omega}\right)^2$$
$$\leq \text{const.}\left(\sum_{j=1}^N \int_{\Sigma_j} |v_j|^2\, dy + \|pu\|_{0,\Omega}^2\right),$$

where $v_j = \sqrt{J_j}\zeta_j u$ is in $C_0^\infty(\Sigma_j \cup \tau_j)$. It was indicated by Hayashida in [2] that for any $\varepsilon > 0$ the inequality

$$\int_{\Sigma_j} |v_j|^2\, dy \leq \varepsilon \int_{\Sigma_j} |y_l||\partial_l v_j|^2\, dy + \frac{1}{\varepsilon}\int_{\Sigma_j} |y_l||v_j|^2\, dy$$

holds. Thus we can establish the proof with the aid of Lemma 2.

Now we introduce an integro-differential bilinear form:

$$Q[u, v] = B[u, qv] + \int_\Gamma (\gamma u + \beta u)\cdot v d\sigma ,$$

where

$$B[u, v] = \int_\Omega \left(\sum_{i,j=1}^l a_{ij}\partial_i u\cdot\partial_j u + \sum_{i=1}^l a_i\partial_i u\cdot v + au\cdot v\right)dx .$$

It is easily seen that $u \in C^2(\bar{\Omega})$ satisfies (1) and (2) if and only if it satisfies

$$Q[u, v] = (qf, v)_\Omega + (\varphi, v)_\Gamma , \qquad v \in C^\infty(\bar{\Omega}) , \tag{7}$$

where $(\ ,\ )_\Omega$ and $(\ ,\ )_\Gamma$ denote the usual inner products in $L^2(\Omega)$ and $L^2(\Gamma)$, respectively. Hence we have only to deal with (7). This idea was used in [4].

Throughout the paper we always assume condition (3).

PROPOSITION 1. *There exist two positive constants c_2, λ_0 such that*

$$Q_\lambda[u, u] \geq c_2(\|p\partial u\|_{0,\Omega}^2 + \|pu\|_{0,\Omega}^2 + \|u\|_{0,\Gamma}^2)$$

holds for every $u \in C^\infty(\bar{\Omega})$ and $\lambda \geq \lambda_0$, where $\|u\|_{0,\Gamma}^2 = (u, u)_\Gamma$ and

$$Q_\lambda[u\cdot v] = Q[u, v] + \lambda(u, qv) .$$

Proof. For $u \in C^\infty(\bar{\Omega})$ we have

$$Q[u, u] = \int_\Omega q\left(\sum_{i,j=1}^l a_{ij}\partial_i u\cdot\partial_j\cdot u + \sum_{i=1}^l a_i\partial_i u\cdot u + auu\right)dx$$
$$+ \frac{1}{2}\int_\Omega \sum_{i,j=1}^l a_{ij}\partial_j q\cdot\partial_i(u^2)dx + \int_\Gamma\left(\frac{1}{2}\gamma(u^2) + \beta u^2\right)d\sigma$$

$$\geq \frac{c_0}{2}\|p\partial u\|_{0,\Omega}^2 - C\|pu\|_{0,\Omega}^2 + \frac{1}{2}\int A_0 q\cdot u^2 dx$$
$$+ \int_\Gamma\Big(\frac{1}{2}\partial_\nu q + \frac{1}{2}\gamma^*(1) + \beta\Big)u^2 d\sigma ,$$

where C is a constant and $A_0 = -\sum_{i,j=1}^{l} \partial_i a_{ij}\partial_j$. Thus, using Lemmas 2 and 3, we can conclude the proposition.

For any ε, $0 < \varepsilon \leq 1$, putting $q_\varepsilon(x) = q(x) + \varepsilon$, we define an integro-differential bilinear form as

$$Q^\varepsilon[u, v] = B[u, q_\varepsilon v] + \int_\Gamma (\gamma u + \beta u)v d\sigma .$$

PROPOSITION 2. *Let $\lambda \geq \lambda_0$ and $t \geq 0$. Then for every $f \in C^\infty(\bar{\Omega})$ and every $\varphi \in C^\infty(\Gamma)$, there exists the unique $u_\varepsilon \in C^\infty(\bar{\Omega})$ which depends also on λ and t, satisfying*

$$Q_{\lambda,t}^\varepsilon[u_\varepsilon, v] = (q_\varepsilon f, v)_\Omega + (\varphi, v)_\Gamma , \qquad v \in C^\infty(\bar{\Omega}) . \tag{8}$$

Moreover it follows that there exists a constant $c_3 > 0$ independent of ε, λ and t such that

$$c_3(\|p_\varepsilon\partial u_\varepsilon\|_{0,\Omega}^2 + \|p_\varepsilon u_\varepsilon\|_{0,\Omega}^2 + (1+t)\|u_\varepsilon\|_{0,\Gamma}^2) \leq \|p_\varepsilon f\|_{0,\Omega}^2 + \|\varphi\|_{0,\Gamma}^2 , \tag{9}$$

where $p_\varepsilon = \sqrt{q_\varepsilon}$ and

$$Q_{\lambda,t}^\varepsilon[u, v] = Q^\varepsilon[u, v] + \lambda(u, q_\varepsilon v) + t(u, v)_\Gamma .$$

Proof. By the same argument as in Proposition 1, we can immediately obtain

$$\begin{aligned} Q_{\lambda,t}^\varepsilon[u, u] &\geq c_2'(\|p_\varepsilon\partial u\|_{0,\Omega}^2 + \|p_\varepsilon u\|_{0,\Omega}^2 + (1+t)\|u\|_{0,\Gamma}^2) \\ &(= c_2'|||u|||_{\varepsilon,t}^2) , \qquad u \in C^\infty(\bar{\Omega}) , \end{aligned} \tag{10}$$

with $c_2' = \min(c_2, 1)$. Clearly we have

$$\begin{cases} Q_{\lambda,t}^\varepsilon[u, u] \geq \varepsilon c_2'\|u\|_{1,\Omega}^2 \\ |Q_{\lambda,t}^\varepsilon[u, v]| \leq \text{const.}\ \|u\|_{1,\Omega}\|v\|_{1,\Omega} , \end{cases}$$

where

$$\|u\|_{1,\Omega}^2 = \sum_{j=1}^{l}\int_\Omega |\partial_j u|^2 dx + \|u\|_{0,\Omega}^2 .$$

Accordingly we can apply the theorem of Riesz-Milgram-Lax which guarantees the existence of the unique solution u_ε of (8) in $H^1(\Omega)$. It is well known that u_ε is really in $C^\infty(\bar{\Omega})$, since the problem is elliptic. In fact u_ε satisfies

$$\begin{cases}(A+\lambda)u_\varepsilon = f & \text{in } \Omega \\ (\alpha+\varepsilon)\partial_\nu u_\varepsilon + \gamma u_\varepsilon + (\beta+t)u_\varepsilon = \varphi & \text{on } \Gamma .\end{cases} \tag{11}$$

Substituting $v = u_\varepsilon$ in (8) and using (10), we obtain

$$\begin{aligned} c_2' |||u_\varepsilon|||_{\varepsilon,t}^2 &\le Q_{\lambda,t}^\varepsilon[u_\varepsilon, u_\varepsilon] = (q_\varepsilon f, u_\varepsilon)_\Omega + (\varphi, u_\varepsilon)_\Gamma \\ &\le \|p_\varepsilon f\|_{0,\Omega}\|p_\varepsilon u_\varepsilon\|_{0,\Omega} + \|\varphi\|_{0,\Gamma}\|u_\varepsilon\|_{0,\Gamma} \\ &\le (\|p_\varepsilon f\|_{0,\Omega} + \|\varphi\|_{0,\Gamma})|||u_\varepsilon|||_{\varepsilon,t} , \end{aligned}$$

which proves (9).

Finally we shall define the Hilbert space $H^k(\Omega\,;p)$ for integer $k \ge 0$. By $H^s(\Omega)$, s real, we denote the Sobolev space with norm $\|\cdot\|_{s,\Omega}$. Then $H^k(\Omega\,;p)$ is a Hilbert space given by the completion of $C^\infty(\bar{\Omega})$ with respect to the norm $\|\cdot, p\|_k$ defined by

$$\|u\,;p\|_k^2 = \|p\partial^k u\|_{0,\Omega}^2 + \|u\|_{k-1/2,\Omega}^2 . \tag{12}$$

2. Proof of Theorem 1.

Setting $U_0 = \Omega - \bigcup_{k=1}^N U_j$, we obtain the partition of unity $\{U_j, \zeta_j\}$, $j = 0, 1, \cdots, N$, of $\bar{\Omega}$. In the following we denote by $U, \zeta, \kappa, \Sigma, \tau, J$ and K one of $U_j, \zeta_j, \kappa_j, \Sigma_j, \tau_j, J_j$ and K_j $(j = 1, \cdots, N)$, respectively, and assume that by the transformation κ the form $Q_{\lambda,t}^\varepsilon[u\cdot v]$ is altered to, λ fixed,

$$\begin{aligned} P_t^\varepsilon[u, v] &= \int_\Sigma \Big(\sum_{i,j=1}^l b_{ij}\partial_i u\cdot\partial_j(q_\varepsilon v) + \sum_{i=1}^l b_i\partial_i u\cdot q_\varepsilon v + buq_\varepsilon v\Big)dy \\ &\quad + \int_\tau \delta u\cdot vKdy' + \int_\tau \beta uvKdy' + t\int_\tau uvKdy' \\ &= \mathrm{I}[u, v] + \mathrm{II}[u, v] + \mathrm{III}[u, v] + \mathrm{IV}[u, v] , \end{aligned}$$

with $b_{ij} = b_{ji}$. It then follows from (10) that there exists a constant $c_2'' > 0$ independent of ε, λ and t such that

$$c_2''(\|p_\varepsilon \partial u\|_{0,\Sigma}^2 + \|p_\varepsilon u\|_{0,\Sigma}^2 + (1+t)\|u\|_{0,\Gamma}^2) \le P_t^\varepsilon[u, u] , \qquad u \in C_0^\infty(U) . \tag{13}$$

For any multi-integers $\rho = (\rho_1, \cdots, \rho_{l-1})$ such that $|\rho| = \rho_1 + \cdots + \rho_{l-1} = r \ge 1$, we set

$$Tu = \partial^\rho(\zeta u) = \partial_1^{\rho_1}\cdots\partial_{l-1}^{\rho_{l-1}}(\zeta u)$$

with $\partial_j = \partial/\partial y_j$. In the following propositions all constants are independent of ε and $t \geq 0$.

PROPOSITION 3. *There exist positive constants* $C_{\mathrm{I}}, C_{\mathrm{II}}$ *and* C_{III} *depending only on the forms* I, II *and* III, *respectively, such that*

$$P_t^\varepsilon[Tu, Tu] - P_t^\varepsilon[u, K^{-1}T^*KTu] \leq C_{\mathrm{I}}(\|u\|_{r,\Sigma}\|\partial(q_\varepsilon Tu)\|_{0,\Sigma} + \|u\|_{r,\Sigma}^2) + C_{\mathrm{II}}\|u\|_{r,\tau}^2 + C_{\mathrm{III}}\|u\|_{r-1,\tau}\|Tu\|_{0,\tau}\,, \qquad u \in C^\infty(\boldsymbol{R}_y^n)\,,$$

where $K(y', y_l) = K(y')$.

Proof. (I) Setting $R = b_{ij}\partial_i$ and $S = \partial_j$, and writting simply $(\,,\,)_\Sigma = (\,,\,)$ and $[A, B] = AB - BA$, we can compute as follows:

$$\begin{aligned}(RTu, Sq_\varepsilon Tu) &= (Ru, T^*Sq_\varepsilon Tu) + ([R, T]u, Sq_\varepsilon Tu)\\ &= (Ru, T^*Sq_\varepsilon K^{-1}KTu) + ([R, T]u, Sq_\varepsilon Tu)\\ &= (Ru, Sq_\varepsilon K^{-1}T^*KTu) + (Ru, [T^*, Sq_\varepsilon K^{-1}]KTu)\\ &\quad + ([R, T]u, Sq_\varepsilon Tu)\\ &= (Ru, Sq_\varepsilon K^{-1}T^*KTu) + (Ru, [T^*, S]q_\varepsilon Tu)\\ &\quad + ([R, T]u, Sq_\varepsilon Tu) + (Ru, S[T^*, q_\varepsilon K^{-1}]KTu)\,.\end{aligned}$$

Thus

$$\mathrm{I}[Tu, Tu] - \mathrm{I}[u, K^{-1}T^*KTu] \leq C(\|u\|_{r,\Sigma}\|\partial(q_\varepsilon Tu)\|_{0,\Sigma} + \|u\|_{r,\Sigma}^2) + \int_\Sigma \sum_{i,j=1}^{l} b_{ij}\partial_i u\cdot\partial_j v\,dy\,, \tag{14}$$

where we put $v = [T^*, q_\varepsilon K^{-1}]KTu$. Now

$$\begin{aligned}(Ru, Sv) + (Rv, Su) &= (Ru, [T^*, q_\varepsilon K^{-1}]KTSu) + (Ru, [S, [T^*, q_\varepsilon K^{-1}]KT]u)\\ &\quad + ([T^*, q_\varepsilon K^{-1}]KTRu, Su) + ([R, [T^*, q_\varepsilon K^{-1}]KT]u, Su)\\ &= (Ru, \{[T^*, q_\varepsilon K^{-1}]KT + T^*K[q_\varepsilon K^{-1}, T]\}Su) + O(\|u\|_{r,\Sigma}^2)\,,\end{aligned}$$

which implies

$$|(Ru, Sv) + (Rv, Su)| \leq C\|u\|_{r,\Sigma}^2\,.$$

This together with (14) and the fact $b_{ij} = b_{ji}$ implies

$$\mathrm{I}[Tu, Tu] - \mathrm{I}[u, K^{-1}T^*KTu] \leq C_{\mathrm{I}}(\|u\|_{r,\Sigma}\|\partial(qTu)\|_{0,\Sigma} + \|u\|_{r,\Sigma}^2)\,.$$

(II) Next

$$\begin{aligned}\mathrm{II}[Tu, Tu] &= (T\delta u, KTu)_\tau + ([\delta, T]u, KTu)_\tau\\ &= (\delta u, KK^{-1}T^*KTu)_\tau + ([\delta, T]u, KTu)_\tau\,.\end{aligned}$$

Therefore we have

(15) $$\mathrm{II}[Tu, Tu] - \mathrm{II}[u, K^{-1}T^*KTu] = ([\delta, T]u, KTu)_\tau \leq C_{\mathrm{II}} \|u\|^2_{r,\tau}$$

(III) By the same way as (II) we have

$$\mathrm{III}[Tu, Tu] - \mathrm{III}[u, K^{-1}T^*KTu] \leq C_{\mathrm{III}} \|u\|_{r-1,\tau} \|Tu\|_{0,\tau} .$$

(IV) Finally

$$\mathrm{IV}[Tu, Tu] - \mathrm{IV}[u, K^{-1}T^*KTu] = 0 .$$

Thus (I), (II), (III) and (IV) conclude the proposition.

Now, by using (8), we shall estimate the term $P_t^\varepsilon[u, K^{-1}T^*KTu]$ with $u = u_\varepsilon$ which was introduced in Proposition 2. That is,

PROPOSITION 4. *We have, with a suitable constant* $C > 0$,

$$|P_t^\varepsilon[u_\varepsilon, K^{-1}T^*KTu_\varepsilon]| \leq C(\|p_\varepsilon \partial^{r-1} f\|_{0,\Sigma} \|p_\varepsilon \partial(KTu_\varepsilon)\|_{0,\Sigma} + \|f\|_{r-2+1/2,\Sigma} \|Tu_\varepsilon\|_{1/2,\Sigma} + \|\varphi\|_{r,\Sigma} \|Tu_\varepsilon\|_{0,\tau}) .$$

Proof. For the sake of simplicity, we write $u_\varepsilon = u$. Then

$$\begin{aligned} P_t^\varepsilon[u, K^{-1}T^*KTu] &= (Jq_\varepsilon f, K^{-1}T^*KTu)_\Sigma + (\varphi, KK^{-1}T^*KTu)_\tau \\ &= (\zeta K^{-1}Jq_\varepsilon f, \partial^\rho KTu)_\Sigma + (\varphi, T^*KTu)_\tau \\ &= (\zeta K^{-1}Jq_\varepsilon(-\partial)^{\rho'} f, \partial KTu)_\Sigma + ([(-\partial)^{\rho'}, \zeta K^{-1}Jq_\varepsilon]f, \partial(KTu))_\Sigma \\ &\quad + (\varphi, T^*KTu)_\tau \qquad (\partial^\rho = \partial^{\rho'}\partial) \\ &= (\zeta K^{-1}Jp_\varepsilon(-\partial)^{\rho'} f, p_\varepsilon \partial(KTu))_\Sigma \\ &\quad - (\partial[(-\partial)^{\rho'}, \zeta K^{-1}Jq_\varepsilon]f, KTu)_\Sigma + (\varphi, T^*KTu)_\tau , \end{aligned}$$

from which we easily obtain the proposition.

PROPOSITION 5. *There exists a constant* $C_0 > 0$ *such that*

$$\|p_\varepsilon \partial^{r+1} u_\varepsilon\|^2_{0,\Omega} + (1 + t)\|u_\varepsilon\|^2_{r,\Gamma} \leq C_0(\|u_\varepsilon\|^2_{r,\Omega} + \sum_{s=0}^{r-1} \|p_\varepsilon \partial^s f\|^2_{0,\Omega} + \|f\|_{r-2+1/2,\Omega} \|u_\varepsilon\|_{r+1/2,\Omega} + \|\varphi\|^2_{r,\Gamma} + C_{\mathrm{II}} \|u_\varepsilon\|^2_{r,\Gamma}) .$$

Proof. Using (13) and Proposition 3 with $u = u_\varepsilon$, we can obtain, with the aid of Proposition 4.

$$\|p_\varepsilon \partial Tu_\varepsilon\|^2_{0,\Sigma} + \|p_\varepsilon Tu_\varepsilon\|^2_{0,\Sigma} + (1 + t)\|Tu_\varepsilon\|^2_{0,\tau} \leq C_1(\|u_\varepsilon\|^2_{r,\Omega} + \sum_{s=0}^{r-1} \|p_\varepsilon \partial^s f\|^2_{0,\Omega} + \|f\|_{r-2+1/2,\Omega} \|u_\varepsilon\|_{r+1/2,\Omega} + \|\varphi\|^2_{r,,\Gamma} + C_{\mathrm{II}}(\|u_\varepsilon\|^2_{r,\Gamma}) \qquad (= C_1 F) .$$

Noting that this remains valid for any $\rho=(\rho_1,\cdots,\rho_{l-1})$ with $|\rho|\le r$, we have, with a suitable constant C_2,

$$\sum_{|\rho|\le r}(\|p_\varepsilon\partial^\rho\partial(\zeta u_\varepsilon)\|_{0,\Sigma}^2+\|p_\varepsilon\zeta u_\varepsilon\|_{0,\Sigma}^2+(1+t)\|\partial^\rho(\zeta u_\varepsilon)\|_{0,\tau}^2)\le C_2F\ .$$

With the aid of (11), we can assert that $\partial_l^2(\zeta u_\varepsilon)$ can be written by a linear conbination of $\partial_j\partial_l(\zeta u_\varepsilon)$, $\partial_j\partial_k(\zeta u_\varepsilon)$ $(j,k=1,\cdots,\ l-1)$, $\partial_j(\zeta u_\varepsilon)$ $(j=1,\cdots,l)$, ζu_ε, ζf and $[A,\zeta]u_\varepsilon$. Hence we have

$$\sum_{|\rho|\le r-1}\|p_\varepsilon\partial^\rho\partial^2(\zeta u_\varepsilon)\|_{0,\Sigma}^2+(1+t)\sum_{|\rho|\le r}\|\partial^\rho(\zeta u_\varepsilon)\|_{0,\tau}^2\le C_3F\ .$$

Repeating this process if $r>1$, we finally obtain

$$\|p_\varepsilon\partial^{r+1}(\zeta u_\varepsilon)\|_{0,\Sigma}^2+(1+t)\sum_{|\rho|\le r}\|\partial^\rho(\zeta u_\varepsilon)\|_{0,\tau}^2\le C_4F\ .$$

Clearly this remains also valid for $\zeta=\zeta_0$. Therefore applying this for $\zeta=\zeta_j$ $(j=0,\cdots,N)$ and using $\sum_{j=0}^N\zeta_j=1$ on $\bar{\Omega}$, we obtain

$$\|p_\varepsilon\partial^{r+1}u_\varepsilon\|_{0,\Omega}^2+(1+t)\|u_\varepsilon\|_{r,\Gamma}^2\le C_5F\ .$$

This completes the proof.

PROPOSITION 6. *For every integer $k\ge 2$, we can find two constant $C_k>0$ and $t_k\ge 0$ such that*

$$\|p_\varepsilon\partial^k u_\varepsilon\|_{0,\Omega}^2+\|u_\varepsilon\|_{k-1/2,\Omega}^2\le C_k(\|p_\varepsilon\partial^{k-2}f\|_{0,\Omega}^2+\|f\|_{k-2-1/2,\Omega}^2+\|\varphi\|_{k-1,\Gamma}^2)$$

is valid for all ε and $t\ge t_k$.

Proof. Using the preceding proposition in the case $k=r+1$ and $t\ge C_0C_{\text{II}}$ $(=t_k)$, we have

$$\begin{aligned}&\|p_\varepsilon\partial^k u_\varepsilon\|_{0,\Omega}^2+\|u_\varepsilon\|_{k-1,\Gamma}^2\\ &\quad\le C(\|u_\varepsilon\|_{k-1,\Omega}^2+\sum_{s=0}^{k-2}\|p_\varepsilon\partial^s f\|_{0,\Omega}^2+\|f\|_{k-2-1/2,\Omega}\|u_\varepsilon\|_{k-1/2,\Omega}+\|\varphi\|_{k-1\Gamma}^2)\ .\end{aligned}\tag{16}$$

From (11) and the coercive inequality for Dirichlet problem it follows

$$C'\|u_\varepsilon\|_{k-1/2,\Omega}^2-\|f\|_{k-2-1/2,\Omega}^2\le\|u_\varepsilon\|_{k-1,\Gamma}^2\ .\tag{17}$$

The interpolation inequality says that for any $\delta>0$ there exists a constant $C_\delta>0$ such that

$$\|u\|_{k-1,\Omega}^2\le\delta\|u\|_{k-1/2,\Omega}^2+C_\delta\|u\|_{0,\Omega}^2\ ,\qquad u\in C^\infty(\bar{\Omega})\ .\tag{18}$$

Thus, the inequalities (16), (17) and (18) together with (9) immediately imply the proposition.

In the below, Theorem 1 will be proved. We begin with the proof in case $f \in C^\infty(\bar{\Omega})$ and $\varphi \in C^\infty(\Gamma)$. So that we can use Propositions 1–6. Proposition 6 becomes, by using the notation (12),

$$\|u_\varepsilon ; p\|_k \leq C_k(\|f ; p_\varepsilon\|_{k-2} + \|\varphi\|_{k-1,\Gamma}) .$$

The theorem of Banach-Sacks guarantees that there exists a sequence $\varepsilon_1 > \varepsilon_2 > \cdots$ converging to zero such that, as $n \to \infty$,

$$v_n = \frac{u_{\varepsilon_1} + \cdots + u_{\varepsilon_n}}{n} \to u \qquad \text{in } H^k(\Omega ; p) .$$

From (8) we have, setting $B_\lambda[u,v] = B[u,v] + \lambda(u,v)$,

$$Q_{\lambda,t}[v_n, v] + B_\lambda\left[\frac{\varepsilon_1 u_{\varepsilon_1} + \cdots + \varepsilon_n u_{\varepsilon_n}}{n}, v\right]$$
$$= (qf, v)_\Omega + (\varphi, v)_\Gamma + \frac{\varepsilon_1 + \cdots + \varepsilon_n}{n}(f, v)_\Omega .$$

Noting that $v_n \to u$ and $\varepsilon_n u_{\varepsilon_n} \to 0$ in $H^{k-\frac{1}{2}}(\Omega)$ as $n \to \infty$, we can derive

$$(19) \qquad Q_{\lambda,t}[u, v] = (qf, v)_\Omega + (\varphi, v)_\Gamma , \qquad v \in C^\infty(\bar{\Omega}) ,$$

and hence the u satisfies (4). Moreover

$$\|v_n ; p\|_k \leq \frac{1}{n}(\|u_{\varepsilon_1} ; p\|_k + \cdots + \|u_{\varepsilon_n} ; p\|_k)$$
$$\leq C_k\left(\|f ; p\|_{k-2} + \|\varphi\|_{k-1,\Gamma} + \frac{\sqrt{\varepsilon_1} + \cdots + \sqrt{\varepsilon_n}}{n}\|\partial^{k-2} f\|_{0,\Omega}\right) .$$

Accordingly, we obtain (5) as $n \to \infty$. It is easily seen that the uniqueness of solution of (4) follows from (19) and Proposition 1.

Suppose now that f and φ are in $H^{k-2}(\Omega ; p)$ and $H^{k-1}(\Gamma)$, respectively. Let $f_j \in C^\infty(\bar{\Omega})$ and $\varphi_j \in C^\infty(\Gamma)$ $(j = 1, 2, \cdots)$ such that $f_j \to f$ in $H^{k-2}(\varphi ; p)$ and $\varphi_j \to \varphi$ in $H^{k-1}(\Gamma)$ as $j \to \infty$. For each j, we can find $u_j \in H^k(\Omega ; p)$ whose existence has just been proved, satisfying (4) and (5) with $f = f_j$ and $\varphi = \varphi_j$. We can immediately see that u_j converges to u in $H^k(\Omega ; p)$ an $j \to \infty$. Thus we finally obtain that u is the unique solution of (4) and satisfies (5).

§ 3. Proof of Corollary.

Assume that there exists an open neighbourhood U_0 of Γ_0 in $\boldsymbol{R}^l$ such that $\gamma = 0$ in $V_0 = \Gamma \cap U_0$, and that $(\Gamma - V_0) \cap U_j$ is transformed by κ_j to $\tau'_j \subset \tau_j$. Then we have instead of (15)

$$(15') \qquad |([\delta, T]u, KTu)_\tau| \leq C_{\text{II}} \|u\|^2_{r,\tau'} .$$

Hence we can change, in Proposition 5, the term $\|u_\varepsilon\|_{r,\Gamma}$ into $\|u_\varepsilon\|_{r,\Gamma - V_0}$. By the well known inequalities:

$$\begin{aligned}\|u\|_{r,\Gamma - V_0} &\leq \text{const.}\, \|u\|_{r+1/2,\Omega - U_0} \\ &\leq \delta \|u\|_{r+1,\Omega - U_0} + C_\delta \|u\|_{r,\Omega} \\ &\leq C(\delta \|p_\varepsilon \partial^{r+1} u\|_{0,\Omega} + C_\delta \|u\|_{r,\Omega}) ,\end{aligned}$$

we obtain Proposition 5 with $C_{\text{II}} = 0$. In this case we have $t_k = 0$ in Proposition 6. Thus we can assert Corollary.

§ 4. Proof of Theorem 2.

We assume that $\Gamma_0 = \{x \in \Gamma ; \alpha(x) = 0\}$ is a C^∞ manifold of dimension $l - 2$ and γ is transversal to Γ_0. Let $U_j, \kappa_j, \Sigma_j, \tau_j, J_j, K_j$ and ζ_j be the same in § 1. Here we further assume that for every j such that $U_j \cap \Gamma_0 \neq \emptyset$, the set $U_j \cap \Gamma_0$ is transformed onto an open portion τ_j^0 of $y_l = 0$, $y_1 = 0$ and γ is altered to $\delta_j = \partial_1$ by κ_j, and $\gamma(\zeta_j(x)) = 0$ in a neighbourhood V_0 of Γ_0.

LEMMA 4. *There exists a positive C^∞ function h on Γ such that*

$$\frac{1}{2}\gamma^*(h) + \beta h > 0 \qquad on\ \Gamma_0 .$$

Proof. By Lemma 1, we have only to find h such that $-\gamma h + (b + 2\beta)h > 0$ on Γ_0. For every j such that $U_j \cap \Gamma_0 \neq \emptyset$, let h_j be satisfying $-\partial_1 h_j + (b + 2\beta)h_j = 1$. Then $h = \Sigma \zeta_j h_j$ is a desired one, since $\gamma \zeta_j = 0$ on Γ_0.

Using this lemma, we can easily prove

LEMMA 2′. *We can find a function $q(x) \in C^\infty(\bar{\Omega})$ satisfying*

(i) *$q > 0$ in Ω and $q = h\alpha$ on Γ.*

(ii) (ii) *of Lemma* 2.

(iii) *There exists a positive constant c_1 such that*

$$\frac{1}{2}\partial_\nu q + \frac{1}{2}\gamma^*(h) + \beta h \geq c_1 \qquad on\ \Gamma .$$

If we define as

$$Q[u, v] = B[u, qv] + \int_{\Gamma} (h\gamma u + h\beta u)v d\sigma ,$$

then Propositions 1 and 2 with $t = 0$ remain valid. We shall now show that Proposition 3 also holds if $P_t^\varepsilon[u, K^{-1}T^*KTu]$ and $C_{\mathrm{II}} \|u\|_{r,\tau}^2$ are replaced with $P_t^\varepsilon[u, (hK)^{-1}T^*hKTu]$ and $C_{\mathrm{II}} \|u\|_{r,\tau'}^2$, where τ' denotes the same notation as in § 3. In (I) of the proof of Proposition 3 we have only to replace K with hK. In this case, the forms II and III become

$$\mathrm{II}[u, v] = \int_{\tau} \partial_1 u \cdot hKv d\sigma$$

and

$$\mathrm{III}[u, v] = \int_{\tau} \beta u \cdot hKv d\sigma .$$

Therefore we have

$$\begin{aligned} \mathrm{II}[Tu, Tu] &= (\partial_1 Tu, hKTu)_\tau \\ &= (T\partial_1 u, hKTu)_\tau + ([\partial_1, T]u, hKTu)_\tau \\ &= (\partial_1 u, hK(hK)^{-1}T^*hKTu)_\tau + ([\partial_1, T]u, hKTu)_\tau \\ &= \mathrm{II}[u, (hK)^{-1}T^*hKTu] + ([\partial_1, T]u, hKTu)_\tau . \end{aligned}$$

Hence

$$\mathrm{II}[Tu, Tu] - \mathrm{II}[u, (hK)^{-1}T^*hKTu] \le C_{\mathrm{II}} \|u\|_{r,\tau'}^2 ,$$

since $\partial_1 \zeta = 0$ in V_0. It is obvious that

$$\mathrm{III}[Tu, Tu] - \mathrm{III}[u, (hK)^{-1}T^*hKTu] \le C_{\mathrm{III}} \|u\|_{r-1,\tau} \|Tu\|_{0,\tau} .$$

Thus, Proposition 3 can be concluded in our case.

By the same argument as in § 3, we obtain Proposition 5 with $C_{\mathrm{II}} = 0$. Finally we can complete the proof of Theorem 2 by the same argument as in the proof of Theorem 1.

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